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BMS Particles in Three Dimensions

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Blagoje Oblak

BMS Particles in Three Dimensions

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Мојој најмилијој породици,
с много љубави.

Supervisor's Foreword

Symmetries play a key role in our understanding of Nature. For instance, all of particle physics crucially relies on the Poincaré symmetry group of Minkowski space-time. Once gravity is switched on, however, the isometry group of generic space-time manifolds is empty and Poincaré symmetry becomes irrelevant. What replaces it are asymptotic symmetries: Those are the symmetries of a space-time manifold seen by a "faraway" observer.

About fifty years ago, Bondi, Metzner, van der Burg, and Sachs studied the asymptotic symmetries at null infinity of Einstein gravity on a Minkowskian background. What they found, quite surprisingly, was that Poincaré symmetry is extended into an infinite-dimensional group now known as the Bondi-Metzner-Sachs (BMS) group. More recent developments confirm that such infinite-dimensional extensions of exact isometries are actually quite common. In addition, there are reasons to believe that BMS symmetry can be extended further so as to contain local conformal transformations; in that picture, BMS consists of infinite-dimensional "superrotations" and "supertranslations" in the same way that the Poincaré group consists of finite-dimensional Lorentz transformations and translations.

Ever since its discovery, the BMS group has been conjectured to play a central role in the quest for a quantum theory of gravity. In the last couple of years, exciting new proposals indicate the existence of hitherto unexplored degrees of freedom, closely connected to and controlled by the BMS group, that may eventually account for the Bekenstein-Hawking entropy of realistic black holes in four dimensions.

The complexity of the four-dimensional problem suggests that a good strategy is to turn to a toy model. A natural candidate is provided by three-dimensional gravity, where beautiful asymptotic symmetry groups have been known to exist ever since the work of Brown and Henneaux in the eighties. Accordingly, this thesis is devoted to BMS symmetry in three space-time dimensions. It addresses the specific problem of classifying the irreducible unitary representations of BMS_3 ("BMS particles"), and relating them to quantum gravity. The material is presented in a self-contained and pedagogical manner, with all the necessary background collected in the first seven chapters. This allows the reader to fully appreciate the original results that have been obtained while learning many fundamental concepts in the field along the way. It makes the present work a perfect point of entry into the matter and a most rewarding read for anyone with a serious interest in BMS symmetry, or asymptotic symmetries in general.

Brussels, Belgium
June 2017

Prof. Glenn Barnich

Abstract

This thesis is devoted to the group-theoretic aspects of three-dimensional quantum gravity on Anti-de Sitter and Minkowskian backgrounds. In particular, we describe the relation between unitary representations of asymptotic symmetry groups and gravitational perturbations around a space-time metric. In the asymptotically flat case, this leads to *BMS particles*, representing standard relativistic particles dressed with gravitational degrees of freedom accounted for by coadjoint orbits of the Virasoro group. Their thermodynamics are described by BMS characters, which coincide with gravitational one-loop partition functions. We also extend these considerations to higher spin theories and supergravity.

Preface

This thesis collects thoughts and results that originate from a four-year-long research project in theoretical physics. The main topic is representation theory and its application to quantum gravity, in particular in the context of BMS symmetry. The text consists of three parts:

Part I: Group theory;
Part II: Virasoro symmetry and AdS_3/CFT_2;
Part III: BMS symmetry in three dimensions.

It is written in such a way that each part can be read more or less independently of the others, although the later parts do depend on background material presented in the earlier ones; the logical flow of chapters is explained in Sect. 1.5. A few sections are marked with an asterisk; they contain somewhat more advanced material that may be skipped without affecting the reading of the main track.

Brussels, Belgium Dr. Blagoje Oblak

The original contributions of this thesis are based on the following publications:

- G. Barnich and B. Oblak, "Holographic positive energy theorems in three-dimensional gravity," *Class. Quant. Grav.* **31** (2014) 152001, 1403.3835.
- G. Barnich and B. Oblak, "Notes on the BMS group in three dimensions: I. Induced representations," *JHEP* **06** (2014) 129, 1403.5803.
- G. Barnich and B. Oblak, "Notes on the BMS group in three dimensions: II. Coadjoint representation," *JHEP* **03** (2015) 033, 1502.00010.
- B. Oblak, "Characters of the BMS Group in Three Dimensions," *Commun. Math. Phys.* **340** (2015), no. 1, 413–432, 1502.03108.
- G. Barnich, H.A. González, A. Maloney, and B. Oblak, "One-loop partition function of three-dimensional flat gravity," *JHEP* **04** (2015) 178, 1502.06185.
- B. Oblak, "From the Lorentz Group to the Celestial Sphere," *Notes de la Septième BSSM*, U.L.B. (2015). 1508.00920.
- A. Campoleoni, H.A. González, B. Oblak, and M. Riegler, "Rotating Higher Spin Partition Functions and Extended BMS Symmetries," *JHEP* **04** (2016) 034, 1512.03353.
- H. Afshar, S. Detournay, D. Grumiller, and B. Oblak, "Near-Horizon Geometry and Warped Conformal Symmetry," *JHEP* **03** (2016) 187, 1512.08233.
- A. Campoleoni, H.A. González, B. Oblak, and M. Riegler, "BMS Modules in Three Dimensions," *Int. J. Mod. Phys.* **A31** (2016), no. 12, 1650068, 1603.03812.

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Professional Community

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Physics and Art

At this point, I would like to thank two professors who have been most important for my development, both as a scientist and as a person. The first is my high school physics teacher, Emmanuel Thiran, who first managed to show me a glimpse of the beauty of Nature and the thrill of lifting its veil. While indirect, his influence pervades the entirety of my approach to physics and guides me to this very day. The second is Michel Laurent, my piano teacher. For more than a decade, he has been showing me the subtleties of music, but in truth, his teaching extends far beyond that. While difficult to express in words, the conceptions of art and beauty that he has conveyed to me have had a great influence on me, and hence on this thesis.

Financial and Logistic Support

To conclude, I wish to thank the institutions who have provided me with financial and logistic support throughout my Ph.D. Above all, I am grateful to the *Fonds de la Recherche Scientifique*—F.N.R.S. for the grant (number FC-95570) that has allowed me to make a living out of the most delightful of occupations. On the other hand, the financial support allowing me to work at the University of Cambridge during the last academic year has been granted to me by the *Fondation Wiener-Anspach* . I gratefully acknowledge their support in this marvelous experience, and wish to thank particularly Nicole Bosmans for her help with logistics.

Contents

Chapter 1
Introduction

The quantization of gravity is one of the long-standing puzzles of theoretical physics. The purpose of this thesis is to study certain aspects of the problem that can be studied on the sole basis of symmetries, without any assumptions on the underlying microscopic theory. In this introduction we describe this strategy in some more detail, starting in Sect. 1.1 with a broad overview of asymptotic symmetries in general and Bondi-Metzner-Sachs (BMS) symmetry in particular. We then introduce the distinction between global and extended BMS groups in Sect. 1.2. Section 1.3 is devoted to a lightning review of AdS/CFT and its putative Minkowskian counterpart. Finally, in Sect. 1.4 we describe the relation between BMS symmetry and soft graviton degrees of freedom. Section 1.5 contains a general presentation of the upcoming chapters and describes their logical flow.

1.1 Asymptotic BMS Symmetry

The notion of symmetry is a cornerstone of physics and mathematics. A system is *symmetric* if there exists a set of transformations that leave it invariant, i.e. that preserve its structure. In physical terms, saying that a system has symmetries is really saying that there exist certain transformations that can be performed without affecting the outcome of experiments. For instance, translational symmetry is the statement that the result of an experiment does not depend on where one carries it out. By construction, the set of symmetry transformations of a system forms a *group*, so the mathematical tool used in the study of symmetries is group theory.

In this thesis we shall be concerned with symmetries of gravitational systems, that is, changes of coordinates that can be applied to space-time and that leave invariant the large-distance behaviour of the gravitational field. They are known as *asymptotic symmetries* and can be thought of as a generalization of Poincaré symmetry for systems endowed with a weak gravitational field. In other words,

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these symmetries are those one would observe by looking at a gravitational system "from far away". In that context, the general type of question that we will ask is the following: given the asymptotic symmetries of a gravitational system, what are their physical implications? In particular, how do these symmetries affect one's intuition about particle physics?

Asymptotic symmetries of gravitational systems have been studied for about 50 years by now. Their first appearance in the literature is also the one that motivates the present work. Indeed, it was observed in the sixties by Bondi, van der Burg, Metzner [1, 2] and Sachs [3, 4] that the presence of gravitation in an asymptotically flat space-time leads to a symmetry group that is much, *much* larger than standard Poincaré. The group that they found turned out to be an infinite-dimensional extension of the Poincaré group, and is known today as the *Bondi-Metzner-Sachs group*, or *BMS group* for short.

The BMS group considered by the authors of [2–4] consists of two pieces: the first is the standard Lorentz group of special relativity, and the second is an infinite-dimensional Abelian group of so-called *supertranslations*.[1] In abstract mathematical notation, its structure can be written symbolically as

$$\text{BMS} = \text{Lorentz} \ltimes \text{Supertranslations}. \tag{1.1}$$

The notation $\ltimes$ used here means that elements of the BMS group are pairs consisting of a Lorentz transformation and a supertranslation, and that Lorentz transformations act non-trivially on supertranslations. In the same way, the Poincaré group is

$$\text{Poincare} = \text{Lorentz} \ltimes \text{Translations}. \tag{1.2}$$

The latter is a subgroup of BMS: the group of space-time translations is contained in the infinite-dimensional group of supertranslations.

Groups of the form (1.1) or (1.2) are known as *semi-direct products*. They are ubiquitous in physics, and many of the conclusions of this thesis rely on this structure.

1.2 Global BMS and Extended BMS

In this section we introduce Bondi coordinates to explain briefly how BMS symmetry emerges from an asymptotic analysis. We then describe the distinction between "global" and "extended" BMS transformations.

Bondi Coordinates

Consider Minkowski space-time, endowed with inertial coordinates x^μ in terms of which the metric reads

[1]The terminology of "super-things" here has nothing to do with supersymmetry: "super-object" simply means that a certain object, which one is familiar with in the finite-dimensional context of special relativity, gets extended in an infinite-dimensional way in the BMS group.

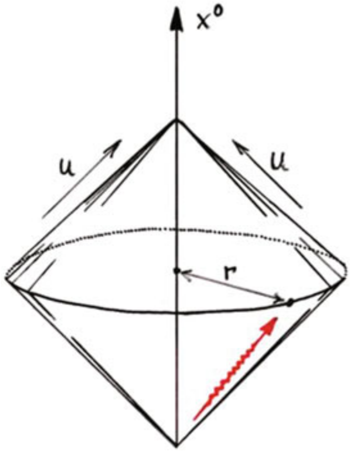

Fig. 1.1 The coordinates u and r in space-time. The time coordinate x^0 points upwards. The wavy *red line* represents an outgoing radial massless particle emitted at $r = 0$ and moving to some non-zero distance r away from the observer at $r = 0$; the particle moves along one of the generators of the light cone given by $u = \text{const}$. The drawing is three-dimensional, so the circle of radius r in this picture would actually be a sphere (spanned by the coordinate z) in a four-dimensional space-time

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu, \qquad \text{with } (\eta_{\mu\nu}) = \text{diag}(-1, +1, +1, +1). \tag{1.3}$$

Now suppose we wish to study, say, outgoing massless particles sent by an observer located at the spatial origin. For this purpose we introduce *retarded Bondi coordinates*

$$r \equiv \left[x^i x^i\right]^{1/2}, \qquad z \equiv \frac{x^1 + ix^2}{r + x^3}, \qquad u \equiv x^0 - r. \tag{1.4}$$

Here r is a space-like radial coordinate, z is a stereographic coordinate on the sphere of radius r (such that the north and south poles respectively correspond to $z = 0$ and $z = \infty$), and u is known as *retarded time*. In these coordinates the Minkowski metric (1.3) reads

$$ds^2 = -du^2 - 2\,du dr + r^2 \frac{4dzd\bar{z}}{(1 + z\bar{z})^2} \tag{1.5}$$

and the world line of an outgoing massless particle (moving away from the origin) is of the form $u = \text{const.}$, $z = \text{const.}$:

In terms of Bondi coordinates, the region reached by massless particles emitted at some moment from the origin $r = 0$ is a sphere at null infinity ($r \to +\infty$) spanned by the complex coordinate z, called a (future) *celestial sphere*. There is one such sphere for each value of retarded time u; the succession of all possible celestial spheres is a manifold $\mathbb{R} \times S^2$ located at $r \to +\infty$ and known as *future null infinity*. It is the region where all outgoing massless radiation "escapes" out of space-time; it is the upper null cone of the Penrose diagram of Minkowski space-time.

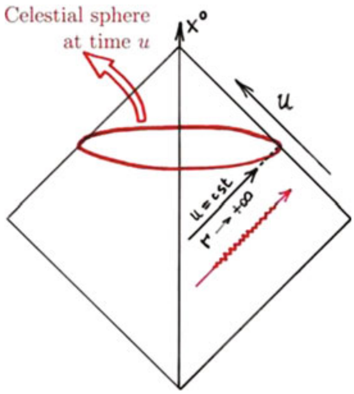

Fig. 1.2 A representation of celestial spheres on the Penrose diagram of Minkowski space. As in Fig. 1.1, the wavy *red line* represents an outgoing radial light ray. The drawing is three-dimensional, so the *red circle* at the *top* of the picture would really be a sphere — a celestial sphere — in a four-dimensional space-time. Future null infinity is the cone $\mathbb{R} \times S^2$ on the upper half of the image, spanned by u and z

There exists a parallel construction of Bondi coordinates which is convenient for the study of *past* null infinity. These are *advanced Bondi coordinates* (r, z, v), defined in terms of inertial coordinates x^μ exactly as in (1.4) up to a sign difference in retarded time: $v = x^0 + r$. The spheres at $r \to +\infty$ then are past celestial spheres and they foliate past null infinity (spanned by v and z) in slices of constant time. In particular, all incoming massless particles originate from past null infinity (Fig. 1.2).

Asymptotic Flatness and the BMS Group

We now have the tools needed to introduce BMS symmetry. First, one declares that a space-time manifold is *asymptotically flat* at, say, future null infinity, if it admits local coordinates (r, z, u) such that, as r goes to infinity with u finite, the metric takes the form (1.5) up to subleading corrections. These coordinates need not be defined globally — all that is needed is that they span a neighbourhood of future null infinity. Also, there is a precise definition of what is meant by "subleading corrections"; these are alterations of the Minkowski metric (1.5) that typically decay as inverse powers of r at infinity, but the allowed powers themselves are constrained in a specific way. These constraints are motivated by physical considerations and they are part of the definition of "asymptotic flatness". (We will not deal with these subtleties for now, but we shall display them in Sect. 9.1 in the three-dimensional case.)

The notion of asymptotic flatness allows one to define the associated asymptotic symmetry group. The latter consists, roughly speaking, of diffeomorphisms of space-time that preserve the asymptotic behaviour of the metric.[2] Bondi et al. [2–4] found that there are two families of such diffeomorphisms:

- The first family consists of Lorentz transformations. Their effect at null infinity is that of conformal transformations of celestial spheres (i.e. Möbius transformations) given in terms of the stereographic coordinate z of (1.4) by

$$z \mapsto \frac{az+b}{cz+d} + \mathcal{O}(1/r), \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{C}). \tag{1.6}$$

  This property relies on the isomorphism[3] $\mathrm{SO}(3,1)^{\uparrow} \cong \mathrm{SL}(2,\mathbb{C})/\mathbb{Z}_2$, which expresses Lorentz transformations in terms of $\mathrm{SL}(2,\mathbb{C})$ matrices. Lorentz transformations also act on the coordinates r and u at infinity by angle-dependent rescalings, but this subtlety is unimportant at this stage.
- The second family consists of angle-dependent translations of retarded time,

$$u \mapsto u + \alpha(z, \bar{z}), \tag{1.7}$$

  where $\alpha(z,\bar{z})$ is any (smooth) real function on the sphere. In this language, Poincaré space-time translations are reproduced by functions α which are linear combinations of the functions

$$1\,, \qquad \frac{1-z\bar{z}}{1+z\bar{z}}\,, \qquad \frac{z+\bar{z}}{1+z\bar{z}}\,, \qquad \frac{i(z-\bar{z})}{1+z\bar{z}}\,.$$

  (In terms of polar coordinates θ and φ, this corresponds to the spherical harmonics $Y_{00}(\theta,\varphi)$ and $Y_{1,m}(\theta,\varphi)$ with $m=-1,0,1$.) This is the main surprise discovered by Bondi et al. It states that asymptotic symmetries (as opposed to isometries) enhance the Poincaré group to an infinite-dimensional group with an infinite-dimensional Abelian normal subgroup consisting of transformations (1.7). These transformations are the *supertranslations* alluded to in Eq. (1.1).

Extended BMS

The group of asymptotic symmetry transformations (1.6) and (1.7) is the original BMS group discovered in [2–4]. It consists of globally well-defined, invertible transformations of null infinity, so from now on we call it the *global* BMS group. This slight terminological alteration is rooted in one of the most intriguing aspects of BMS symmetry. Indeed, in their work, Bondi et al. observed that asymptotic symmetries include conformal transformations (1.6), but in principle one may even include transformations generated by arbitrary (generally singular) conformal Killing vector fields on the celestial spheres. Only six of those vector fields generate the invertible Möbius

[2]More precisely, the asymptotic symmetry group is the quotient of the group of diffeomorphisms that preserve the asymptotic behaviour of the metric by its normal subgroup consisting of so-called trivial diffeomorphisms. We will return to this in Sect. 8.1.

[3]$\mathrm{O}(3,1)$ is the Lorentz group in four dimensions and $\mathrm{SO}(3,1)^{\uparrow}$ is its largest connected subgroup.

transformations (1.6); the remaining ones are singular. Upon including these extra generators, the global conformal transformations (1.6) are enhanced to arbitrary local conformal transformations

$$z \mapsto f(z) + \mathcal{O}(1/r), \tag{1.8}$$

where $f(z)$ is any meromorphic function. Despite their singularities, these transformations do preserve the asymptotic behaviour of the metric and may therefore qualify as asymptotic symmetries, at least infinitesimally.

The extension of the BMS group obtained by replacing Lorentz transformations by local conformal transformations (1.8) is called the *extended* BMS group.[4] In that context, local conformal transformations of celestial spheres are known as *superrotations* and should be thought of as an infinite-dimensional extension of Lorentz transformations, in the same way that supertranslations extend space-time translations. In the notation of (1.1) and (1.2), the extended BMS group looks like

$$\boxed{\text{Extended BMS} = \text{Superrotations} \ltimes \text{Supertranslations}} \tag{1.9}$$

where now both factors of the semi-direct product are infinite-dimensional.

It was recently suggested by Barnich and Troessaert [5, 6] that extended (as opposed to global) BMS symmetry is the true, physically relevant symmetry of asymptotically flat gravitational systems in four dimensions (see also footnote 17 of [7]). This proposal is motivated by a similar symmetry enhancement occurring in two-dimensional conformally-invariant systems: while their global symmetry algebra is finite-dimensional, they turn out to enjoy a much richer infinite-dimensional symmetry. This observation first appeared in a seminal paper by Belavin, Polyakov and Zamolodchikov [8] and triggered the development of two-dimensional conformal field theory (CFT).

Thus, the truly thrilling aspect of extended BMS symmetry is the prospect of applying conformal field-theoretic techniques to gravitational phenomena in four dimensions. This reduction from four to two dimensions is reminiscent of holograms, and indeed the notion of "holography" in quantum gravity is one of the main motivations that led to these considerations.

1.3 Holography

The elementary concept of holography in quantum gravity is simple: it is the statement that gravitational phenomena occurring in a certain space-time manifold can be described equivalently in terms of some lower-dimensional, "dual" theory. This

[4]Strictly speaking, local conformal transformations do not span a group but a semi-group, and the same applies to extended BMS. This abuse of terminology is pretty common, and it will be inconsequential for the discussion of this introduction.

idea is originally due to 't Hooft [9] and Susskind [10], who were led to it by model-independent considerations. In particular, holography is compatible with the Bekenstein-Hawking entropy [11, 12] formula according to which the entropy of a black hole is proportional to the area of its horizon. (The keyword here is "area", as opposed to the "volume" expected on the basis of standard thermodynamics.)

AdS/CFT

In practice, the first genuine illustration of holography in a concrete model of quantum gravity — namely string theory — was exhibited by Maldacena [13], initiating what has come to be known as the Anti-de Sitter/Conformal Field Theory (AdS/CFT) correspondence. The latter states, in a nutshell, that (quantum) gravity on a D-dimensional asymptotically Anti-de Sitter space-time is dual to a $(D-1)$-dimensional conformal field theory. The CFT may be seen as living on the boundary of AdS, that is, at spatial infinity, and is supposed to capture all the information on gravitational observables. This is a statement of *duality*, where two completely different theories contain the same physical information. While there is (as yet) no proof of the full equivalence, a substantial amount of checks have been carried out to confirm that gravity on AdS and a suitable CFT on its boundary do indeed produce the same physical predictions.

The case of a three-dimensional bulk space-time ($D = 3$) is especially important for our purposes. In that context the first hint of a holographic duality actually dates back to the eighties, when Brown and Henneaux [14] noticed that the asymptotic symmetries of AdS_3 gravity are infinite-dimensional. Analogously to the Minkowskian setting studied two decades earlier by Bondi et al., Brown and Henneaux found that asymptotic symmetries enhance the usual AdS_3 isometry algebra $\mathfrak{so}(2, 2)$ in an infinite-dimensional way and span the algebra of local conformal transformations (1.8) in two dimensions. In addition, the conserved charges generating these symmetries turn out to satisfy a centrally extended algebra, with a central charge proportional to the AdS radius measured in Planck units, now known as the *Brown-Henneaux central charge*. The latter is the one key parameter specifying the putative two-dimensional CFT dual to gravity on AdS_3. For instance, it was used in [15] to show that the entropy of black holes in three dimensions [16, 17] can be reproduced by a purely conformal field-theoretic computation.

The proposal that BMS symmetry might account for gravitational physics in asymptotically flat space-times is similar in spirit to AdS/CFT. The problem is that most known holographic constructions rely on the key assumption that the bulk space-time is endowed with a negative cosmological constant, i.e. that it is of the AdS type. This leads to a natural question: how should one deal with holography in flat space?

Flat Space Holography

In the context of AdS/CFT, the dual theory of gravity is a CFT; in particular, even without full knowledge of the dual theory, one can at least hope to make sense of it by relying on the well understood consequences of conformal invariance. By contrast, in asymptotically flat space-times, the concept of a "dual theory" is unclear, partly due to poorly understood symmetries and partly because (in dimension four or higher) gravitational waves cross the null boundary of space-time; in fact, one may ask whether flat space holography makes any sense to begin with. In view of this pessimistic omen, a safe approach to the problem is to avoid unnecessary assumptions and rely solely on the one known property of asymptotically flat gravity, namely BMS symmetry. Indeed, whatever flat space holography means, if a dual theory exists, then it must be invariant under a certain version of BMS.

The most interesting incarnation of BMS symmetry is the four-dimensional one, since it is relevant to macroscopic gravitational waves. Unfortunately, the structure of the extended BMS (semi-)group in four dimensions is very poorly understood (despite recent progress [18, 19]). In short, this structure appears to be such that standard group theory fails to apply. One is thus led to study toy models that capture the key features of BMS symmetry without the complications of a four-dimensional world.

In this thesis we argue that the *BMS group in three dimensions* [20], or BMS_3, provides such a toy model. We shall see that it displays the extended structure (1.9) in a simplified and controlled setting, and successfully accounts for many aspects of three-dimensional asymptotically flat gravity, both classically and quantum-mechanically. The BMS_3 group is the main actor of this work and we will use it to develop our intuition on flat space holography in general, including the four-dimensional case.

Holography as an Erlangen Programme

Aside from the study of quantum gravity in asymptotically flat space-times, this thesis puts a strong accent on the relation between group theory and physics. Most, if not all, of the topics that we will encounter in both $\mathrm{AdS}_3/\mathrm{CFT}_2$ and flat space holography follow from the properties of suitable groups — Virasoro and BMS_3, respectively. For instance, the phase space of gravity will turn out to coincide with the space of the coadjoint representation of its asymptotic symmetry group, and its quantization will produce families of unitary representations of that group. In this sense, three-dimensional gravity and its "holographic" properties can be reformulated as statements in group theory.

In hindsight this observation is not too surprising. Indeed, Klein's Erlangen programme [21] posits that geometric statements can be recast in the language of group theory. This point of view has led to numerous developments in mathematics throughout the twentieth century (including e.g. the work of Poincaré on special relativity). Since general relativity is essentially the dynamics of pseudo-Riemannian geometry, it is natural that the programme should apply to it as well provided one identifies the correct symmetry group. In particular, holography may sometimes be seen as an Erlangen programme in disguise.[5]

[5] See e.g. the Wikipedia page https://en.wikipedia.org/wiki/Erlangen_program.

1.4 BMS Particles and Soft Gravitons

The symmetry representation theorem of Wigner [22] states that the symmetry group of any quantum-mechanical system acts on the corresponding Hilbert space by unitary transformations. Accordingly, a natural first step in the study of BMS symmetry is the construction and classification of its irreducible, unitary representations.

Since the Poincaré group is a subgroup of BMS, it provides a first rough picture of what one should expect from BMS representations. Indeed, irreducible unitary representations of Poincaré are, by definition, *particles*: they are classified by their mass and spin [23] and their Hilbert space accounts for the available one-particle states. The BMS group is expected to generalize this notion in a way that incorporates certain gravitational effects. It should describe the quantum states of a particle, plus some extra degrees of freedom accounting for the fact that BMS is only an *asymptotic*, rather than an exact, symmetry group. Guided by this picture, we introduce the following terminology:

A BMS particle is an irreducible, unitary representation of the BMS group.

This thesis is devoted to the description and classification of such particles in three space-time dimensions.

Recent developments in the study of BMS symmetry provide a simple interpretation for BMS particles. Indeed, it was observed in [19, 24] that, in four dimensions, the statement of supertranslation-invariance of the gravitational S-matrix is equivalent to Weinberg's soft graviton theorem [25]. This result was subsequently generalized to include superrotations [26, 27], producing a subleading term in the soft graviton expansion of the S-matrix. The bottom line of these considerations is that BMS symmetry describes the soft sector of gravity, that is, the one consisting of infinite-wavelength gravitational degrees of freedom. The interpretation of BMS particles follows: they are particles (in the standard sense) dressed with soft gravitons. Dressed particles are indeed ubiquitous in the quantization of gauge theories [28–36], and this in itself is not a new result. What *is* new, however, is the fact that this dressing is accounted for by a symmetry principle that generalizes Poincaré; this is the key content of the relation between BMS symmetry and soft theorems.

Accordingly, the classification of BMS particles that we expose in this thesis may be thought of as a classification of all possible ways to dress a Poincaré particle with soft gravitons. A word of caution is in order: since we will be working in three space-time dimensions, the gravitational field will have no local degrees of freedom so there will be no genuine gravitons. In particular, the name "soft graviton" is ambiguous, as there is no actual graviton whose zero-energy limit would be a soft particle. However, asymptotic symmetries precisely account for soft graviton degrees of freedom, so we shall adopt the viewpoint that any system with non-trivial asymptotic symmetries does indeed have non-trivial soft degrees of freedom. This amounts to using the words "soft graviton" as a synonym for the more standard "topological" or "boundary degree of freedom". In particular, three-dimensional gravitational systems generally do have highly non-trivial asymptotic symmetries [14, 37] and therefore possess soft degrees of freedom in this sense. In this language, the statement that three-dimensional gravity has no bulk degrees of freedom turns into the fact that the *only* non-trivial degrees of freedom of three-dimensional gravity are soft.

Remark Unitary representations of the globally well-defined BMS group (1.1) have already been classified by McCarthy and others in [38–40], and it was indeed suggested in [41] that BMS symmetry is relevant to particle physics in that it provides a better definition of the notion of "particle". However, these representations appear to miss the fact that supertranslations create soft gravitons when acting on the vacuum, which is crucial for the application of BMS symmetry to soft theorems. In this sense the understanding of BMS particles in four dimensions is still an open problem; it suggests that some extension of (1.1) is necessary if representations of BMS are to reflect reality. We shall comment further on this issue in Sect. 10.1.

1.5 Plan of the Thesis

We now describe the topics studied in this thesis. The latter is divided in three parts, devoted respectively to group theory in quantum mechanics, to the Virasoro group, and to the BMS_3 group.

Quantum Symmetries

The first part of the thesis deals with the implementation of symmetries in quantum mechanics through projective unitary representations, which are worked out in detail for the Poincaré groups and the Bargmann groups. It consists of four chapters.

Quantum symmetries generally act in a projective way, which is to say that the group operation of the underlying symmetry group is represented up to certain constant phases. The presence of such phases is captured by *central extensions* of the symmetry group. Accordingly, Chap. 2 is devoted to central extensions and to the more general notion of group and Lie algebra cohomology. Chapter 3 then explains how one can build Hilbert spaces of wavefunctions on a homogeneous space endowed with a unitary action of a symmetry group. This involves the important notion of *induced representations*, which we discuss in detail.

As an application, in Chap. 4 we describe the irreducible unitary representations of semi-direct products of the general form (1.1) or (1.2). As it turns out, *all* these representations are induced representations and consist of wavefunctions on a *momentum orbit*. This provides an exhaustive classification of unitary representations for such groups. We illustrate these considerations with the Poincaré group (in any space-time dimension) and with its non-relativistic counterpart, the Bargmann group, corresponding respectively to relativistic and Galilean particles.

Finally, Chap. 5 describes the relation between classical and quantum symmetries through geometric quantization. In a nutshell this relation is obtained by defining a space of wavefunctions on what is known as a *coadjoint orbit* of a symmetry group.

For semi-direct products this approach reproduces the classification of representations by momentum orbits and leads to a group-theoretic version of the world line formalism.

Remark The tools used in Chaps. 2–4 rely on elementary group theory; we refer for instance to [42] for an introduction. The language of Chap. 5, on the other hand, relies more heavily on differential and symplectic geometry; see e.g. [43, 44] for some background material.

Virasoro Symmetry and AdS_3 gravity

The second part of the thesis deals with the Virasoro group and its application to three-dimensional gravity on Anti-de Sitter backgrounds. It consists of three chapters. The material exposed in part II relies in a crucial way on Chap. 2 and to a lesser extent on Chap. 5, but is independent of the considerations of Chaps. 3 and 4.

Chapter 6 is devoted to the construction of the Virasoro group as a central extension of the group of diffeomorphisms of the circle and introduces its coadjoint representation. The latter coincides with the transformation law of stress tensors in two-dimensional conformal field theory. In Chap. 7 we classify the orbits of this action, i.e. the coadjoint orbits of the Virasoro group, and observe that they look roughly like infinite-dimensional cousins of Poincaré momentum orbits.

In Chap. 8 we show how Virasoro symmetry emerges in AdS_3 gravity with Brown-Henneaux boundary conditions, after explaining some basic notions on asymptotic symmetries in general. We also show that the phase space of AdS_3 gravity is embedded as a hyperplane at constant central charge in the space of the coadjoint representation of two copies of the Virasoro group. As an application we relate highest-weight representations of the Virasoro algebra to the quantization of gravitational boundary degrees of freedom.

BMS_3 Symmetry and Gravity in Flat Space

The third and last part of the thesis is devoted to three-dimensional BMS symmetry and contains most of the original contributions of this work. It consists of three chapters, plus a conclusion. The material presented in part III relies crucially on the content of parts I and II.

In Chap. 9 we introduce BMS_3 symmetry by way of an asymptotic analysis of Brown-Henneaux type applied to Minkowskian backgrounds, and show that the resulting algebra of surface charges has a classical central extension. We then put this observation on firm mathematical ground by defining rigorously the BMS_3 group and its central extension. We also show that the phase space of asymptotically flat gravity is a hyperplane at fixed central charges embedded in the space of the coadjoint representation of BMS_3 [45, 46].

Chapter 10 is devoted to the quantization of BMS_3 symmetry, i.e. to its irreducible unitary representations [47]. In the language introduced above, each representation is a *BMS_3 particle*. We show that the supermomentum orbits that classify these particles coincide with coadjoint orbits of the Virasoro group and describe the resulting Hilbert spaces of one-particle states. This leads in particular to the interpretation of BMS_3 particles as particles dressed with gravitational degrees of freedom.

Finally, Chap. 11 deals with rotating one-loop partition functions of quantum fields in flat space at finite temperature. Each partition function takes the form of an exponential of Poincaré characters. In three space-time dimensions and for a massless field with spin two, the combination of characters is precisely such that the whole partition function coincides with the character of a unitary representation of the BMS_3 group [48, 49]. For higher spins in three dimensions we similarly obtain characters of flat non-linear $\mathcal{W}_N$ algebras [50]. Along the way we describe unitary representations of these algebras [51] and show that they differ qualitatively from earlier proposals in the literature. We end by describing certain supersymmetric extensions of the BMS_3 group, their representations, and their characters.

Remark The group-theoretic methods developed in this thesis apply to essentially any symmetry group involving the Virasoro group. In particular one can use this approach to derive the transformation laws of the stress tensor of a warped conformal field theory for all values of its three central charges. Since these considerations are somewhat out of our main line of thought we will not review them in this thesis and refer instead to [52], where they were used to derive a Cardy-like formula for the entropy of Rindler backgrounds.

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Part I
Quantum Symmetries

In this part, we describe symmetry groups in quantum mechanics along three related lines of thought. First, we argue that the action of symmetry transformations in quantum mechanics is unitary up to phases, which leads to central extensions. Then, we show how to build concrete unitary representations using the method of induced representations, which we apply to the description of relativistic particles. Finally, we describe the general relation between unitary representations and homogeneous spaces through geometric quantization.

Chapter 2
Quantum Mechanics and Central Extensions

In this short chapter we discuss the implementation of symmetries in a quantum-mechanical context. For definiteness and simplicity we assume throughout that these symmetries span a Lie group. We start in Sect. 2.1 with a brief review of the symmetry representation theorem of Wigner and show how quantum mechanics gives rise to projective unitary representations. The problem of classifying such representations then leads to Sects. 2.2 and 2.3, respectively devoted to Lie algebra cohomology and group cohomology. The presentation is inspired by [1–4]; see also [5].

2.1 Symmetries and Projective Representations

In this section we review the interplay between quantum mechanics and symmetries. After a brief general reminder on the formalism of quantum theory, we state the symmetry representation theorem which justifies the study of unitary representations of groups and Lie algebras. We also show how the fact that quantum states are rays in a Hilbert space (rather than individual vectors) leads to projective representations, hence to central extensions. We end with a discussion of topological central extensions, while algebraic central extensions are postponed to Sect. 2.2.

2.1.1 Quantum Mechanics

Definition A (complex) *Hilbert space* $\mathscr{H}$ is a vector space over $\mathbb{C}$ endowed with a Hermitian form

$$\langle\cdot|\cdot\rangle : \mathscr{H} \times \mathscr{H} \to \mathbb{C} : (\Phi, \Psi) \mapsto \langle\Phi|\Psi\rangle, \tag{2.1}$$

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such that the norm of a vector Ψ be $\sqrt{\langle\Psi|\Psi\rangle}$, and such that the resulting normed vector space be complete.[1] We take the scalar product (2.1) to be linear in its second argument and antilinear in the first one.

Note that our notation is *not* the standard Dirac notation of bras and kets: a vector in $\mathscr{H}$ is denoted as Ψ (not $|\Psi\rangle$), and its dual is the linear form $\langle\Psi|\cdot\rangle$ on $\mathscr{H}$. Accordingly, the Hermitian conjugate $\hat{A}^\dagger$ of a linear operator $\hat{A}$ is defined by

$$\langle\Phi|\hat{A}^\dagger\Psi\rangle \equiv \langle\hat{A}\Phi|\Psi\rangle \quad \text{for all } \Phi, \Psi \in \mathscr{H}. \tag{2.2}$$

An operator $\hat{A}$ is Hermitian (or self-adjoint)[2] if $\hat{A}^\dagger = \hat{A}$.

Now consider a quantum system whose space of states is a Hilbert space $\mathscr{H}$. A pure *quantum state* of the system is a ray in $\mathscr{H}$, that is, a one-dimensional subspace

$$[\Psi] = \{z\Psi | z \in \mathbb{C}\} \tag{2.3}$$

where Ψ is some non-zero state vector. The vanishing vector does not represent a quantum state, so the set of mutually inequivalent pure states is the projective space $\mathbb{P}\mathscr{H} = (\mathscr{H}\backslash 0)/\mathbb{C}$. It is the set of one-dimensional subspaces of $\mathscr{H}$. Stated differently, the set of distinct states in $\mathscr{H}$ is the quotient of the unit sphere in $\mathscr{H}$ by the equivalence relation

$$\Psi \sim e^{i\theta}\Psi \quad \text{for all } \theta \in \mathbb{R}. \tag{2.4}$$

We shall denote by $[\Psi]$ the resulting equivalence class of Ψ. For example, in a two-level system where $\mathscr{H} = \mathbb{C}^2$, the set of inequivalent states is $\mathbb{C}P^1 \cong S^2$.

Now let the system be in a state $[\Psi]$. If $\hat{A}$ is an observable and if λ is one of its eigenvalues with eigenvector Φ say, the probability of finding the value λ is

$$\text{Prob}(\lambda, \hat{A}, [\Psi]) = \frac{|\langle\Phi|\Psi\rangle|^2}{\langle\Phi|\Phi\rangle\langle\Psi|\Psi\rangle}. \tag{2.5}$$

(We are assuming for simplicity that the eigenvalue λ is not degenerate.) Note that this expression is independent, as it should, of the choice of both the representative Ψ of the state $[\Psi]$, and the eigenvector Φ.

Remark In quantum mechanics, one generally assumes that the Hilbert space is *separable*, i.e. that it admits a countable basis. Any such space is isometric to the space $\ell^2(\mathbb{N})$ of square-integrable sequences of complex numbers — so there really exists only *one* infinite-dimensional separable Hilbert space. This is not to say that all separable Hilbert spaces describe the same quantum system, because the definition of a system also involves the set of observables that act on it — and identical Hilbert spaces may well come with very different operator algebras.

[1] Recall that a metric space is *complete* if any Cauchy sequence converges.

[2] We will not take into account issues related to the domains of operators.

2.1.2 *Symmetry Representation Theorem*

Symmetry Groups

A *symmetry* is a transformation of a system that leaves it invariant. In particular, the set of symmetries of a system always contains the identity transformation, and any symmetry transformation is invertible. In addition the composition of any two symmetry transformations is itself a symmetry, and composition is associative. Put together, these properties imply that

the set of symmetries of any system forms a group.

Accordingly, the framework suited for the study of symmetries is *group theory*.

In this thesis we will be concerned with *Lie groups*, consisting of symmetry transformations that depend smoothly on a certain number of real parameters. This number is the *dimension* of the group. In part I of the thesis, all Lie groups are finite-dimensional.

Remark The notion of symmetry can be relaxed in such a way that not all pairs of symmetry transformations are allowed to be composed together. The resulting set of symmetry transformations then spans a *groupoid* rather than a group (see e.g. [6, 7]). This relaxed notion of symmetry is relevant to gauge theories [8], and in particular to BMS symmetry in four dimensions [9]. However, standard group theory suffices for all symmetry considerations in three-dimensional gravity (and in particular for BMS_3), so we will not deal with groupoids in this thesis.

Symmetries in Quantum Mechanics

Consider a quantum Hilbert space of states $\mathscr{H}$. In these terms a symmetry is a bijection $\mathbb{P}\mathscr{H} \to \mathbb{P}\mathscr{H} : [\Psi] \mapsto \mathcal{S}([\Psi])$ that preserves the probabilities (2.5). Equivalently, if we represent rays in $\mathscr{H}$ by normalized vectors subject to the identification (2.4), a symmetry transformation $\mathcal{S}$ must be such that

$$|\langle\Phi|\Psi\rangle| = \left|\langle\Phi'|\Psi'\rangle\right| \tag{2.6}$$

for all normalized vectors Φ, Ψ, Φ', Ψ' such that $\Phi' \in \mathcal{S}([\Phi])$ and $\Psi' \in \mathcal{S}([\Psi])$. The key result on symmetries in quantum mechanics is the following [10]:

Symmetry Representation Theorem Let $\mathcal{S} : \mathbb{P}\mathscr{H} \to \mathbb{P}\mathscr{H}$ be an invertible transformation satisfying property (2.6). Then it takes the form $\mathcal{S}([\Psi]) = [\hat{U} \cdot \Psi]$, where $\hat{U}$ is either a linear, unitary operator so that

$$\hat{U} \cdot (\lambda\Phi + \mu\Psi) = \lambda\,\hat{U} \cdot \Phi + \mu\,\hat{U} \cdot \Psi \quad \text{and} \quad \langle\hat{U} \cdot \Phi|\hat{U} \cdot \Psi\rangle = \langle\Phi|\Psi\rangle\,,$$

or an antilinear, antiunitary operator so that

$$\hat{U} \cdot (\lambda\Phi + \mu\Psi) = \bar{\lambda}\,\hat{U} \cdot \Phi + \bar{\mu}\,\hat{U} \cdot \Psi \quad \text{and} \quad \langle\hat{U} \cdot \Phi|\hat{U} \cdot \Psi\rangle = \langle\Psi|\Phi\rangle$$

for all $\lambda, \mu \in \mathbb{C}$ and all $\Phi, \Psi \in \mathscr{H}$. A proof of this theorem can be found in Chap. 2 (Appendix A) of [1].

Note that symmetries represented by antiunitary operators only arise when the symmetry group is disconnected. For example, in Lorentz-invariant theories, time-reversal is always represented in an antiunitary way (see e.g. [11]). In this work we will restrict attention to connected symmetry groups, in which case all symmetry operators are linear and unitary. In particular they satisfy $\hat{U}^\dagger = \hat{U}^{-1}$, where Hermitian conjugation is defined by (2.2).

2.1.3 Projective Representations

The symmetry representation theorem implies that all (connected) symmetry groups are represented unitarily in a quantum-mechanical system, and thus motivates the study of unitary representations in general. Let us first recall the basics:

Definition A *representation* of a group G in a vector space $\mathscr{H}$ is a homomorphism[3]

$$\mathcal{T} : G \to \mathrm{GL}(\mathscr{H}) : g \mapsto \mathcal{T}[g]$$

where $\mathrm{GL}(\mathscr{H})$ is the group of invertible linear transformations of $\mathscr{H}$. When $\mathscr{H}$ is a Hilbert space, the representation is *unitary* if $\mathcal{T}[g]$ is a unitary operator for each $g \in G$.

In quantum mechanics the notion of symmetry as a transformation that satisfies (2.6) leads to a key subtlety. Let us call $\mathcal{T}[f]$ the unitary operator that represents a symmetry transformation f belonging to some group G. Then, because a quantum state is really an equivalence class (2.3) of vectors in $\mathscr{H}$, there is no need to require $\mathcal{T}$ to be a homomorphism; rather, all we need is that the ray of $\mathcal{T}[f] \cdot \mathcal{T}[g] \cdot \Phi$ coincides with that of $\mathcal{T}[f \cdot g] \cdot \Phi$ (for all $f, g \in G$ and any $\Phi \in \mathscr{H}$). Accordingly, $\mathcal{T}$ must really be a unitary representation *up to a phase*,

$$\mathcal{T}[f] \cdot \mathcal{T}[g] = e^{i\mathsf{C}(f,g)}\, \mathcal{T}[f \cdot g] \qquad \text{for } f, g \in G, \tag{2.7}$$

where C is some real function on $G \times G$. In more abstract terms, $\mathcal{T}$ must define a group action on the projective space $\mathbb{P}\mathscr{H}$, which is to say that the map

$$[\mathcal{T}] : G \to \mathrm{GL}(\mathscr{H})/\mathbb{C}^* : f \mapsto \big[\mathcal{T}[f]\big] \tag{2.8}$$

is a homomorphism. Here $\mathrm{GL}(\mathscr{H})/\mathbb{C}^*$ is the projective group of $\mathscr{H}$, i.e. the quotient of the linear group of $\mathscr{H}$ by its normal subgroup consisting of multiples of the identity. For any operator $\mathcal{O}$ in $\mathrm{GL}(\mathscr{H})$, the symbol $[\mathcal{O}]$ denotes its class in the projective group. Throughout this thesis, any map $\mathcal{T}$ satisfying this property will be called

[3]Throughout this thesis representations of groups are denoted by the letters $\mathcal{R}, \mathcal{S}, \mathcal{T}$, etc. The letter G denotes a group whose elements are written f, g, h, etc. The identity in G is denoted e.

a *projective representation*. In quantum mechanics, symmetries are represented by *unitary* projective representations, i.e. projective representations whose operators are unitary.

From now on, if we wish to stress that a representation is *not* projective, we will call it *exact*. Quantum mechanics tells us that exact representations are overrated: the truly important ones are generally projective. This seemingly anecdotal observation is at the core of the richest aspects of the representation theory of the Virasoro algebra, and it will also play a key role for BMS_3 particles. For instance, all interesting two-dimensional conformal field theories are such that the conformal group is represented projectively in their Hilbert space, and how exactly this phenomenon takes place is measured by the central charge. For this reason, this whole chapter is devoted to the various ways in which projective effects occur; they are accounted for by group and Lie algebra cohomology.

Remark Since we are focussing on Lie groups, the representations of interest are *continuous* in the sense that the map $G \times \mathscr{H} \to \mathscr{H} : (f, \Psi) \mapsto \mathcal{T}[f] \cdot \Psi$ is continuous. From now on it is understood that all representations are continuous.

2.1.4 Central Extensions

The function C appearing in (2.7) is not completely arbitrary. Indeed, the product (2.7) must be associative in the sense that $\mathcal{T}[f]\cdot(\mathcal{T}[g] \cdot \mathcal{T}[h]) = (\mathcal{T}[f] \cdot \mathcal{T}[g])\cdot\mathcal{T}[h]$ for all group elements f, g, h, so that

$$\mathsf{C}(f, gh) + \mathsf{C}(g, h) = \mathsf{C}(fg, h) + \mathsf{C}(f, g) \quad \text{for all } f, g, h \in G. \tag{2.9}$$

Any function $\mathsf{C} : G \times G \to \mathbb{R}$ satisfying this requirement is known as a (real) *two-cocycle*, and the condition itself is known as the *cocycle condition*. Given any such function one can define a new group

$$\widehat{G} \equiv G \times \mathbb{R} \tag{2.10}$$

whose elements are pairs (f, λ), endowed with a group operation

$$\boxed{(f, \lambda) \cdot (g, \mu) = \big(f \cdot g, \lambda + \mu + \mathsf{C}(f, g)\big).} \tag{2.11}$$

The group (2.10) is called a *central extension* of the group G. We will study this notion in much greater detail in Sect. 2.3. For now let us only work out the basic consequences of this structure and its relation to representation theory.

Projective Versus Exact Representations

Property (2.7) says that $\mathcal{T}$ is an exact unitary representation of the centrally extended group (2.10), provided one represents the pair (f, λ) by $e^{i\lambda}\mathcal{T}[f]$. In other words, exact representations are not overrated after all: we may view any projective representation of G as an exact (i.e. non-projective) representation of a central extension $\widehat{G}$ of G, and the problem of classifying projective unitary representations of G boils down to that of classifying *exact* unitary representations of its central extensions.

The question then is whether G admits central extensions to begin with. For any group, an obvious type of central extension always exists. Namely, suppose K is a real function on G and define $\mathsf{C} : G \times G \to \mathbb{R}$ by

$$\mathsf{C}(f, g) \equiv \mathsf{K}(fg) - \mathsf{K}(f) - \mathsf{K}(g). \tag{2.12}$$

This automatically satisfies condition (2.9). A two-cocycle of that form is said to be *trivial*. In particular, if the cocycle in (2.7) is trivial, it can be absorbed by defining $\tilde{\mathcal{T}}[f] \equiv e^{i\mathsf{K}(f)}\mathcal{T}[f]$, which is an exact representation of G. Thus, what we wish to know is not quite whether G admits two-cocycles at all (since trivial ones are always available), but rather whether it admits *non-trivial* two-cocycles. If yes, it admits genuine projective representations, whose phases cannot be absorbed by a mere redefinition.

This question leads to group (and Lie algebra) cohomology, studied in detail in Sects. 2.2 and 2.3. For now we simply point out that central extensions may arise via two distinct mechanisms. The first is *algebraic* in that it follows from the local group structure of G, or equivalently from the commutation relations of its Lie algebra. In short, in some cases, the Lie algebra $\mathfrak{g}$ of G can be enlarged into a bigger algebra $\widehat{\mathfrak{g}}$ which contains extra generators commuting with those of $\mathfrak{g}$ (see Eq. (2.27) below). The group corresponding to this enlarged algebra then is a central extension of G. The second mechanism is *topological* in the sense that it is due to the global structure of G. We now describe this topological mechanism in some more detail.

2.1.5 Topological Central Extensions

If the group G is not simply connected (i.e. its fundamental group is non-trivial), there exist closed paths in G that cannot be continuously deformed into a point. Let $\gamma : [0, 1] \to G$ be such a path, starting and ending at some group element f so that $\gamma(0) = \gamma(1) = f$. Suppose we are given a (continuous) projective unitary representation $\mathcal{T}$ of G, and consider the path

$$\mathcal{T} \circ \gamma : [0, 1] \to \mathrm{GL}(\mathscr{H}) : t \mapsto \mathcal{T}[\gamma(t)]$$

in the space of unitary operators on $\mathscr{H}$. Since $\mathcal{T}$ is projective, the fact that γ is a closed path does *not* imply that $\mathcal{T} \circ \gamma$ is closed: in general $\mathcal{T}[\gamma(0)]$ and $\mathcal{T}[\gamma(1)]$ differ by a γ-dependent phase, $\mathcal{T}[\gamma(1)] = e^{i\phi(\gamma)}\mathcal{T}[\gamma(0)]$.

Owing to the fact that the map $\mathcal{T}$ is continuous, the phase $\phi(\gamma)$ only depends on the homotopy class of γ. In addition, if γ_1 and γ_2 are two closed paths starting at f, we can concatenate them into a single path $\gamma_1 \cdot \gamma_2$ (which is γ_1 at double speed followed by γ_2 at double speed); the phase ϕ must be compatible with this operation in the sense that $e^{i\phi(\gamma_1)} \cdot e^{i\phi(\gamma_2)} = e^{i\phi(\gamma_1 \cdot \gamma_2)}$. Thus, any one-dimensional unitary representation of the fundamental group of G, multiplying an exact unitary representation of G, produces a projective unitary representation of G.

This is the topological notion of central extensions that we wanted to exhibit: if G is multiply connected, it admits genuine projective representations (whose phases cannot be removed by redefinitions) due to one-dimensional unitary representations of its fundamental group.[4] Projective representations of that type may equivalently be seen as exact representations of the universal cover $\widetilde{G}$ of G, which is the unique connected and simply connected group locally isomorphic to G.

Remark One might be worried by the fact that only *one-dimensional* unitary representations of the fundamental group are allowed to appear in this construction. Indeed, if the fundamental group was non-Abelian, it would generally admit no non-trivial one-dimensional unitary representation. Fortunately, it turns out that the fundamental group of any finite-dimensional Lie group is a discrete commutative group, whose irreducible unitary representations are necessarily one-dimensional.

Rotations and Anyons

The simplest example of topological projective representations arises with the group U(1). The latter is diffeomorphic to a circle and has a fundamental group isomorphic to $\mathbb{Z}$ (see Fig. 2.1). Any exact irreducible, unitary representation of U(1) takes the form

$$\mathcal{T} : \text{U}(1) \to \mathbb{C}^* : \theta \mapsto e^{is\theta} \tag{2.13}$$

where θ is identified with $\theta + 2\pi$, as a consequence of which the "spin" s is an integer. For example, when $s = 2$, a rotation by $\theta = \pi$ is represented by the identity. (We will see in Sect. 4.3 that the label s actually *is* the spin of a particle in certain representations of the Poincaré groups.) But there is a subtlety: U(1) is multiply connected and admits topological projective representations, which from the viewpoint of quantum mechanics are just as acceptable as exact ones. For example, the map (2.13) with $s = 1/2$ definitely isn't an exact representation because a full rotation by 2π is now represented by an inversion, $\mathcal{T}[2\pi] = e^{i\pi} = -1$. Nevertheless, in quantum mechanics, the vectors Ψ and $\mathcal{T}[2\pi] \cdot \Psi$ define the same state by virtue of the identification (2.4), so in this sense $\mathcal{T}[2\pi]$ acts as an "almost-identity" operator. More generally, formula (2.13) is a projective representation of U(1) for *any* real value of the spin s.

The example just described occurs in Nature. Indeed, fermions provide a well-known example of projective representations, as already suggested above by the case $s = 1/2$. By the spin-statistics theorem, all fermions have half-integer spins, and therefore transform according to a projective representation of the Lorentz group. The latter is multiply connected (its fundamental group is $\mathbb{Z}_2$), which is why it admits projective representations in the first place. We will return to the representation theory

[4]Beware: a manifold being *multiply connected* means that it has a non-trivial fundamental group, and *not* that it has several connected components.

Fig. 2.1 The group U(1) is diffeomorphic to a circle S^1, whose universal cover is the real line $\mathbb{R}$. The projection $\mathbb{R} \to S^1 \cong \mathbb{R}/\mathbb{Z}$ is obtained by identifying points of $\mathbb{R}$ that differ by some periodicity, typically $\theta \sim \theta + 2\pi$. In particular, paths in $\mathbb{R}$ which are not closed may be projected on closed paths in S^1. As an application we can picture topological projective representations: if $\mathcal{T}$ is projective and if γ is a closed path in the circle, the sequence $\mathcal{T}[\gamma(t)]$ may not be a closed path in the space of operators

of the Lorentz group (as a subgroup of Poincaré) in much greater detail in Sect. 4.2. In the cases where arbitrary real values of spin are allowed by quantum mechanics, as for example in three space-time dimensions, the particles whose spin is neither an integer nor a half-integer are known as *anyons*. We will encounter this phenomenon in Sect. 10.1 when dealing with BMS_3 particles.

2.1.6 Classifying Projective Representations

Given a group G, suppose we wish to find all its projective unitary representations. The above considerations provide an algorithm that allows us, in principle, to solve that problem:

- First find the universal cover $\widetilde{G}$ of G to take care of topological central extensions.
- Then find the most general central extension $\widehat{\widetilde{G}}$ of $\widetilde{G}$ in order to take care of differentiable central extensions. (We will deal with the actual definition of these extensions in the next section.)
- Finally, consider an *exact* unitary representation of $\widehat{\widetilde{G}}$; any projective unitary representation of G may be seen as a representation of that type.

Thus we now have a systematic procedure allowing us to build arbitrary projective unitary representations of symmetry groups in quantum mechanics. We will apply it later to the Virasoro algebra (Sect. 8.4) and the BMS_3 group (Sect. 10.1), where central extensions play a crucial role.

2.2 Lie Algebra Cohomology

This section is devoted to a thorough investigation of the concept of central extensions at the Lie-algebraic level. In fact, we shall describe the more general framework of Lie algebra cohomology and we will show how statements on algebraic central extensions can be recast in that language. The group-theoretic analogue of this construction is relegated to Sect. 2.3.

2.2.1 Cohomology

Let $\mathfrak{g}$ be a Lie algebra with Lie bracket $[\cdot,\cdot]$. We recall that a *representation* of $\mathfrak{g}$ in a vector space $\mathbb{V}$ is a linear map $\mathscr{T}:\mathfrak{g}\to\text{End}(\mathbb{V})$ such that $\mathscr{T}[X]\circ\mathscr{T}[Y]-\mathscr{T}[Y]\circ\mathscr{T}[X]=\mathscr{T}\big[[X,Y]\big]$ for all Lie algebra elements X, Y.[5]

Definition Let k be a non-negative integer, $\mathscr{T}$ a representation of $\mathfrak{g}$ in $\mathbb{V}$. Then a $\mathbb{V}$-valued *k-cochain* on $\mathfrak{g}$ is a continuous, multilinear, completely antisymmetric map[6]

$$\mathsf{c}:\underbrace{\mathfrak{g}\times\cdots\times\mathfrak{g}}_{k\text{ times}}\to\mathbb{V}:(X_1,...,X_k)\mapsto\mathsf{c}(X_1,...,X_k). \tag{2.14}$$

In other words, a $\mathbb{V}$-valued k-cochain on $\mathfrak{g}$ is a k-form on $\mathfrak{g}$ with values in $\mathbb{V}$; note that $0\leq k\leq\dim(\mathfrak{g})$. A zero-cochain on $\mathfrak{g}$ is a vector in $\mathbb{V}$ while a $\dim(\mathfrak{g})$-cochain is a volume form on $\mathfrak{g}$. We denote the space of $\mathbb{V}$-valued k-cochains on $\mathfrak{g}$ by $\mathcal{C}^k(\mathfrak{g},\mathbb{V})$ and we define the associated cochain complex $\mathcal{C}^*(\mathfrak{g},\mathbb{V})\equiv\oplus_{k=0}^{\dim(\mathfrak{g})}\mathcal{C}^k(\mathfrak{g},\mathbb{V})$. The latter is sometimes called the *Chevalley-Eilenberg complex*.

Definition The *Chevalley-Eilenberg differential* $\mathsf{d}:\mathcal{C}^*(\mathfrak{g},\mathbb{V})\to\mathcal{C}^*(\mathfrak{g},\mathbb{V})$ is defined by $\dim(\mathfrak{g})$ linear maps

$$\mathsf{d}_k:\mathcal{C}^k(\mathfrak{g},\mathbb{V})\to\mathcal{C}^{k+1}(\mathfrak{g},\mathbb{V}):\mathsf{c}\mapsto\mathsf{d}_k\mathsf{c}$$

where k runs from 0 to $\dim(\mathfrak{g})-1$ and the $(k+1)$-cochain $\mathsf{d}_k\mathsf{c}$ is given by

$$\begin{aligned}(\mathsf{d}_k\mathsf{c})(X_1,...,X_{k+1})\equiv&\sum_{1\leq i<j\leq k+1}(-1)^{i+j-1}\mathsf{c}\big([X_i,X_j],X_1,...,\widehat{X_1},...,\widehat{X_j},...,X_{k+1}\big)\\&+\sum_{1\leq i\leq k+1}(-1)^i\mathscr{T}[X_i]\cdot\mathsf{c}\big(X_1,...,\widehat{X_i},...,X_{k+1}\big)\end{aligned} \tag{2.15}$$

for all $X_1, ..., X_{k+1}$ in $\mathfrak{g}$; the hat denotes omission. Note that the representation $\mathscr{T}$ of $\mathfrak{g}$ in $\mathbb{V}$ appears explicitly in this definition. In particular, when $\mathscr{T}$ is trivial, formula (2.15) simplifies since its last line disappears.

Cocycles and Coboundaries

Using the fact that $\mathscr{T}$ is a representation, one can verify that the Chevalley-Eilenberg differential (2.15) is nilpotent:

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[5]Throughout this thesis the elements of a Lie algebra $\mathfrak{g}$ will be denoted as X, Y, etc. Representations of Lie algebras will be denoted by script capital letters such as $\mathscr{R}, \mathscr{S}, \mathscr{T}$.

[6]Cochains on Lie algebras will be denoted by lowercase sans serif letters such as c, s, etc.

$$\mathsf{d}_k \circ \mathsf{d}_{k-1} = 0 \qquad \forall\, k = 0, ..., \dim(\mathfrak{g}) \tag{2.16}$$

where it is understood that the "extreme differentials" are $\mathsf{d}_{-1} : 0 \to \mathbb{V} : 0 \mapsto 0$ and $\mathsf{d}_{\dim \mathfrak{g}} : \mathcal{C}^{\dim \mathfrak{g}}(\mathfrak{g}, \mathbb{V}) \to 0 : \mathsf{c} \mapsto 0$. Accordingly, one adapts the standard terminology of differential forms to cochains on a Lie algebra: a *k-cocycle* is a k-cochain c such that $\mathsf{d}_k\mathsf{c} = 0$; a *k-coboundary* is a k-cochain c of the form $\mathsf{c} = \mathsf{d}_{k-1}\mathsf{b}$, where b is some $(k-1)$-cochain. By virtue of property (2.16), one has $\mathrm{Im}(\mathsf{d}_{k-1}) \subseteq \mathrm{Ker}(\mathsf{d}_k)$ for each k (any coboundary is a cocycle). One can therefore define the k^{th} *cohomology space* of $\mathfrak{g}$ with coefficients in $\mathbb{V}$ as the quotient of the space of k-cocycles by the space of k-coboundaries:

$$\mathcal{H}^k(\mathfrak{g}, \mathbb{V}) \equiv \mathrm{Ker}(\mathsf{d}_k)/\mathrm{Im}(\mathsf{d}_{k-1}). \tag{2.17}$$

A k-cocycle is said to be *trivial* if its equivalence class vanishes in $\mathcal{H}^k$, i.e. if the cocycle is a coboundary; the cocycle is *non-trivial* otherwise. When $\mathbb{V} = \mathbb{R}$ with $\mathscr{T}$ the trivial representation of $\mathfrak{g}$, we write $\mathcal{H}^k(\mathfrak{g}, \mathbb{R}) \equiv \mathcal{H}^k(\mathfrak{g})$.

Isomorphic Lie algebras have the same cohomology for any choice of the representation $\mathscr{T}$. Thus, cohomology is a way to associate invariants with Lie algebras: if two algebras have different cohomology spaces, then they cannot be isomorphic. This is analogous to, say, de Rham cohomology in differential geometry, as manifolds with different de Rham cohomologies cannot be diffeomorphic.

Low Degree Cohomologies

There is a simple interpretation for the lowest cohomology spaces. For example, zero-cocycles are vectors $v \in \mathbb{V}$ that are invariant under $\mathfrak{g}$ in the sense that

$$\mathscr{T}[X] \cdot v = 0 \qquad \text{for all } X \in \mathfrak{g}\,, \tag{2.18}$$

so the zeroth cohomology space of $\mathfrak{g}$ classifies the invariants of the representation $\mathscr{T}$. Similarly, one-cocycles are known as *derivations* of $\mathfrak{g}$ and are classified by the first cohomology space $\mathcal{H}^1(\mathfrak{g}, \mathbb{V})$. In the particular case where $\mathscr{T}$ is trivial and $\mathbb{V} = \mathbb{R}$, a one-cocycle is a linear map $\mathsf{c} : \mathfrak{g} \to \mathbb{R}$ such that $\mathsf{c}([X, Y]) = 0$ for all Lie algebra elements X, Y. Hence the first real cohomology space of $\mathfrak{g}$ can be written as

$$\mathcal{H}^1(\mathfrak{g}) \cong \mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]\,, \tag{2.19}$$

which motivates the following definition:

Definition A Lie algebra $\mathfrak{g}$ is *perfect* if $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$, i.e. if any Lie algebra element can be written as the bracket of two other elements.

It follows from (2.19) that $\mathfrak{g}$ is perfect if and only if $\mathcal{H}^1(\mathfrak{g})$ vanishes. We will use this property in Sect. 2.2.2 when defining central extensions.

By the definitions above, a two-cochain is an antisymmetric map $\mathsf{c} : \mathfrak{g} \times \mathfrak{g} \to \mathbb{V}$. It is a coboundary if

$$\mathsf{c}(X, Y) \overset{(2.15)}{=} (\mathsf{d}_1\mathsf{k})(X, Y) = \mathsf{k}([X, Y]) - \mathscr{T}[X] \cdot \mathsf{k}(Y) + \mathscr{T}[Y] \cdot \mathsf{k}(X) \quad (2.20)$$

for some one-cochain k; and it is a cocycle if

$$\begin{aligned} &\mathsf{c}([X, Y], Z) + \mathsf{c}([Y, Z], X) + \mathsf{c}([Z, X], Y) = \\ &= \mathscr{T}[X] \cdot \mathsf{c}(Y, Z) + \mathscr{T}[Y] \cdot \mathsf{c}(Z, X) + \mathscr{T}[Z] \cdot \mathsf{c}(X, Y). \end{aligned} \quad (2.21)$$

As we shall see shortly, when $\mathscr{T}$ is trivial, a two-cocycle defines a *central extension* of $\mathfrak{g}$. Thus the second cohomology of $\mathfrak{g}$ classifies its extensions. More generally, cohomology may be seen as a measure of flexibility: Lie algebras with high-dimensional cohomology groups can be "deformed" in many inequivalent ways; by contrast, Lie algebras with trivial cohomology are "rigid" in the sense that any deformation is equivalent to no deformation at all.

Remark Here we have been using the word "deformation" in a vague way, but there is an exact definition of the notion of deformations. Namely, a (true) *deformation* of a Lie algebra $\mathfrak{g}$ is a Lie algebra $\tilde{\mathfrak{g}}$ that coincides with $\mathfrak{g}$ as a vector space, but whose brackets are

$$\tilde{[}X, Y\tilde{]} = [X, Y] + \mathsf{c}(X, Y) \quad (2.22)$$

where $[\cdot, \cdot]$ is the bracket in $\mathfrak{g}$ while c is a $\mathfrak{g}$-valued two-cocycle on $\mathfrak{g}$,[7] such that the image of c belongs to its kernel. The latter condition means that $\mathsf{c}\big(X, \mathsf{c}(Y, Z)\big) = 0$ for all Lie algebra elements X, Y, Z; together with the fact that c is a cocycle, this ensures that (2.22) is a Lie bracket.

Examples

For finite-dimensional semi-simple Lie algebras, cohomology is trivial:

Whitehead's Lemma Let $\mathfrak{g}$ be a finite-dimensional semi-simple Lie algebra, $\mathscr{T}$ an irreducible, finite-dimensional representation of $\mathfrak{g}$ in a space $\mathbb{V}$. Then

$$\mathcal{H}^k(\mathfrak{g}, \mathbb{V}) = 0 \quad \text{for all } k > 0. \quad (2.23)$$

Despite this result, examples of non-trivial cohomologies do exist in physics. For instance, let c be an arbitrary non-vanishing antisymmetric bilinear form on $\mathbb{R}^2$, and view the latter as an Abelian Lie algebra. Then c defines a non-trivial, real-valued two-cocycle on $\mathbb{R}^2$, so the real-valued second cohomology of $\mathbb{R}^2$ is non-trivial; in fact one can prove that

$$\mathcal{H}^2(\mathbb{R}^2) \cong \mathbb{R}. \quad (2.24)$$

We shall see below that this property is related to the (three-dimensional) Heisenberg algebra, which is crucial for quantum mechanics. Other important examples of algebras with non-trivial cohomology spaces include the Galilei algebra (Sect. 4.4), the Virasoro algebra (Chap. 6) and the $\mathfrak{bms}_3$ algebra (Chap. 9).

[7]It is understood that the relevant representation of $\mathfrak{g}$ in this case is the adjoint, $\mathscr{T}[X] \cdot Y \equiv [X, Y]$.

2.2.2 Central Extensions

Definition Let $\mathfrak{g}$ be a (real) Lie algebra and let $\mathsf{c} \in \mathcal{C}^2(\mathfrak{g}, \mathbb{R})$ be a real two-cocycle on $\mathfrak{g}$. Then c defines a *central extension* $\widehat{\mathfrak{g}}$ of $\mathfrak{g}$, which is a Lie algebra whose underlying vector space

$$\widehat{\mathfrak{g}} = \mathfrak{g} \oplus \mathbb{R} \qquad \text{(as vector spaces)} \tag{2.25}$$

is endowed with the centrally extended Lie bracket

$$\big[(X, \lambda), (Y, \mu)\big] \equiv \big([X, Y], \mathsf{c}(X, Y)\big). \tag{2.26}$$

In particular, elements of $\widehat{\mathfrak{g}}$ are pairs (X, λ) where $X \in \mathfrak{g}$ and $\lambda \in \mathbb{R}$, so that $\mathbb{R}$ is an Abelian subalgebra of $\widehat{\mathfrak{g}}$. The bracket (2.26) satisfies the Jacobi identity on account of the fact that c is a two-cocycle with respect to a trivial representation of $\mathfrak{g}$ (so that the right-hand side of Eq. (2.21) vanishes).

In (2.26) we displayed the definition of central extensions in intrinsic terms thanks to the two-cocycle c. The same definition can be written in terms of Lie algebra generators: let $\{t_a | a = 1, ..., n\}$ be a basis of $\mathfrak{g}$ with brackets $[t_a, t_b] = f_{ab}{}^c\, t_c$. Then a central extension $\widehat{\mathfrak{g}}$ of $\mathfrak{g}$ is a Lie algebra generated by the basis elements $T_a \equiv (t_a, 0)$ together with a central element $\mathcal{Z} = (0, 1)$, whose Lie brackets read

$$[T_a, T_b] = f_{ab}{}^c\, T_c + c_{ab}\, \mathcal{Z} \tag{2.27}$$

where $c_{ab} = \mathsf{c}(t_a, t_b)$, while all brackets with $\mathcal{Z}$ vanish. The cocycle condition on c then becomes the requirement

$$f_{ab}{}^d c_{dc} + f_{bc}{}^d c_{da} + f_{ca}{}^d c_{db} = 0$$

for the coefficients c_{ab}. Note that this construction can be readily generalized to multiple central extensions $\widehat{\mathfrak{g}} = \mathfrak{g} \oplus \mathbb{R}^N$, in which case there are N central generators $\mathcal{Z}_1, ..., \mathcal{Z}_N$.

Non-Trivial Central Extensions

When the two-cocycle c is trivial in the sense of Lie algebra cohomology, it takes the form (2.20) in terms of some one-cocycle k and the map

$$\mathfrak{g} \to \widehat{\mathfrak{g}} : X \mapsto \big(X, \mathsf{k}(X)\big) \tag{2.28}$$

is an injective homomorphism of Lie algebras. The central extension is then said to be *trivial*: the cocycle c can be absorbed by the "redefinition" (2.28), and $\widehat{\mathfrak{g}}$ is isomorphic to the direct sum $\mathfrak{g} \oplus \mathbb{R}$ as a Lie algebra. By contrast, when c is non-trivial, it defines a non-zero element in the second cohomology space $\mathcal{H}^2(\mathfrak{g})$; such a two-cocycle cannot be removed by a mere redefinition, and the central extension is *non-trivial*.

For example, as on p.xx, consider the Abelian Lie algebra $\mathbb{R}^2$ and let c be a non-zero antisymmetric bilinear form on $\mathbb{R}^2$. We then define the three-dimensional *Heisenberg algebra* as the algebra $\mathbb{R}^3 = \mathbb{R}^2 \oplus \mathbb{R}$ whose elements are pairs (X, λ), endowed with the Lie bracket (2.26). Since c is non-trivial, so is the central extension. If we choose a basis $\{Q, P\}$ of $\mathbb{R}^2$ such that $\mathsf{c}(Q, P) = 1$ and if we call Z the central element $(0, 1)$, the commutation relations of the Heisenberg algebra take the form

$$[Q, P] = Z. \tag{2.29}$$

Property (2.24) says that there is only one linearly independent central extension of $\mathbb{R}^2$, i.e. that Heisenberg algebras built using different (non-zero) two-cocycles c are mutually isomorphic. This can be generalized to higher dimensions: by seeing $\mathbb{R}^{2n}$ as an Abelian Lie algebra and taking c an arbitrary non-zero $2n$-form on $\mathbb{R}^{2n}$, the Lie algebra defined by the bracket (2.26) is the $(2n + 1)$-dimensional Heisenberg algebra.

Universal Central Extensions

It is important to know how many inequivalent central extensions an algebra may possess. This leads to the following notion:

Definition A central extension $\widehat{\mathfrak{g}}$ of $\mathfrak{g}$ is *universal* if, for any other central extension $\widehat{\mathfrak{g}}'$ of $\mathfrak{g}$, there exists a unique isomorphism of Lie algebras $\widehat{\mathfrak{g}}' \cong \widehat{\mathfrak{g}}$.

As it turns out, any perfect Lie algebra admits a universal central extension. By virtue of (2.19), this is to say that any algebra such that $\mathcal{H}^1(\mathfrak{g}) = 0$ admits a universal central extension. For example we will see in Chap. 6 that the Virasoro algebra is the universal central extension of the Lie algebra of vector fields on the circle.

2.3 Group Cohomology

This section is devoted to the group-theoretic analogue of the considerations of the previous pages. We start by discussing generalities on group cohomology before focussing on central extensions of groups.

2.3.1 Cohomology

Let G be a Lie group, $\mathcal{T} : G \to \mathrm{GL}(\mathbb{V})$ a representation of G in a vector space $\mathbb{V}$.

Definition Let $k \geq 0$ be an integer. A $\mathbb{V}$-valued *k-cochain* on G is a smooth map[8]

[8]Cochains on a group will be denoted by capital sans serif symbols such as C, S, etc.

$$\mathsf{C} : \underbrace{G \times \cdots \times G}_{k \text{ times}} \to \mathbb{V} : (g_1, ..., g_k) \mapsto \mathsf{C}(g_1, ..., g_k). \tag{2.30}$$

Note that, in contrast to the Lie-algebraic definition (2.14), there is no restriction on k. The new ingredient in the group-theoretic context is the requirement that the map (2.30) be smooth. As in the case of Lie algebras, we denote by $\mathcal{C}^k(G, \mathbb{V})$ the vector space of $\mathbb{V}$-valued k-cochains on G and we let $\mathcal{C}^*(G, \mathbb{V}) = \oplus_{k=0}^{+\infty} \mathcal{C}^k(G, \mathbb{V})$ be the associated cochain complex. The space of zero-cochains is just $\mathbb{V}$.

Definition The *differential* $\mathsf{d} : \mathcal{C}^*(G, \mathbb{V}) \to \mathcal{C}^*(G, \mathbb{V})$ is defined by the maps

$$\mathsf{d}_k : \mathcal{C}^k(G, \mathbb{V}) \to \mathcal{C}^{k+1}(G, \mathbb{V}) : \mathsf{C} \mapsto \mathsf{d}_k\mathsf{C}$$

where $k \in \mathbb{N}$ and the $(k+1)$-cochain $\mathsf{d}_k\mathsf{C}$ is given by

$$\begin{aligned}(\mathsf{d}_k\mathsf{C})(g_1, ..., g_{k+1}) \equiv \mathcal{T}[g_1] \cdot \mathsf{C}(g_2, ..., g_{k+1}) + (-1)^{k+1}\, \mathsf{C}(g_1, ..., g_k) \\ + \sum_{i=1}^{k} (-1)^i\, \mathsf{C}(g_1, ..., g_i g_{i+1}, ..., g_{k+1})\end{aligned} \tag{2.31}$$

for all $g_1, ..., g_{k+1}$ in G.

The differential (2.31) satisfies the key property (2.16), so the usual machinery of homological algebra applies: one defines a *k-cocycle* as a closed k-cochain, that is, a cochain C such that $\mathsf{d}_k\mathsf{C} = 0$. One also defines a *k-coboundary* to be an exact k-cochain, i.e. one that can be written as the differential of a $(k-1)$-cochain. As before any coboundary is trivially a cocycle, so one defines the k^{th} *cohomology space* of G with values in $\mathbb{V}$ as the quotient of the space of k-cocycles by the space of k-coboundaries:

$$\mathcal{H}^k(G, \mathbb{V}) \equiv \text{Ker}(\mathsf{d}_k)/\text{Im}(\mathsf{d}_{k-1}).$$

A k-cocycle is *trivial* if its class in $\mathcal{H}^k(G, \mathbb{V})$ vanishes; it is non-trivial otherwise. When $\mathbb{V} = \mathbb{R}$ with $\mathcal{T}$ the trivial representation, we write $\mathcal{H}^k(G, \mathbb{R}) \equiv \mathcal{H}^k(G)$.

Interpretation

As in the case of Lie algebras, cohomology spaces are invariants that measure the flexibility of a group structure; isomorphic Lie groups have the same cohomology. This interpretation is simplest to illustrate with the cohomology spaces of lowest degree.

A $\mathbb{V}$-valued zero-cocycle on G is a vector $v \in \mathbb{V}$ such that $(\mathsf{d}_0 v)(f) = \mathcal{T}[f] \cdot v - v = 0$ for any group element f. Accordingly, the zeroth cohomology space of G classifies vectors $v \in \mathbb{V}$ that are left invariant by G. This is the group-theoretic analogue of (2.18).

A $\mathbb{V}$-valued one-cocycle is a (smooth) map $\mathsf{S} : G \to \mathbb{V}$ satisfying the property

$$\mathsf{S}(fg) = \mathcal{T}[f] \cdot \mathsf{S}(g) + \mathsf{S}(f) \qquad \forall\, f, g \in G. \tag{2.32}$$

Given a one-cocycle S, one defines the associated *affine module* as the space $\mathbb{V} \oplus \mathbb{R}$ acted upon by the following representation $\widehat{\mathcal{T}}$ of G:

$$\widehat{\mathcal{T}}[f] \cdot (v, \lambda) \equiv \big(\mathcal{T}[f] \cdot v + \lambda\, \mathsf{S}(f), \lambda\big). \tag{2.33}$$

The cocycle condition (2.32) ensures that $\widehat{\mathcal{T}}$ is indeed a representation. In addition one can show that affine modules defined using different one-cocycles are equivalent if (and only if) their cocycles differ by a coboundary. Thus $\mathcal{H}^1(G, \mathbb{V})$ classifies affine G-modules based on $\mathbb{V}$. For example, in Sect. 6.3 we will see that the Schwarzian derivative is a one-cocycle on the group of diffeomorphisms of the circle; this is why we denote the cocycle in (2.33) by S. The corresponding affine module will be the coadjoint representation of the Virasoro group and the parameter λ left invariant by (2.33) will be a Virasoro central charge. More generally one can think of the term $\lambda \mathsf{S}[f]$ in (2.33) as an anomaly that adds an inhomogeneous term to the otherwise homogeneous transformation law of v under G.

Two-cocycles lead to the notion of group extensions; in particular, when $\mathbb{V} = \mathbb{R}$ with $\mathcal{T}$ the trivial representation, $\mathcal{H}^2(G)$ classifies central extensions of G. Indeed, when C is a real two-cocycle on G, the requirement $\mathsf{d}_2\mathsf{C} = 0$ becomes the cocycle condition (2.9); the central extension is trivial when C is a coboundary, i.e. if it takes the form (2.12) for some one-cochain K. We will return to central extensions of groups in Sect. 2.3.2.

Relation to Lie Algebra Cohomology

One may ask how group and Lie algebra cohomology are related. The following result provides a first answer:

Proposition Let G be a Lie group, $\mathfrak{g}$ its Lie algebra. Let $\mathbb{V}$ be a vector space, $\mathcal{T}$ a smooth representation of G in $\mathbb{V}$, and $\mathscr{T}$ the representation of $\mathfrak{g}$ corresponding to $\mathcal{T}$ by differentiation. Then, for any non-negative integer k, there is a homomorphism

$$\mathcal{H}^k(G, \mathbb{V}) \rightarrow \mathcal{H}^k(\mathfrak{g}, \mathbb{V}) : [\mathsf{C}] \mapsto \big[\delta \mathsf{C}\big] \tag{2.34}$$

given by

$$\delta\mathsf{C}(X_1, ..., X_k) \equiv \frac{\partial^k}{\partial t_1 ... \partial t_k} \left[\sum_{1 \le i_1 < ... < i_k \le k} \epsilon_{i_1 ... i_k} \mathsf{C}\big(e^{t_{i_1} X_{i_1}}, ..., e^{t_{i_k} X_{i_k}}\big) \right]\Bigg|_{t_1=0, ..., t_k=0}$$

for all $X_1, ..., X_k$ in $\mathfrak{g}$, with e^X the exponential of $X \in \mathfrak{g}$ and $\epsilon_{i_1 ... i_k}$ the Levi-Civita symbol with k indices (and $\epsilon_{12...k} \equiv +1$). For $k = 2$ this can be rewritten as

$$\delta\mathsf{C}(X, Y) = \frac{\partial^2}{\partial t\, \partial s} \left[\mathsf{C}\left(e^{tX}, e^{sY}\right) - \mathsf{C}\left(e^{sY}, e^{tX}\right) \right]\Big|_{t=0,\, s=0}. \tag{2.35}$$

The fact that (2.34) is a homomorphism ensures that, if $\delta\mathsf{C}$ is a non-trivial cocycle, then C itself is non-trivial. The converse is not true since the map need not be injective: a non-trivial cocycle C may well be such that $\delta\mathsf{C}$ is trivial.

We will use formula (2.35) in Sect. 6.2 to relate the Virasoro algebra to the Virasoro group. The key point here is that any differentiable group cocycle C admits an algebraic analogue $\delta\mathsf{C}$. The converse problem is to start from a Lie algebra cocycle, say c, and ask whether there exists a group cocycle whose differential is c. This is the problem of integrating Lie algebra cocycles to group cocycles, and it is generally much more complicated than differentiation. However, for "sufficiently connected" Lie groups, the Van Est theorem states that integration is trivial because group and Lie algebra cohomologies coincide (see e.g. [4]). In particular, when the universal cover of a group is homotopic to a point, the cohomology of the universal cover coincides with that of the Lie algebra.

2.3.2 Central Extensions

Here we return in more detail to the notion of centrally extended groups, already outlined around (2.11). For simplicity we deal only with simply connected groups, so as to avoid the topological complications of Sect. 2.1.5. Including these subtleties would lead to a definition of central extensions somewhat more general (see e.g. [4]) than the one given here:

Definition Let G be a Lie group, C a real two-cocycle on G. Then the associated *centrally extended group* $\widehat{G}$ is topologically a product $G \times \mathbb{R}$ whose elements are pairs (f, λ) with $f \in G$ and $\lambda \in \mathbb{R}$, endowed with a group operation (2.11).

It is straightforward to generalize this definition to the case where $\mathbb{R}$ is replaced by an arbitrary (additive) Abelian group such as $\mathbb{R}^N$.

Non-Trivial Central Extensions

As in the Lie-algebraic case, a central extension of G is *trivial* if the two-cocycle C defining the group operation (2.11) is a coboundary (2.12) for some one-cochain K. Then the map $G \to \widehat{G} : f \mapsto \big(f, \mathsf{K}(f)\big)$ is an injective homomorphism whose Lie-algebraic analogue is (2.28), and $\widehat{G}$ is isomorphic, as a group, to the direct product $G \times \mathbb{R}$. Thus any trivial central extension can be absorbed by a redefinition of the group, and is irrelevant as regards projective representations. By contrast, when the cohomology class of C is a non-zero vector in $\mathcal{H}^2(G)$, the central extension cannot be removed by a redefinition and is said to be *non-trivial*.

Example Let us find the group corresponding to the $(2n+1)$-dimensional Heisenberg algebra. Consider the Abelian additive group $G = \mathbb{R}^n \times \mathbb{R}^n$ (whose elements are pairs of column vectors (α, β)) and define the *Heisenberg group* as

$$\widehat{G} \equiv \left\{ \begin{pmatrix} 1 & \alpha^t & \lambda \\ 0 & \mathbb{I}_n & \beta \\ 0 & 0 & 1 \end{pmatrix} \middle| \alpha, \beta \in \mathbb{R}^n,\ \lambda \in \mathbb{R} \right\} \tag{2.36}$$

where $\mathbb{I}_n$ denotes the $n \times n$ identity matrix and α^t is the transpose of α. The group operation in $\widehat{G}$ is given by matrix multiplication and can be written as

$$(\alpha, \beta, \lambda) \cdot (\alpha', \beta', \lambda') = \big(\alpha + \alpha', \beta + \beta', \lambda + \lambda' + \alpha^t \cdot \beta'\big) \tag{2.37}$$

where $\alpha^t \cdot \beta' \equiv \alpha^i \beta'^i$ is the Euclidean scalar product of α and β'. Thus the Heisenberg group is a central extension of $\mathbb{R}^{2n}$ defined by the two-cocycle

$$\mathsf{C}\big((\alpha, \beta), (\alpha', \beta')\big) = \alpha^t \cdot \beta'. \tag{2.38}$$

By differentiation, one can associate with C a Lie algebra cocycle given by (2.35). For example, when $n = 1$ (and writing elements of the Lie algebra $\mathbb{R}^2$ as pairs $X = (x, y)$),

$$\delta\mathsf{C}\big((x, y), (x', y')\big) \overset{(2.35)}{=} \frac{\partial^2}{\partial t\, \partial s} \big(tx \cdot sy' - sx' \cdot ty\big)\big|_{t=0,\, s=0} = xy' - yx'.$$

This is a non-zero antisymmetric bilinear form on $\mathbb{R}^2$, hence defining the Heisenberg algebra of (2.29). Note that this is an example of "cocycle integration": we have found the explicit group two-cocycle whose differential defines the Heisenberg Lie algebra.

Universal Central Extensions

Universal central extensions of groups can be defined exactly as for Lie algebras. A central extension $\widehat{G}$ of G is *universal* if, for any other central extension $\widehat{G}'$ of G by A, there exists a unique isomorphism $\widehat{G} \to \widehat{G}'$.

As in the algebraic case, there is a simple criterion for knowing when a group admits a universal central extension. A group is said to be *perfect* if it coincides with the group of its commutators, i.e. if any $f \in G$ can be written as $f = ghg^{-1}h^{-1}$ for some $g, h \in G$. It turns out that any perfect group admits a universal central extension. In Chaps. 6 and 9 we will see that both $\text{Diff}(S^1)$ and BMS_3 are perfect groups, so that their central extensions are universal.

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Chapter 3
Induced Representations

In the previous chapter we learned how to deal with projective representations: given a symmetry group, we are to find its universal cover and its most general central extension. Exact representations of this central extension then account for all projective representations of the original group. The remaining problem then is to write down explicit representations, so our goal in this chapter is to build Hilbert spaces of wavefunctions acted upon by a group of unitary transformations. Guided by group actions on homogeneous spaces, we will be led to the method of *induced representations*. Their basic principle is very simple: starting from a representation of some subgroup H of a group G, one induces a representation of G that acts on wavefunctions which live on the quotient space G/H.

Induced representations are ubiquitous in mathematics and physics:

- The irreducible unitary highest-weight representations of any compact, simple Lie group are induced from those of its maximal torus, i.e. its largest Abelian subgroup (whose Lie algebra is the Cartan subalgebra).
- Highest-weight representations of $\mathfrak{sl}(2,\mathbb{R})$ and of the Virasoro algebra are induced from representations of their $\mathfrak{u}(1)$ subalgebra generated by L_0 (see Sect. 8.4).
- All irreducible unitary representations of the Euclidean groups, the Bargmann groups, the Poincaré groups and the BMS_3 group are induced from those of their translation subgroups combined with "little groups" (see Chaps. 4 and 10).

The plan of this chapter is as follows. In Sect. 3.1 we review some basics of measure theory and the ensuing construction of Hilbert spaces of square-integrable wavefunctions. Section 3.2 is concerned with measures on homogeneous spaces and introduces quasi-regular representations — the simplest examples of induced representations. In Sect. 3.3 we display the basic formulas of induced representations and list some of their elementary properties. Along the way we define a basis of plane waves, later to be interpreted as particles with definite momentum. This basis is then used in Sect. 3.4 to compute characters. Finally, Sect. 3.5 is devoted to systems of imprimitivity. All these notions are crucial prerequisites for Chap. 4.

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It would be illusory to present a complete account of the rich theory of induced representations, so we refer to Barut and Raczka [1] or Mackey [2] for a more thorough exposition. For some background on measure theory, see e.g. [3, 4].

3.1 Wavefunctions and Measures

Here we start with general considerations on measure theory before reviewing the construction of Hilbert spaces of square-integrable wavefunctions, independently of group theory. We also define Radon–Nikodym derivatives and show that Hilbert spaces of wavefunctions built with equivalent measures are isomorphic. For the record, our approach will not be mathematically rigorous, and is merely intended to give a rough picture of the actual mathematical theory.

3.1.1 Measures

When defining a quantum-mechanical system, one of the key ingredients is a prescription for computing scalar products. For the spaces of wavefunctions that we wish to consider, this requires being able to evaluate integrals of functions on a manifold. Integration, in turn, relies on the existence of a measure.

Measures

Let $\mathcal{M}$ be a set. Roughly speaking, a measure is a function μ that associates a non-negative number with essentially any subset U of $\mathcal{M}$. That number, denoted $\mu(U)$, "measures" the size of U. Strictly speaking, not all subsets of $\mathcal{M}$ can be measured: there exists a family of subsets of $\mathcal{M}$, called "measurable sets", and only those subsets can actually be measured. The measure μ then is a map

$$\mu : \{\text{measurable subsets of } \mathcal{M}\} \to \bar{\mathbb{R}}^+ : U \mapsto \mu(U) \tag{3.1}$$

where $\bar{\mathbb{R}}^+$ denotes the set of non-negative real numbers supplemented with $+\infty$. In order to qualify as a measure, this map needs to satisfy certain conditions; in particular, it must be *σ-additive*: if U_1, U_2, etc. are disjoint measurable sets, then

$$\mu\left(\bigcup_{i=1}^{+\infty} U_i\right) = \sum_{i=1}^{+\infty} \mu(U_i) \qquad \text{when} \;\; U_i \cap U_j = \emptyset \; \forall\, i, j. \tag{3.2}$$

In other words, the total measure of a set consisting of several disconnected components must be the sum of the measures of the individual components. A measure μ on $\mathcal{M}$ is said to be *finite* if $\mu(\mathcal{M})$ is finite; it is *σ-finite* if $\mathcal{M}$ is a countable union of measurable sets with finite measure (any finite measure is trivially σ-finite).

For instance, the standard translation-invariant Lebesgue measure on the real line $\mathbb{R}$ is defined so that $\mu([a, b]) = b - a$ for any closed interval $[a, b] \subset \mathbb{R}$; the measure takes the same value for open or half-open intervals. In particular, $\mathbb{R}$ is a countable union of intervals of finite length, so the Lebesgue measure is σ-finite. This definition is readily generalized to $\mathbb{R}^n$.

If $\mathcal{N}$ is a topological space, a function $\mathcal{F} : \mathcal{M} \to \mathcal{N}$ is said to be *measurable* if $\mathcal{F}^{-1}(V)$ is a measurable subset of $\mathcal{M}$ for any open set V in $\mathcal{N}$. In other words, measurable functions are those that "preserve the structure of measurable sets". Those are the functions that we will be allowed to integrate later on.

Borel Measures

Throughout this chapter and the next ones, we systematically endow $\mathcal{M}$ with a topology. One can take advantage of this structure when defining a measure:

Definition Let $\mathcal{M}$ be a topological space. A *Borel set* in $\mathcal{M}$ is a subset $U \subseteq \mathcal{M}$ which is either an open set, or a closed set, or a union or an intersection of countably many open or closed sets. A *Borel measure* on $\mathcal{M}$ is a measure whose measurable sets are the Borel sets of $\mathcal{M}$.

Thus, Borel measures are compatible with the topology of $\mathcal{M}$. In particular, any continuous function $\mathcal{F} : \mathcal{M} \to \mathcal{N}$ is Borel-measurable. From now on, all measures are understood to be Borel. When $\mathcal{M}$ is a smooth manifold, the data of a Borel measure is equivalent to that of a volume form on $\mathcal{M}$. For simplicity, we always assume that $\mathcal{M}$ is a manifold.

Integrals

Measures can be used to integrate functions.[1] Let μ be a Borel measure on $\mathcal{M}$ and $U \subseteq \mathcal{M}$ a Borel set. When $\mathbb{V}$ is a topological vector space and $\mathcal{F} : \mathcal{M} \to \mathbb{V} : q \mapsto \mathcal{F}(q)$ is a measurable function, the (Lebesgue) integral of $\mathcal{F}$ over U relative to the measure μ is written as[2]

$$\int_U \mathcal{F}(q)\, d\mu(q) \qquad \text{or} \qquad \int_U \mathcal{F}\, d\mu\,.$$

In these terms, the measure $\mu(U)$ of a Borel set U is the integral of the function $\mathcal{F}(q) = 1$ over U:

$$\mu(U) = \int_U d\mu(q). \tag{3.3}$$

The word "measure" often also refers to the quantity $d\mu$ appearing in this expression.

For example, the standard translation-invariant Lebesgue measure on $\mathbb{R}^n$ is denoted $d\mu(x) \equiv d^n x$, with the usual rules for integration. One can generate infinitely

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[1]The concrete definition of integrals relies on a limiting procedure where the integrand is approximated by a sequence of locally constant functions, but we will not review these details here.

[2]We denote points of $\mathcal{M}$ as p, q, etc. to suggest thinking of them as possible momenta of a particle.

many other measures on $\mathbb{R}^n$ by multiplying the Lebesgue measure by an arbitrary function: for any non-negative measurable map $\rho : \mathbb{R}^n \to \mathbb{R} : x \mapsto \rho(x)$, the quantity $d\mu(x) = \rho(x)d^n x$ is a Borel measure on $\mathbb{R}^n$. Another example is provided by the sphere S^2, which admits the rotation-invariant measure $\sin\theta\, d\theta\, d\varphi$ in terms of polar coordinates θ, φ. Finally, in Sect. 4.2 we will use the Lorentz-invariant measure

$$d\mu(\mathbf{q}) = \frac{d^{D-1}\mathbf{q}}{\sqrt{M^2 + \mathbf{q}^2}} \tag{3.4}$$

where M^2 is a positive parameter (the mass squared) while $\mathbf{q} = (q_1, \ldots, q_{D-1})$ is the spatial momentum in D space-time dimensions.

3.1.2 Hilbert Spaces of Wavefunctions

We now have the tools needed to define Hilbert spaces of square-integrable functions. For the sake of generality we consider wavefunctions taking values in a complex Hilbert space $\mathcal{E}$ endowed with a scalar product

$$(\cdot|\cdot) : \mathcal{E} \times \mathcal{E} \to \mathbb{C} : v, w \mapsto (v|w)\,,$$

which we take to be linear in its second argument and antilinear in the first one. When $\mathcal{E} = \mathbb{C}$ we simply set $(v|w) = v^* w$.

Wavefunctions

Definition Let $\mathcal{M}$ be a topological space, μ a Borel measure on $\mathcal{M}$, $\mathcal{E}$ a complex Hilbert space with scalar product $(\cdot|\cdot)$. Then an $\mathcal{E}$-valued *square-integrable wavefunction* is a measurable map $\Psi : \mathcal{M} \to \mathcal{E}$ such that

$$\int_{\mathcal{M}} d\mu(q)\big(\Psi(q)|\Psi(q)\big) < +\infty.$$

We denote by $\mathcal{L}^2(\mathcal{M}, \mu, \mathcal{E})$ the vector space of such functions.

It is tempting to turn $\mathcal{L}^2(\mathcal{M}, \mu, \mathcal{E})$ into a Hilbert space by declaring that the scalar product of two wavefunctions is the integral of their product over $\mathcal{M}$, but there is a problem: wavefunctions need not be continuous. In particular, functions that vanish everywhere on $\mathcal{M}$ except at some countable number of points, are strictly speaking non-zero vectors in $\mathcal{L}^2$ even though all their would-be scalar products vanish. In the language of conformal field theory, those are "null states". In order to cure this pathology, one introduces the following notion:

Definition Let μ be a Borel measure on $\mathcal{M}$. A property is said to be true *almost everywhere* on $\mathcal{M}$ if there exists a Borel set $U \subset \mathcal{M}$ such that $\mu(U) = 0$ and such that the property be true on each point of $\mathcal{M}\backslash U$.

For example, when $\mathcal{F}$ and $\mathcal{G}$ are functions $\mathcal{M} \to \mathcal{N}$, we say that $\mathcal{F} = \mathcal{G}$ almost everywhere on $\mathcal{M}$ and write $\mathcal{F} \sim \mathcal{G}$ if $\mathcal{F}$ and $\mathcal{G}$ differ only on a set of measure zero. The relation $\sim$ is an equivalence relation. This solves the pathology of $\mathcal{L}^2$ spaces, as one can show that integrals of functions that coincide almost everywhere are equal. In particular, any function $\mathcal{F} \sim 0$ is said to *vanish almost everywhere*; such a function belongs to $\mathcal{L}^2$ (the integral of its square vanishes) and can now be identified with the function that vanishes identically on $\mathcal{M}$. More precisely, let us denote by $N(\mathcal{M}, \mu, \mathcal{E})$ the space of $\mathcal{E}$-valued measurable functions on $\mathcal{M}$ that vanish almost everywhere; it is a subspace of $\mathcal{L}^2$ and may be seen as the set of null states (hence the notation N) in $\mathcal{L}^2$. This leads to the following notion:

Definition The *space of square-integrable wavefunctions* on $\mathcal{M}$ with values in $\mathcal{E}$ relative to the measure μ is the quotient of $\mathcal{L}^2$ by N:

$$L^2(\mathcal{M}, \mu, \mathcal{E}) \equiv \mathcal{L}^2(\mathcal{M}, \mu, \mathcal{E}) \big/ N(\mathcal{M}, \mu, \mathcal{E}). \tag{3.5}$$

This space is also simply called the ($\mathcal{E}$-valued) L^2 space on $\mathcal{M}$ relative to the measure μ.

Elements of L^2 are thus equivalence classes of functions $\Psi : \mathcal{M} \to \mathcal{E}$, two functions being identified if they coincide almost everywhere. With this identification, one can endow L^2 with a norm $\| \cdot \|$ defined by

$$\|\Psi\|^2 \equiv \int_{\mathcal{M}} d\mu(q) \big(\Psi(q)|\Psi(q)\big). \tag{3.6}$$

Strictly speaking we should write the left-hand side of this definition as $\|[\Psi]\|^2$, where $[\Psi] \in L^2$ is the class[3] of $\Psi \in \mathcal{L}^2$. However, we will systematically abuse notation by choosing arbitrarily a representative Ψ of a class $[\Psi]$, and we use the word "wavefunction" to refer both to actual functions $\Psi : \mathcal{M} \to \mathcal{E}$ and to the corresponding equivalence classes in L^2.

Formula (3.6) is a well-defined norm on L^2: it is independent of the chosen representative for the class $[\Psi]$, and it satisfies the properties required for a norm. In particular, a function has zero norm if it vanishes almost everywhere, i.e. if its class is the zero vector in L^2. This is indeed the solution of the pathology we encountered in $\mathcal{L}^2$ spaces.

It can be shown that the space L^2 is a complete normed vector space, i.e. a *Banach space*, with respect to the norm (3.6). In addition the space of (equivalence classes of) smooth functions with compact support is dense in $L^2(\mathcal{M}, \mu, \mathcal{E})$, so any wavefunction can be approximated with arbitrary precision by a smooth function.

Hilbert Spaces of Wavefunctions

Definition Let μ be a Borel measure on $\mathcal{M}$, $\mathcal{E}$ a Hilbert space with scalar product $(\cdot|\cdot)$. Let Φ and Ψ be two $\mathcal{E}$-valued square-integrable wavefunctions on $\mathcal{M}$. Then the *scalar product* of Φ and Ψ is

[3]This class has nothing to do with the ray (2.3) despite the identical notation.

$$\langle \Phi | \Psi \rangle \equiv \int_{\mathcal{M}} d\mu(q) \big(\Phi(q) | \Psi(q) \big), \tag{3.7}$$

where the integrand reduces to $\Phi^*(q)\Psi(q)$ when $\mathcal{E} = \mathbb{C}$. The space $L^2(\mathcal{M}, \mu, \mathcal{E})$ is a Hilbert space with respect to this scalar product.

With this definition we can start interpreting $L^2(\mathcal{M}, \mu, \mathcal{E})$ as the space of states of some quantum system. In Dirac notation we would write wavefunctions as $\Psi \equiv |\Psi\rangle$, which is indeed suggested by the notation (3.7). The quantum state defined by such a wavefunction is a ray (2.3) consisting of all functions $\mathcal{M} \to \mathcal{E}$ that are equal almost everywhere to some constant multiple of Ψ. (Again, the notation $[\cdot]$ in (2.3) does *not* mean the same thing as the class of a wavefunction in (3.5)!)

To interpret $\mathcal{E}$-valued wavefunctions, we note the isomorphism

$$L^2(\mathcal{M}, \mu, \mathcal{E}) \cong L^2(\mathcal{M}, \mu, \mathbb{C}) \otimes \mathcal{E}. \tag{3.8}$$

For example suppose $\mathcal{E} = \mathbb{C}^2$ is the Hilbert space of a two-state system (as will be the case, say, for the spin-$1/2$ representation of the Poincaré group in Sect. 4.2). In Dirac notation, we can define an orthonormal basis $\{|+\rangle, |-\rangle\}$ of $\mathcal{E}$ such that a generic (normalized) state of $L^2(\mathcal{M}, \mu, \mathcal{E})$ takes the form

$$|\Phi\rangle = \frac{1}{\sqrt{2}} \Big(|\phi\rangle \otimes |+\rangle + |\psi\rangle \otimes |-\rangle \Big) \tag{3.9}$$

where $|\phi\rangle$ and $|\psi\rangle$ are normalized complex-valued wavefunctions on $\mathcal{M}$. If we think of $\mathcal{E}$ as a space of spin degrees of freedom and if $\mathcal{M}$ is a space of momenta, then the wavefunction (3.9) describes the propagation of two spin states with generally different momentum distributions accounted for by ϕ and ψ. Note that the state (3.9) is typically entangled with respect to the splitting (3.8); it is unentangled if and only if $|\psi\rangle = e^{i\lambda}|\phi\rangle$ for some real number λ. The generalization of (3.9) to higher-dimensional spaces $\mathcal{E}$ is straightforward.

Remark When dealing with unitary representations of the BMS_3 group in part III, we will need to describe square-integrable wavefunctions on infinite-dimensional manifolds (see Sect. 10.1). Until then we will not discuss this issue.

3.1.3 Equivalent Measures and Radon–Nikodym Derivatives

The definition of Hilbert spaces of square-integrable wavefunctions relies on the measure μ used to define the scalar product (3.7). Naively, one might therefore expect that the Hilbert spaces $L^2(\mathcal{M}, \mu, \mathcal{E})$ and $L^2(\mathcal{M}, \nu, \mathcal{E})$ differ if the measures μ and ν do not coincide. However it is easy to show that the space $L^2(\mathcal{M}, \mu, \mathcal{E})$ is essentially independent of the measure μ. Here we prove this statement while introducing the notion of equivalent measures and their Radon–Nikodym derivative.

Definition Let μ, ν be two Borel measures on a manifold $\mathcal{M}$. We say that μ and ν are *equivalent* if they have the same sets of zero measure.

Equivalent measures can be vastly different, yet they are still pretty much the same with regard to measure theory:

Radon–Nikodym theorem Let μ and ν be equivalent σ-finite measures. Then there exists a measurable function $\rho : \mathcal{M} \to \mathbb{R}^+$ such that

$$\nu(U) = \int_U \rho(q) d\mu(q) \qquad \text{for any Borel set } U. \tag{3.10}$$

This relation is often written in infinitesimal form

$$d\nu(q) = \rho(q) d\mu(q) \qquad \text{or} \qquad \rho(q) = \frac{d\nu(q)}{d\mu(q)}. \tag{3.11}$$

In addition, any other function $\tilde{\rho}$ satisfying this property coincides with ρ almost everywhere on $\mathcal{M}$. The function ρ is called the *Radon–Nikodym derivative* of ν with respect to μ.[4] A proof of this theorem can be found in [5].

For example, we mentioned below (3.3) that when $d^n x$ is the Lebesgue measure on $\mathbb{R}^n$, any non-negative function ρ gives rise to a new measure $\rho(x) d^n x$. The Radon–Nikodym derivative of that measure with respect to the Lebesgue measure then coincides with the function ρ. In particular, when $\rho(x)$ only vanishes on a set of Lebesgue measure zero, the measures $d^n x$ and $\rho(x) d^n x$ are equivalent.

Remark When μ and ν are equivalent measures, one has $d\mu(q)/d\nu(q) \sim [d\nu(q)/d\mu(q)]^{-1}$, i.e. the Radon–Nikodym of μ with respect to ν is (almost everywhere) the inverse of the Radon–Nikodym of ν with respect to μ.

Isomorphic L^2 spaces

The notion of equivalent measures allows us to address the question raised above, namely whether the Hilbert spaces $L^2(\mathcal{M}, \mu, \mathcal{E})$ and $L^2(\mathcal{M}, \nu, \mathcal{E})$ differ if the measures μ and ν differ.

Proposition Let μ and ν be equivalent Borel measures on $\mathcal{M}$, $\mathcal{E}$ a Hilbert space; we write $L^2(\mathcal{M}, \mu, \mathcal{E}) \equiv L^2(\mu)$ and similarly for ν. Then there is an isometry

$$\mathcal{U} : L^2(\mu) \to L^2(\nu) : \Psi \mapsto \mathcal{U} \cdot \Psi \qquad \text{with} \qquad (\mathcal{U} \cdot \Psi)(q) \equiv \sqrt{\frac{d\mu(q)}{d\nu(q)}}\, \Psi(q) \tag{3.12}$$

so the spaces $L^2(\mathcal{M}, \mu, \mathcal{E})$ and $L^2(\mathcal{M}, \nu, \mathcal{E})$ are isomorphic as Hilbert spaces.

---

[4] There exist infinitely many functions that all represent equally well the Radon–Nikodym derivative; the theorem ensures that these functions agree, except possibly on a set of zero measure. Accordingly, we call "the" Radon–Nikodym derivative any function that satisfies (3.10).

Proof The map (3.12) is manifestly linear and invertible, since the measures μ and ν are equivalent so that the Radon–Nikodym derivative $\rho = d\nu/d\mu$ is strictly positive almost everywhere. It only remains to prove that $\mathcal{U}$ preserves the scalar products (3.7); let us denote them by $\langle\cdot|\cdot\rangle_\mu$ and $\langle\cdot|\cdot\rangle_\nu$ in $L^2(\mu)$ and $L^2(\nu)$, respectively. For any two μ-square-integrable wavefunctions Φ and Ψ, the definitions (3.11) and (3.12) readily yield $\langle\mathcal{U}\cdot\Phi|\mathcal{U}\cdot\Psi\rangle_\nu = \langle\Phi|\Psi\rangle_\mu$, which proves that $\mathcal{U}$ is an isometry. ■

This proposition says that the structure of the Hilbert space $L^2(\mathcal{M}, \mu, \mathcal{E})$ does not depend on the measure μ, since any other equivalent measure would give rise to an isomorphic Hilbert space. A similar phenomenon will occur in Sect. 3.2.2, where induced representations built with different scalar products will turn out to be equivalent.

3.2 Quasi-regular Representations

In the previous pages we have seen how to build spaces of wavefunctions. Our goal now is to endow such Hilbert spaces with a unitary group action. The strategy will be to take the manifold $\mathcal{M}$ (on which wavefunctions are defined) to be homogeneous with respect to some group action, then use this action to define unitary operators. We now describe this approach after recalling some basic properties of group actions and measures on homogeneous spaces. This will lead to the notion of quasi-regular representations, which provides the simplest example of induced representations.

3.2.1 Quasi-invariant Measures on Homogeneous Spaces

Group Actions and Orbits

Definition Let $\mathcal{M}$ be a manifold, G a Lie group.[5] An *action* of G on $\mathcal{M}$ is a smooth map $G \times \mathcal{M} \to \mathcal{M} : (f, q) \mapsto f \cdot q$ such that $e \cdot q = q$ and $f \cdot (g \cdot q) = (fg) \cdot q$ for all group elements f, g and any $q \in \mathcal{M}$. Equivalently, an action of G on $\mathcal{M}$ is a homomorphism from G to the group $\mathrm{Diff}(\mathcal{M})$ of diffeomorphisms of $\mathcal{M}$.

There exist many important examples of group actions in physics: the space $\mathbb{R}^n$ can be seen as an Abelian group acting on itself by the addition of vectors; the sphere S^2 is acted upon by rotations. More generally, any group representation is a linear action of a group on a vector space; in particular the energy-momentum of a particle in Minkowski space is acted upon linearly by the Lorentz group.

Consider an action of G on $\mathcal{M}$, and pick a point $p \in \mathcal{M}$. The *orbit* of p is the submanifold of $\mathcal{M}$ consisting of all points that can be reached by acting on p with G:

$$\mathcal{O}_p \equiv \{f \cdot p | f \in G\}\,. \tag{3.13}$$

[5] As before elements of G are written as f, g, etc. and the identity is denoted e.

The orbit is independent of the choice of p in the sense that, whenever $q \in \mathcal{O}_p$, we have $\mathcal{O}_p = \mathcal{O}_q$. The *stabilizer* of $p \in \mathcal{M}$ is the subgroup of G that leaves it invariant,

$$G_p \equiv \{ f \in G | f \cdot p = p \} . \tag{3.14}$$

If q is another point in $\mathcal{O}_p$, and if $g \in G$ is such that $g \cdot p = q$, then the stabilizer of q is $g\,G_p\,g^{-1}$, which is isomorphic to G_p. (In particular, one often abuses terminology by saying "the stabilizer of an orbit" instead of the stabilizer of a *point* on the orbit.) The stabilizer is a (closed) subgroup of G and the orbit (3.13) is diffeomorphic to the coset space

$$\mathcal{O}_p \cong G/G_p . \tag{3.15}$$

This diffeomorphism is explicitly given by the bijection $G/G_p \to \mathcal{O}_p : f\,G_p \mapsto f \cdot p$.

Homogeneous Spaces

Definition An action of a group G on $\mathcal{M}$ is said to be *transitive* when for any two points $p, q \in \mathcal{M}$ there exists a group element f such that $f \cdot p = q$. The space $\mathcal{M}$ is then said to be a *homogeneous space* for this action.

In particular a homogeneous space coincides with the orbit of any of its points under the group action: $\mathcal{M} = \mathcal{O}_p$ for any $p \in \mathcal{M}$. It follows that any homogeneous space can be written as a coset space (3.15).

The simplest example of a G-homogeneous space is the group G itself, with the action given by left multiplication:

$$g \longmapsto L_f(g) = fg. \tag{3.16}$$

The stabilizer in that case is trivial. Note that right multiplication

$$g \longmapsto R_f(g) = gf \tag{3.17}$$

is not quite a group action since $R_f \circ R_g = R_{gf}$ does not coincide with R_{fg}. This can be cured by considering right multiplication by *inverse* elements, i.e. $g \mapsto R_{f^{-1}}g = gf^{-1}$. In the aforementioned example of $\mathbb{R}^n$, seen as an Abelian group acting on itself by the addition of vectors, left and right multiplications coincide. (This is true for any Abelian group.) The sphere S^2 is a more interesting example of homogeneous space, since it is acted upon transitively by the group of rotations SO(3) but has a non-trivial stabilizer SO(2), and is therefore diffeomorphic to the quotient SO(3)/SO(2). More generally, one has a family of diffeomorphisms $S^n \cong \mathrm{SO}(n+1)/\mathrm{SO}(n)$. Homogeneous spaces will play a central role in representation theory, so we will encouter many more examples of transitive actions later in this thesis.

The Haar Measure

We now initiate the study of measure theory on homogeneous spaces.

Definition Let $\mathcal{M}$ be a homogeneous space with respect to the action of a group G; let μ be a Borel measure on $\mathcal{M}$. We say that the measure is *invariant* under G if $\mu(f \cdot U) = \mu(U)$ for all $g \in G$ and for any Borel set U.

For example, the measure $\sin\theta\, d\theta\, d\varphi$ on a sphere S^2 is invariant under rotations, while the momentum measure (3.4) is invariant under Lorentz transformations. When the homogeneous space $\mathcal{M}$ is the group manifold G itself, one has the following result:

Haar's theorem Let G be a finite-dimensional Lie group. Then, up to a positive multiplicative constant, there exists a unique Borel measure on G invariant under left multiplication (3.16), known as the *left Haar measure* on G.

Proof Any left-invariant volume form on G is the pull-back by left multiplication of a volume form on the tangent space T_eG at the identity. Since G is finite-dimensional the volume form on T_eG is unique up to a positive multiplicative constant, so the theorem follows. (See e.g. [6] for details.) ■

The same theorem would hold for right multiplications, although the resulting *right* Haar measure generally differs from the left one. If the group G is Abelian, any left-invariant measure is also right-invariant. For instance, the standard measure $d^n x$ on $\mathbb{R}^n$ is the translation-invariant Haar measure for the Abelian group $\mathbb{R}^n$.

Quasi-invariant Measures

Given a homogeneous space $\mathcal{M}$, we wish to integrate functions over it and we ask whether there exists an invariant measure. It turns out that this is not always the case (see e.g. [1]), so one introduces the following weaker notion of invariance:

Definition Let $\mathcal{M}$ be a homogeneous space with respect to the action of a group G. A Borel measure μ on $\mathcal{M}$ is said to be *quasi-invariant* under G if, for any group element f, the measure μ_f defined by

$$\mu_f(U) \equiv \mu(f \cdot U) \qquad \text{for any Borel set } U \tag{3.18}$$

is equivalent to μ.

Equivalent measures are related through (3.11) by their Radon–Nikodym derivative. Accordingly, when μ is a quasi-invariant measure on a homogeneous space $\mathcal{M}$, we denote the Radon–Nikodym derivative of μ_f with respect to μ by

$$\rho_f(q) \equiv \frac{d\mu_f(q)}{d\mu(q)} = \frac{d\mu(f \cdot q)}{d\mu(q)} \tag{3.19}$$

for any $f \in G$ and any $q \in \mathcal{M}$. We shall refer to it as "the" Radon–Nikodym derivative of μ under the action of G. Note that it satisfies the important property

$$\rho_{fg}(q) = \rho_f(g \cdot q)\rho_f(q). \tag{3.20}$$

The measure μ is invariant if and only if its Radon–Nikodym derivative ρ_f is equal ot one (almost everywhere) for any group element f.

Intuitively one can think of the Radon–Nikodym derivative (3.19) as an anomaly: since μ is defined on a homogeneous space, one naively expects it to be invariant under the group action. The Radon–Nikodym derivative measures the extent to which invariance is spoiled. Taking again the example of $\mathbb{R}^n$, the Lebesgue measure $d^n x$ is invariant under translations and rotations, but *not* under arbitrary diffeomorphisms. Indeed, for $f : x \mapsto f(x)$ a diffeomorphism of $\mathbb{R}^n$, the Lebesgue measure transforms as

$$d\mu_f(x) = d^n[f(x)] = \left|\frac{\partial f}{\partial x}\right| d^n x, \tag{3.21}$$

where $|\partial f/\partial x|$ is the Jacobian of f. Thus the Radon–Nikodym derivative of a quasi-invariant measure can also be seen as a generalization of the Jacobian.

Since we motivated quasi-invariant measures by the observation that invariant measures do not always exist, one might worry that a similar problem arises for quasi-invariant measures. Fortunately one can show that, in contrast to invariant measures, quasi-invariant measures *do* always exist on any finite-dimensional homogeneous space (see e.g. [1]). We shall discuss the infinite-dimensional generalization of that statement in Sect. 10.1. Note that the existence of one quasi-invariant measure μ on $\mathcal{M}$ implies the existence of infinitely many of them, since multiplying μ by any positive function yields another quasi-invariant measure.

3.2.2 *The Simplest Induced Representations*

We are now in position to describe quasi-regular representations. Let $\mathcal{M}$ be a manifold acted upon by a group G, and consider a vector space of wavefunctions $\Psi : \mathcal{M} \to \mathbb{C}$. We then readily define a representation $\mathcal{T}$ of G in that space by writing

$$\boxed{(\mathcal{T}[f] \cdot \Psi)(q) \equiv \Psi(f^{-1} \cdot q).} \tag{3.22}$$

Each operator $\mathcal{T}[f]$ is manifestly linear, and the fact that this is indeed a representation follows from the fact that the map $q \mapsto f \cdot q$ is a group action. The interpretation of formula (3.22) is simple: if the wavefunction Ψ is sharply centred around some point k of $\mathcal{M}$, then the operator $\mathcal{T}[f]$ maps Ψ on a new wavefunction, now centred around the point $f \cdot k$. In Chap. 4 the space $\mathcal{M}$ will consist of the allowed momenta of a particle, Ψ will be the particle's wavefunction (in momentum space), and the map $q \mapsto f \cdot q$ will be an action by boosts or rotations (Fig. 3.1).

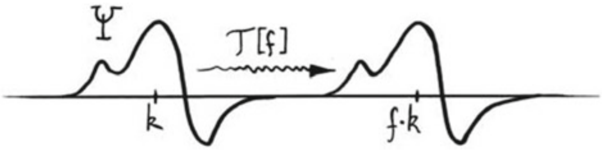

Fig. 3.1 A wavefunction Ψ on $\mathcal{M} = \mathbb{R}$ centred around some point k is acted upon by a unitary operator $\mathcal{T}[f]$ that implements the transformation $k \mapsto f \cdot k$. The resulting transformed wavefunction $\mathcal{T}[f] \cdot \Psi$ is the old one, translated by f

In order to interpret formula (3.22) as the action of a symmetry group on a Hilbert space of wavefunctions, we need to make sure that each operator $\mathcal{T}[f]$ is unitary. If $\mathcal{M}$ is a homogeneous space and μ is a quasi-invariant measure on $\mathcal{M}$, the scalar product of wavefunctions is (3.7) with $(\Phi(q)|\Psi(q)) = \Phi^*(q)\Psi(q)$. Now it is easy to verify that the representation (3.22) is generally *not* unitary for this scalar product:

$$\langle \mathcal{T}[f]\Phi|\mathcal{T}[f]\Psi\rangle = \int_{\mathcal{M}} d\mu(q)\Phi^*(f^{-1}\cdot q)\Psi(f^{-1}\cdot q) \overset{(3.19)}{=} \int_{\mathcal{M}} d\mu(q)\rho_f(q)\Phi^*(q)\Psi(q). \tag{3.23}$$

The far right-hand side generally does not coincide with the original scalar product (3.7) because it involves the Radon–Nikodym derivative (3.19). Thus, in order to ensure unitarity, we need to correct formula (3.22) by a factor that compensates the non-trivial transformation law of μ:

Definition Let G be a Lie group acting transitively on a manifold $\mathcal{M}$. Let μ be a quasi-invariant measure on $\mathcal{M}$ and let $L^2(\mathcal{M}, \mu, \mathbb{C})$ be the space of square-integrable wavefunctions on $\mathcal{M}$. The *quasi-regular representation* $\mathcal{T}$ of G acts on this space according to

$$(\mathcal{T}[f]\cdot\Psi)(q) \equiv \sqrt{\rho_{f^{-1}}(q)}\ \Psi(f^{-1}\cdot q) \tag{3.24}$$

for any wavefunction Ψ, where ρ_f is the Radon–Nikodym derivative (3.19) of μ. If μ is invariant, the quasi-regular representation boils down to (3.22).

Proposition The quasi-regular representation defined by (3.24) is a unitary representation of G in $L^2(\mathcal{M}, \mu, \mathbb{C})$.

Proof First we need to check that (3.24) actually defines a representation, i.e. that $\mathcal{T}[f\cdot g] = \mathcal{T}[f]\circ\mathcal{T}[g]$ for all $f, g \in G$. As in (3.22), linearity of $\mathcal{T}[f]$ is obvious. Now pick a wavefunction $\Psi \in L^2(\mathcal{M}, \mu, \mathbb{C})$. At some point $q \in \mathcal{M}$, we find

$$(\mathcal{T}[fg]\Psi)(q) \overset{(3.24)}{=} \left[\rho_{g^{-1}f^{-1}}(q)\right]^{1/2}\Psi\big(g^{-1}\cdot(f^{-1}\cdot q)\big)$$

where we relied on the fact that $q \mapsto f \cdot q$ is a group action. Now using (3.20) and the definition (3.24), we can rewrite this as

$$(\mathcal{T}[fg]\Psi)(q) = \left[\rho_{g^{-1}}(f^{-1} \cdot q)\rho_{f^{-1}}(q)\right]^{1/2} \Psi\big(g^{-1} \cdot (f^{-1} \cdot q)\big) \tag{3.25}$$

$$= \left[\rho_{f^{-1}}(q)\right]^{1/2} (\mathcal{T}[g]\Psi)\,(f^{-1} \cdot q) = \Big(\big(\mathcal{T}[f] \circ \mathcal{T}[g]\big) \cdot \Psi\Big)(q)\,, \tag{3.26}$$

which proves that (3.24) is indeed a representation. To complete the proof we also have to show that $\mathcal{T}$ is unitary for the scalar product (3.7) with $(\Phi(q)|\Psi(q)) = \Phi^*(q)\Psi(q)$. Repeating the computation (3.23) we now find that the Radon–Nikodym derivative in (3.24) yields an extra term in the integrand. Using (3.20) and the fact that $\rho_e = 1$, this term cancels the Radon–Nikodym derivative in (3.23) so (3.24) is indeed unitary. ■

Remark When the homogeneous space $\mathcal{M}$ coincides with the group G and is endowed with the invariant Haar measure, formula (3.22) defines a unitary representation of G known as the *regular representation*. Quasi-regular representations extend this concept by trading the base manifold G for an arbitrary homogeneous space $\mathcal{M}$.

Equivalence of Quasi-regular Representations

Recall that L^2 spaces defined with equivalent measures are isometric via the map (3.12). One may wonder how that statement affects quasi-regular representations: is it true that two representations of the form (3.24) are equivalent if they are defined using different but equivalent measures? The answer is yes: if μ and ν are equivalent quasi-invariant measures on $\mathcal{M}$ and if we denote the corresponding quasi-regular representations by $\mathcal{T}_\mu$ and $\mathcal{T}_\nu$ respectively, then the isometry (3.12) is an *intertwiner*:

$$\mathcal{U} \circ \mathcal{T}_\mu[f] = \mathcal{T}_\nu[f] \circ \mathcal{U} \qquad \text{for all } f \in G. \tag{3.27}$$

Accordingly, the representations $\mathcal{T}_\mu$ and $\mathcal{T}_\nu$ are unitarily equivalent. This is to say that the quasi-regular representation (3.24) is essentially independent of the measure μ.

*3.2.3 Radon–Nikodym Is a Cocycle**

Here we show that property (3.20) is a cohomological statement: it says that the Radon–Nikodym derivative is a one-cocycle with respect to the representation (3.22). This is an anecdotal observation, so the hasty reader may go directly to Sect. 3.3.

Proposition Let μ be a quasi-invariant measure on a homogeneous space $\mathcal{M}$ and let (3.19) be its Radon–Nikodym derivative. Then the map

$$\log \rho : G \to C^\infty(\mathcal{M}) : f \mapsto \log(\rho_{f^{-1}}) \tag{3.28}$$

is a one-cocycle with respect to the representation (3.22), with the understanding that $\log(\rho_{f^{-1}})$ is the function on $\mathcal{M}$ mapping q on $\log\big(\rho_{f^{-1}}(q)\big)$.

Proof We need to show that the Chevalley–Eilenberg differential (2.31) of the map (3.28) vanishes. Using (3.22) we find

$$(\mathsf{d}\log\rho)_{fg}(q) = \log\rho_{g^{-1}}(f^{-1}\cdot q) + \log\rho_{f^{-1}}(q) - \log\rho_{g^{-1}f^{-1}}(q)\,,$$

which vanishes by virtue of property (3.20). ■

Let us discuss the measure-theoretic interpretation of this cohomological statement. For example, suppose the map (3.28) is a trivial one-cocycle. Then

$$\log\rho_{f^{-1}}(q) = (\mathsf{d}\Psi)_f(q) \overset{(2.31)}{=} (\mathcal{T}[f]\cdot\Psi)(q) - \Psi(q) \overset{(3.22)}{=} \Psi(f^{-1}\cdot q) - \Psi(q)$$

for some function $\Psi(q)$. Equivalently,

$$\frac{d\mu(f\cdot q)}{d\mu(q)} = e^{\Psi(f\cdot q)-\Psi(q)}, \qquad \text{i.e.} \qquad e^{-\Psi(q)}d\mu(q) = e^{-\Psi(f\cdot q)}d\mu(f\cdot q)\,, \tag{3.29}$$

which says that the quasi-invariant measure μ is actually an *invariant* measure in disguise! Indeed, the measure ν defined by $d\nu(q) = e^{-\Psi(q)}d\mu(q)$ is invariant by virtue of (3.29). In other words, the first cohomology of G with values in the space of functions on $\mathcal{M}$ classifies the inequivalent quasi-invariant measures on $\mathcal{M}$, two measures being equivalent if they are related to one another by a function that multiplies them. In particular, the first cohomology vanishes if all quasi-invariant measures on $\mathcal{M}$ are equivalent to an invariant measure.

We can also rephrase this in the language of representation theory: the quasi-regular representation (3.24) is a (multiplicative) affine module (2.33) on top of the original representation (3.22). Two such modules are equivalent if the corresponding measures are equivalent.

Remark The cohomological properties of the Radon–Nikodym derivative have applications in physics: we mentioned below (3.19) that the Radon–Nikodym derivative may be thought of as an anomaly, and indeed anomalies in quantum field theory are one-cocycles for the BRST differential, valued in a suitable space of functions [7]. Our observation on the Radon–Nikodym derivative may be seen as a baby version of that general statement.

3.3 Defining Induced Representations

We now extend the construction of quasi-regular representations. Suppose we have a group G with some (closed) subgroup H. Given a representation $\mathcal{S}$ of H, we wish to induce a corresponding representation $\mathcal{T}$ of G. To describe this mechanism we first need to study in more detail the homogeneous manifold

$$\mathcal{M} \cong G/H\,. \tag{3.30}$$

We will then define an action $\mathcal{T}$ of G on wavefunctions that live on $\mathcal{M}$.

3.3.1 Standard Boosts

In Sect. 3.2.1 we introduced homogeneous spaces and observed that they can be written as coset spaces G/G_p, where G_p is the stabilizer of some point $p \in \mathcal{M}$. Now note that, conversely, any coset space G/H determines a homogeneous manifold (provided H is a closed subgroup of G). Indeed the elements of G/H are left cosets gH, where g spans G, and the action of G on G/H is given by left multiplication: $gH \mapsto f \cdot (gH) = (fg)H$. In particular one can think of G/H as a manifold $\mathcal{M}$ where the coset gH corresponds to the point $q = g \cdot p$, where p is identified with the coset eH at the identity. The stabilizer of $g \cdot p$ is gHg^{-1}, as noted below (3.14). Thus from now on we describe the space G/H with the same notation as in Sect. 3.2.1.

Now consider the point $p \in \mathcal{M}$, identified with the identity coset eH in G/H. Since $\mathcal{M}$ is a homogeneous space one can map p on any other point $q \in \mathcal{M}$, with a group element $g \in G$ such that $g \cdot p = q$. Given q, this group element is only defined up to multiplication from the right by an element of H, since $h \cdot p = p$ for any $h \in H$.

Definition Let G act transitively on $\mathcal{M} \cong G/H$ and let $p \in \mathcal{M}$. Then a *family of standard boosts* for p on $\mathcal{M}$ is a map

$$\mathcal{M} \to G : q \mapsto g_q \qquad \text{such that} \qquad g_q \cdot p = q\,. \tag{3.31}$$

Any homogeneous manifold admits a family of standard boosts. For example, in special relativity, the map (3.31) would typically be an assignment of a Lorentz boost g_q for each possible energy-momentum vector q of a massive particle. If p is the energy-momentum of the particle at rest, g_q would map this momentum p on the boosted momentum q. In fact, in the latter case this assignment can be chosen in such a way that g_q depends continuously on q, so the family of standard boosts is *continuous*. In all cases of interest below, continuous families of standard boosts will exist, so from now on we always assume that the map (3.31) is continuous.

Remark The existence of a continuous family of standard boosts is equivalent to that of a global section for the principal bundle $G \to G/H$, which in turn amounts to saying that this bundle is trivial (see e.g. [8]). In general this is not the case — the typical example is the non-trivial bundle $\mathrm{SO}(n+1) \to S^n$, which is relevant to the Euclidean group in $(n+1)$ dimensions. However, all relativistic symmetry groups as well as BMS_3 are such that continuous families of standard boosts do exist,[6] so we do not need to dwell on this subtlety any further.

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[6]This actually follows from the fact that typical momentum orbits for such groups are homotopic to a point, which then implies that the corresponding bundles $G \to G/H$ are trivial.

3.3.2 Induced Representations

Definition Let G be a group acting transitively on a manifold $\mathcal{M}$ endowed with a quasi-invariant measure μ. Let $p \in \mathcal{M}$ be stabilized by a closed subgroup H of G and let the map (3.31) be a continuous family of standard boosts on $\mathcal{M}$. Then the representation $\mathcal{T}$ of G *induced* by $\mathcal{S}$ acts in the Hilbert space $\mathscr{H} = L^2(\mathcal{M}, \mu, \mathcal{E})$ according to

$$\boxed{(\mathcal{T}[f] \cdot \Psi)(q) \equiv \sqrt{\rho_{f^{-1}}(q)}\, \mathcal{S}[g_q^{-1}\, f\, g_{f^{-1}\cdot q}]\, \Psi(f^{-1} \cdot q)} \tag{3.32}$$

for any wavefunction Ψ, where ρ denotes the Radon–Nikodym derivative (3.19). It is common to write

$$\mathcal{T} = \mathrm{Ind}_H^G(\mathcal{S}). \tag{3.33}$$

As given here, formula (3.32) comes a bit out of the blue, so it is worth analysing its elementary features. First note that, if we denote the trivial representation of a group by the symbol $\mathbf{1}$, then $\mathrm{Ind}_H^G(\mathbf{1})$ is the quasi-regular representation (3.24) of G on G/H while $\mathrm{Ind}_{\{e\}}^G(\mathbf{1})$ is the regular representation. Formula (3.32) extends these constructions by including a non-trivial action of H in an internal space $\mathcal{E}$. Before interpreting (3.32) any further, we now verify that it is a consistent definition.

Consistency

Up to the term involving $\mathcal{S}$, the right-hand side of (3.32) coincides with the quasi-regular representation (3.24), so the only potential problem could arise from the insertion of $\mathcal{S}$. But since the map (3.31) is a family of standard boosts, we have

$$\left(g_q^{-1}\, f\, g_{f^{-1}\cdot q}\right) \cdot p = g_q^{-1} \cdot \Big(f \cdot \left(g_{f^{-1}\cdot q} \cdot p\right)\Big) = g_q^{-1} \cdot \Big(f \cdot \left(f^{-1} \cdot q\right)\Big) = g_q^{-1} \cdot q = p\,,$$

so the combination $g_q^{-1}\, f\, g_{f^{-1}\cdot q}$ belongs to the stabilizer H of p, as it should. Since by assumption Ψ takes its values in the carrier space $\mathcal{E}$ of $\mathcal{S}$, we conclude that the right-hand side of (3.32) is well-defined. For each $f \in G$, it defines a linear operator $\mathcal{T}[f]$ acting on $\mathcal{E}$-valued functions on $\mathcal{M}$.

Proposition Let $\mathscr{H} = L^2(\mathcal{M}, \mu, \mathcal{E})$. Then the map $\mathcal{T} : G \to \mathrm{GL}(\mathscr{H})$ defined by (3.32) is a unitary representation of G.

Proof If it were not for the representation $\mathcal{S}$, formula (3.32) would coincide with (3.24); since the latter is a unitary representation of G, we only have to convince ourselves that this feature is not spoiled by the presence of $\mathcal{S}$. Noting that

$$g_q^{-1}\, f g\, g_{(fg)^{-1}\cdot q} = \left(g_q^{-1}\, f\, g_{f^{-1}\cdot q}\right) \cdot \left(g_{f^{-1}\cdot q}^{-1}\, g\, g_{(fg)^{-1}\cdot q}\right) \tag{3.34}$$

and using the fact that $\mathcal{S}$ is a representation of H, one can mimick the sequence of Eqs. (3.25) and (3.26) for $\mathcal{T}$ given by (3.32), which implies that it is indeed a representation. As for unitarity, it follows from the fact that $\mathcal{S}$ is unitary: $\big(\mathcal{S}[h]\Phi(q)\big|\mathcal{S}[h]\Psi(q)\big) = \big(\Phi(q)\big|\Psi(q)\big)$ for all wavefunctions Φ, Ψ, any point $q \in \mathcal{M}$ and any $h \in H$. ■

Interpretation

The basic interpretation of the induced representation (3.32) is the same as for (3.24): $\Psi(f^{-1} \cdot q)$ represents the fact that the wavefunction Ψ is "boosted" by f, while the factor $\sqrt{\rho_{f^{-1}}}$ ensures unitarity. The new ingredient is the combination

$$\mathcal{S}\left[g_q^{-1}\, f\, g_{f^{-1}\cdot q}\right] \equiv W_q[f]. \tag{3.35}$$

Its appearance represents the fact that, in contrast to the quasi-regular representation, wavefunctions take their values not in $\mathbb{C}$, but in some more general "internal" Hilbert space $\mathcal{E}$ carrying a representation $\mathcal{S}$ of H.

When interpreting induced representations as particles, H is typically a group of spatial rotations combined with space-time translations, the space $\mathcal{E}$ consists of spin degrees of freedom, and $\mathcal{S}$ determines the value of spin. In that context the operator (3.35) is known as the *Wigner rotation* associated with f at momentum q. The quasi-regular representation (3.24) can thus be seen as a "scalar" induced representation, as opposed to the spinning case (3.32). Wigner rotations are trivial for scalar particles. Note that because $\mathcal{S}$ is a representation, Wigner rotations satisfy the property $W_q[fg] = W_q[f]\, W_{f^{-1}\cdot q}[g]$.

Remark We mentioned on p. xxx that generic homogeneous manifolds do not admit continuous families of standard boosts, which invalidates the global well-definiteness of the Wigner rotation (3.35). This problem can be cured by reformulating induced representations in terms of wavefunctions defined on the group manifold G rather than G/H (see e.g. [1]). In this thesis we systematically use the homogeneous space viewpoint (3.32), as it will suffice for all cases of interest below. This being said, note that the reformulation in terms of wavefunctions on G is useful for certain applications of three-dimensional higher-spin theories [9, 10] due to the relation between induced representations and harmonic analysis on homogeneous spaces [11, 12].

3.3.3 Properties of Induced Representations

Induced representations have a number of important properties that we now explore. We first show that the definition (3.32) is "robust" in that it depends neither on the choice of the measure μ, nor on the choice of standard boosts (3.31). Then we turn to the behaviour of induced representations under operations such as direct sums and tensor products.

Robustness

Formula (3.32) depends not only on the inducing data (the group G, its subgroup H and a spin representation $\mathcal{S}$), but also on the measure μ on $\mathcal{M} \cong G/H$ and on the choice of a family of standard boosts g_q. Naively, one expects all these parameters to affect the representation. We now show that this is not the case.

As far as the measure is concerned, one readily verifies that two induced representations defined with the same inducing data and the same standard boosts but different, though equivalent, quasi-invariant measures, are unitarily equivalent. The proof is essentially the same as for quasi-regular representations (see Eq. (3.27)), and the intertwiner is the map (3.12). As regards standard boosts, a similar result holds:

Proposition Let $\mathcal{M} \cong G/H$, $\mathcal{S}$ a spin representation of H, $\mathscr{H} = L^2(\mathcal{M}, \mu, \mathcal{E})$. Let $g : \mathcal{M} \to G : q \mapsto g_q$ and $g' : \mathcal{M} \to G : q \mapsto g'_q$ be two continuous families of standard boosts and call $\mathcal{T}, \mathcal{T}'$ (respectively) the corresponding induced representations of G. Then there is a unitary operator

$$\mathcal{V} : \mathscr{H} \to \mathscr{H} : \Psi \mapsto \mathcal{V} \cdot \Psi \qquad \text{with} \qquad (\mathcal{V} \cdot \Psi)(q) = \mathcal{S}\left[g_q^{-1} \cdot g'_q\right] \Psi(q) \tag{3.36}$$

that intertwines $\mathcal{T}$ and $\mathcal{T}'$, which are therefore unitarily equivalent:

$$\mathcal{T}[f] \circ \mathcal{V} = \mathcal{V} \circ \mathcal{T}'[f] \qquad \forall f \in G. \tag{3.37}$$

Proof The fact that $\mathcal{V}$ is a unitary operator follows from unitarity of $\mathcal{S}$. Property (3.37) then follows from the definitions (3.32) and (3.36). ■

A corollary of these observations on robustness is that one may unambiguously say "the" representation of G induced by the representation $\mathcal{S}$ of H, without any reference to the measure or to the choice of standard boosts.

Remark The transformation (3.36) may be seen as a gauge transformation with gauge group H. Indeed the combination $g_q^{-1} g'_q \in H$ can depend on q in an arbitrary way, owing to one's freedom in the choice of standard boosts. It acts on wavefunctions as a momentum-dependent transformation $\Psi \mapsto \mathcal{V} \cdot \Psi$ given by the representation $\mathcal{S}$, and each such transformation maps the system on a unitarily equivalent one. In fact, the differentiation of the operator $\mathcal{S}[g_q^{-1} f g_{f^{-1} \cdot q}]$ defines a gauge field on G/H valued in the Lie algebra $\mathfrak{h}$ of H, and the Wigner rotation itself may be seen as a holonomy (see e.g. [13, 14]).

Operations on Induced Representations

We now study the behaviour of induced representations under standard operations such as conjugation, direct sums and the like. The proofs are omitted and we refer to [1] for details.

Let $\mathcal{E}$ be a Hilbert space with scalar product $(\cdot|\cdot)$. We call *conjugation* the map $C : \mathcal{E} \to \mathcal{E}^*_{\text{cts}} : v \mapsto (v|\cdot)$, where $\mathcal{E}^*_{\text{cts}}$ denotes the space of continuous linear functionals[7]

[7] Recall that any continuous linear functional on a Hilbert space $\mathcal{E}$ is a scalar product $(v|\cdot)$ for some fixed vector $v \in \mathcal{E}$.

on $\mathcal{E}$. Then, if $\mathcal{S}$ is a unitary representation acting on $\mathcal{E}$, its *conjugate representation* is $\overline{\mathcal{S}} \equiv C \circ \mathcal{S} \circ C^{-1}$. In the context of induced representations, one can then show that[8] $\text{Ind}_H^G(\overline{\mathcal{S}}) \sim \overline{\text{Ind}_H^G(\mathcal{S})}$, i.e. the representation induced by the conjugate of $\mathcal{S}$ is unitarily equivalent to the conjugate of the representation induced by $\mathcal{S}$.

One can similarly show that induced representations behave well under direct sums and tensor products thanks to the unitary equivalences

$$\begin{aligned} \text{Ind}_H^G(\mathcal{S}_1 \oplus \mathcal{S}_2) &\sim \text{Ind}_H^G(\mathcal{S}_1) \oplus \text{Ind}_H^G(\mathcal{S}_2)\,, \\ \text{Ind}_H^G(\mathcal{S}_1 \otimes \mathcal{S}_2) &\sim \text{Ind}_H^G(\mathcal{S}_1) \otimes \text{Ind}_H^G(\mathcal{S}_2)\,. \end{aligned} \tag{3.38}$$

As a corollary, if $\mathcal{S}$ is reducible, then $\text{Ind}_H^G(\mathcal{S})$ is reducible. The converse is not true; for instance, if $H = \mathbf{1}$ is the trivial subgroup with $\mathcal{S}$ the irreducible trivial representation, then $\mathcal{T}$ is the regular representation, which is generally reducible.

One should also check that induction itself is a "good" operation on representations. This is guaranteed by the theorem of *induction in stages*: let H_1 be a closed subgroup of H_2, which itself is a closed subgroup of G. Let $\mathcal{S}$ be a unitary representation of H_1. Then one has the following unitary equivalence of representations:

$$\text{Ind}_{H_1}^G(\mathcal{S}) \sim \text{Ind}_{H_2}^G\big(\text{Ind}_{H_1}^{H_2}(\mathcal{S})\big).$$

In other words, inducing directly from H_1 to G, or from H_1 to H_2 and then to G, are the same operations.

3.3.4 Plane Waves

For practical purposes it is convenient to rewrite formula (3.32) in a basis of *plane wave states*. In the relativistic context, they represent particles with definite momentum.

Delta Functions

Let $\mathcal{M} \cong G/H$ be a homogeneous space, μ a quasi-invariant measure on $\mathcal{M}$. Pick a point $k \in \mathcal{M}$. We define the *Dirac distribution* δ_k at k associated with μ as the distribution such that $\langle \delta_k, \varphi \rangle \equiv \varphi(k)$ for any test function φ on $\mathcal{M}$. Equivalently, we introduce a "Dirac delta function" $\delta(k, \cdot)$ such that

$$\langle \delta_k, \varphi \rangle = \int_{\mathcal{M}} d\mu(q)\delta(k, q)\varphi(q) \equiv \varphi(k). \tag{3.39}$$

Thus the distribution δ_k acts on a test function $\varphi(\cdot)$ by integrating it against the delta function $\delta(k, \cdot)$.

[8] The symbol $\sim$ denotes unitary equivalence of representations.

Note that the definition of the delta function δ relies on the measure μ since the combination $d\mu(q)\delta(k, q)$ is G-invariant by design. To make this explicit, let us denote by δ_μ the delta function associated with μ. If ρ is some positive function on $\mathcal{M}$ and $d\nu(q) = \rho(q)d\mu(q)$ is a new measure, then the delta function δ_ν associated with ν differs from δ_μ by a factor ρ:

$$\delta_\nu(k, q) = \frac{\delta_\mu(k, q)}{\rho(q)}. \tag{3.40}$$

In particular, since μ is quasi-invariant under G, for any $f \in G$ we have

$$\delta_\mu(f \cdot k, f \cdot q) = \frac{\delta_\mu(k, q)}{\rho_f(q)} \tag{3.41}$$

where ρ_f is the Radon–Nikodym derivative (3.19). Thinking of the latter as a kind of Jacobian, Eq. (3.41) is a restatement of the transformation law of the Dirac distribution under changes of coordinates. In what follows we shall not indicate explicitly the dependence of δ on the measure μ.

The best known Dirac distribution is the one associated with the translation-invariant Lebesgue measure on $\mathbb{R}^n$. We will encounter this delta function repeatedly so we denote it by $\delta^{(n)}$ to distinguish it from other Dirac distributions. With that notation, the delta function associated with the Lorentz-invariant measure (3.4) is

$$\delta(\mathbf{k}, \mathbf{q}) = \sqrt{M^2 + \mathbf{q}^2}\; \delta^{(D-1)}(\mathbf{k} - \mathbf{q}). \tag{3.42}$$

Plane Wave States

Definition Let $\{e_1, \ldots, e_N\}$ be a countable orthonormal basis[9] of $\mathcal{E}$. For $\ell \in \{1, \ldots, N\}$ and $k \in \mathcal{M}$, we call *plane wave state* with spin ℓ and momentum k the wavefunction

$$\Psi_{k,\ell}(q) \equiv e_\ell\, \delta(k, q) \tag{3.43}$$

where δ is the Dirac distribution associated with the measure μ on $\mathcal{M}$.

In non-relativistic quantum mechanics, if we were describing a particle on the line $\mathcal{M} = \mathbb{R}$, a plane wave would typically be one of the states $\Psi_x = |x\rangle$ representing a particle located at the point $x \in \mathbb{R}$ (with infinite momentum uncertainty). In the dual, momentum-space picture, a plane wave would be a state $\Psi_k = |k\rangle$ with definite momentum k (but infinite position uncertainty). Our terminology is motivated by the latter viewpoint. In the Poincaré case a wavefunction (3.43) will describe a particle with definite spin projection ℓ and energy-momentum k, i.e. a typical asymptotic state in a scattering experiment. With the notation of (3.9), for instance, the space $\mathcal{E}$ of spin degrees of freedom is two-dimensional and has a basis $\{|+\rangle, |-\rangle\} = \{e_1, e_2\}$ consisting of states with definite spin along the vertical axis.

[9] We are assuming that $\mathcal{E}$ is a separable Hilbert space; N may be infinite.

Scalar products of plane waves can be evaluated thanks to the definition (3.7). Using the fact that the basis of e_k's is orthonormal and the definition (3.39) of the delta function, one finds

$$\langle \Psi_{k,\ell} | \Psi_{k',\ell'} \rangle = \delta(k, k') \delta_{\ell\ell'} \,. \tag{3.44}$$

This property allows us to see in which sense plane waves form a "basis" of the Hilbert space $\mathscr{H} = L^2(\mathcal{M}, \mu, \mathcal{E})$. Indeed, any wavefunction $\Phi : \mathcal{M} \to \mathcal{E}$ can be written as

$$\Phi(q) = \sum_{\ell=1}^{N} \Phi_\ell(q)\, e_\ell \qquad \text{where} \qquad \Phi_\ell(k) = \langle \Psi_{k,\ell} | \Phi \rangle \,.$$

Removing the argument q, this says that any wavepacket Φ is a superposition of plane waves:

$$\Phi = \int_{\mathcal{M}} d\mu(k) \sum_{\ell=1}^{N} \Phi_\ell(k) \Psi_{k,\ell} = \int_{\mathcal{M}} d\mu(k) \sum_{\ell=1}^{N} \langle \Psi_{k,\ell} | \Phi \rangle \, \Psi_{k,\ell} \,. \tag{3.45}$$

Note that this can be interpreted as the completeness relation

$$\mathbb{I} = \int_{\mathcal{M}} d\mu(k) \sum_{\ell=1}^{N} \langle \Psi_{k,\ell} | \cdot \rangle \, \Psi_{k,\ell} \tag{3.46}$$

where $\mathbb{I}$ is the identity operator. In the more common (but less precise) Dirac notation this would be a sum of projectors $|\Psi_{k,\ell}\rangle\langle\Psi_{k,\ell}|$. For example, for a particle on the real line, the Dirac form of this completeness relation would read $\mathbb{I} = \int_{\mathbb{R}} dx |x\rangle\langle x|$ in position space, or $\mathbb{I} = \int_{\mathbb{R}} dk |k\rangle\langle k|$ in momentum space. We will show in Sect. 3.5 that the existence of a family of projectors associated with $\mathcal{M}$ is one of the key properties of induced representations.

From the construction of plane waves we see that induced representations are just an upgraded version of one-particle quantum mechanics, with extra freedom in the choice of the space $\mathcal{M}$, the group G, and the spin states contained in $\mathcal{E}$. This observation will guide us in developing our intuition of induced representations, especially in part III of the thesis.

Remark The quantity (3.43) is not a square-integrable function on $\mathcal{M}$ and therefore does not, strictly speaking, belong to the Hilbert space. The same problem arises in standard quantum mechanics, where the states $|x\rangle$ or $|k\rangle$ form a "basis" only in a weak sense. Intuitively one can think of plane waves (3.43) as idealizations of Gaussian wavefunctions centred at k in the limit where their spread goes to zero. A more rigorous way to include such states is to work with so-called *rigged Hilbert spaces* (see e.g. [15, 16]), which are designed so as to include both standard square-integrable functions and distributions.

Boosted Plane Waves

We can now rewrite formula (3.32) for induced representations in terms of plane waves. Choosing a plane wave (3.43), for $f \in G$ and $q \in \mathcal{M}$ we find

$$\Psi_{k,\ell}(f^{-1} \cdot q) = \rho_f(k)\, \Psi_{f \cdot k,\ell}(q) \tag{3.47}$$

where we used (3.41) and the property (3.20). Note what we have achieved: in the original definition (3.32) the argument of Ψ changes between the left and the right-hand sides; here, by contrast, the argument will be the same, but what changes is the label specifying the momentum of the plane wave. Indeed, using (3.47) in formula (3.32), we find

$$\big(\mathcal{T}[f] \cdot \Psi_{k,\ell}\big)(q) = \sqrt{\rho_{f^{-1}}(q)\rho_f(k)}\, \mathcal{S}\left[g_q^{-1}\, f\, g_{f^{-1}\cdot q}\right] \cdot \Psi_{f\cdot k,\ell}(q).$$

Since the plane wave on the right-hand side contains a delta function $\delta(f \cdot k, q)$, we can replace all q's in this expression by $f \cdot k$ and remove the argument from both sides. Using once more (3.20) in $\sqrt{\rho_{f^{-1}}(q)\rho_f(k)} = \sqrt{\rho_f(k)}$, we end up with

$$\mathcal{T}[f] \cdot \Psi_{k,\ell} = \sqrt{\rho_f(k)}\, \mathcal{S}\left[g_{f\cdot k}^{-1}\, f\, g_k\right] \cdot \Psi_{f\cdot k,\ell} \tag{3.48}$$

This formula is the simplest rewriting of the induced representation (3.32). The only extra improvement we could still add is to write as $(\mathcal{S}[\cdots])_{\ell\ell'}$ the matrix element of the operator $\mathcal{S}[\cdots]$ between the states e_ℓ and $e_{\ell'}$, whereupon (3.48) becomes

$$\mathcal{T}[f] \cdot \Psi_{k,\ell} = \sqrt{\rho_f(k)}\, \left(\mathcal{S}\left[g_{f\cdot k}^{-1} \cdot f \cdot g_k\right]\right)_{\ell',\ell} \cdot \Psi_{f\cdot k,\ell'} \tag{3.49}$$

with implicit summation over ℓ'.

Formula (3.48) gives a geometric picture of the states of an induced representation. Indeed, the label k spans all points of $\mathcal{M}$, so we can now view each point of $\mathcal{M}$ as a quantum state. (More precisely, a point of $\mathcal{M}$ is a family of $\dim(\mathcal{E})$ linearly independent states.) Two different points of $\mathcal{M}$, say k and k', then correspond to two linearly independent states Ψ_k and $\Psi_{k'}$, and a transformation f of $\mathcal{M}$ mapping k on $k' = f \cdot k$ gives rise to a unitary operator $\mathcal{T}[f]$ relating the corresponding plane waves. This is a *geometrization* of representation theory: we can "see" each linearly independent state of the representation $\mathcal{T}$ as a point of G/H. This observation is at the core of the *orbit method*, which consists in quantizing suitable homogeneous manifolds to obtain unitary group representations (see Chap. 5). In three-dimensional gravity, the phase space of gravitational perturbations will turn out to be precisely such a homogeneous manifold, and its quantization will produce a Hilbert space of "soft" or "boundary gravitons". We will address these questions in Chap. 8 and in part III of the thesis.

3.4 Characters

In this section we describe the characters associated with induced representations. We start by motivating and defining characters in general terms, before proving the Frobenius formula. We end by discussing the relation between characters and fixed point theorems.

3.4.1 Characters Are Partition Functions

Unitary representations may be seen as general models of symmetric quantum systems: any system invariant under a certain symmetry group G forms a (generally reducible, generally projective) unitary representation of G. Accordingly, symmetry generators provide natural observables in the system, and one may ask about the properties of these observables — typically, about their spectrum.

When a system is invariant under time translations, for instance, the corresponding symmetry generator is the Hamiltonian operator H. The information about its spectrum is captured by the canonical *partition function*[10]

$$Z(\beta) = \mathrm{Tr}\left(e^{-\beta H}\right), \tag{3.50}$$

where β is the inverse of the temperature. If the system admits extra symmetries such as, say, rotations, one can look for the maximal set of mutually commuting symmetry generators Q_a and switch on their chemical potentials μ_a.[11] The spectrum of these new operators, together with H, is then contained in the grand canonical partition function

$$Z(\beta, \mu_1, \ldots, \mu_r) = \mathrm{Tr}\Bigg(\exp\Big[-\beta\Big(H - \sum_{a=1}^{r} \mu_a Q_a\Big)\Big]\Bigg). \tag{3.51}$$

Now suppose we take β to be purely imaginary (while keeping the μ_a's real) in this expression. Then the operator inside the trace is unitary, since it is an exponential of anti-Hermitian operators. In fact, it is a symmetry transformation acting in the Hilbert space according to some unitary representation $\mathcal{T}$, so we can write

$$Z(\beta, \mu_1, \ldots, \mu_r) = \mathrm{Tr}\,(\mathcal{T}[f])$$

for some element f belonging to the symmetry group G. This motivates the following definition:

[10]The notation "Z" stands for the German word *Zustandssumme*, meaning "sum over states."

[11]Here the index a runs from one to r, the latter being essentially the rank of the symmetry group.

Definition Let $\mathcal{T}$ be a representation of a group G in a complex vector space $\mathscr{H}$. The *character* of that representation is the map[12]

$$\chi : G \to \mathbb{C} : f \mapsto \chi[f] \equiv \mathrm{Tr}\,(\mathcal{T}[f])\,. \tag{3.52}$$

This definition ensures that $\chi[f]$ is independent of the basis of $\mathscr{H}$ used to evaluate it. As an application, recall that two group elements f and f' are conjugate if there exists an element $g \in G$ such that $f' = gfg^{-1}$, and that the conjugacy class of f is

$$[f] \equiv \left\{g\,f\,g^{-1}\middle|g \in G\right\}.$$

Thus, formula (3.52) ensures that characters are *class functions* in the sense that $\chi[f]$ only depends on the conjugacy class of f, and not on f itself:

$$\chi[f] = \chi[g\,f\,g^{-1}]. \tag{3.53}$$

Remark The definition (3.52) suggests that characters are functions on G. While this is true for finite-dimensional representations, it is *not* true in infinite-dimensional ones. In fact, characters should not be seen as functions, but rather as *distributions* [19]. Similarly to our dealing with Dirac distributions as if they were "delta functions", we will not take such mathematical subtleties into account.

3.4.2 The Frobenius Formula

Our derivation of the character formula for induced representations is inspired by [20], although the formula itself appears in many textbooks on group theory; see e.g. [21].

Theorem The character of the induced representation $\mathcal{T} = \mathrm{Ind}_H^G(\mathcal{S})$ defined by (3.32) is given by the *Frobenius formula*

$$\chi[f] = \boxed{\mathrm{Tr}\,(\mathcal{T}[f]) = \int_{\mathcal{M}} d\mu(k)\,\delta(k,\,f\cdot k)\,\chi_{\mathcal{S}}[g_k^{-1}\,f\,g_k]} \tag{3.54}$$

where μ is a quasi-invariant measure on $\mathcal{M} \cong G/H$, δ is the associated Dirac distribution, the g_k's are standard boosts, and $\chi_{\mathcal{S}}$ is the character of $\mathcal{S}$.

Proof Let $f \in G$ and let $\mathcal{T}[f]$ be the associated unitary operator (3.32). We work in the basis of plane wave states (3.43) so that the trace of $\mathcal{T}[f]$ reads

[12]The terminology of "characters" is due to Weber and Frobenius, and stems from the fact that irreducible representations of finite groups are wholly characterized by their character (see e.g. [17, p.117] or [18, p.783]).

$$\chi[f] = \mathrm{Tr}\,(\mathcal{T}[f]) = \int_{\mathcal{M}} d\mu(k) \sum_{\ell=1}^{N} \big\langle \Psi_{k,\ell} \big| \mathcal{T}[f] \cdot \Psi_{k,\ell} \big\rangle$$

where we "sum over momenta" thanks to the measure μ on $\mathcal{M}$. Now using (3.48) and the scalar products (3.44), we find

$$\chi[f] = \int_{\mathcal{M}} d\mu(k) \sqrt{\rho_f(k)} \sum_{\ell=1}^{N} \langle \psi_{k,\ell} | \mathcal{S}[g_{f\cdot k}^{-1} \cdot f \cdot g_k] \psi_{f\cdot k,\ell} \rangle \tag{3.55}$$

$$\overset{(3.44)}{=} \int_{\mathcal{M}} d\mu(k) \sqrt{\rho_f(k)}\, \delta(k, f\cdot k) \sum_{\ell=1}^{N} \Big(\mathcal{S}[g_{f\cdot k}^{-1} \cdot f \cdot g_k] \Big)_{\ell\ell} .$$

Here the delta function $\delta(k, f \cdot k)$ allows us to trade $f \cdot k$ for k. In particular the Radon–Nikodym derivative $\rho_f(k) = d\mu(f \cdot k)/d\mu(k)$ reduces to unity. One then recognizes the sum $\sum_{\ell=1}^{N} (\mathcal{S}[\cdots])_{\ell\ell} \equiv \chi_{\mathcal{S}}[\cdots]$ as the character of $\mathcal{S}$, and Eq. (3.54) follows. ■

The Frobenius formula (3.54) embodies the geometrization of representation theory mentioned at the end of Sect. 3.3.4: the trace of an operator has now become an integral over a (subset of a) homogeneous space. That integral can be interpreted as a sum of characters of $\mathcal{S}$. Before studying this formula further, we need to check that it satisfies the basic properties of a character.

First, since it is the character of an induced representation and since the latter is independent (up to unitary equivalence) of the choice of the measure μ, the same should be true of expression (3.54). To see that this is indeed the case, recall that the combination $d\mu(k)\delta(k, \cdot)$ is invariant under changes of measures (as follows from the definition (3.39) of the Dirac distribution), which then implies invariance of the character. Note in particular that the Radon–Nikodym derivative of the measure μ does not appear in (3.54). Secondly, induced representations are independent of the choice of standard boosts g_q; using the fact that the character of $\mathcal{S}$ is a class function, one readily verifies that (3.54) is also independent of that choice. Finally, recall from (3.53) that characters are class functions; using (3.20) one verifies that this is indeed the case with formula (3.54). Note one crucial implication of this fact: because of the term $\chi_{\mathcal{S}}[g_k^{-1} f g_k]$ in (3.54), the character $\chi[f]$ vanishes if f is not conjugate to an element of H. In other words the character of the induced representation $\mathrm{Ind}_H^G(\mathcal{S})$ is supported on the points of G whose conjugacy class intersects H.

Remark Since the character $\chi_{\mathcal{S}}$ of the spin representation is a class function, one is naively tempted to pull the term $\chi_{\mathcal{S}}[g_k^{-1} f g_k]$ out of the integral (3.54), as the notation suggests that f is conjugate to $g_k^{-1} f g_k$. This is *not* true, because for generic $g, g' \in G$ one has $\chi_{\mathcal{S}}[g^{-1} f g] \neq \chi_{\mathcal{S}}[g'^{-1} f g']$. As a consequence, the integral (3.54) is generally non-trivial.

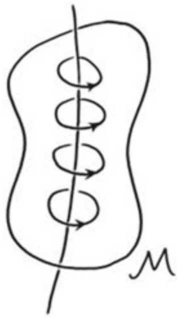

Fig. 3.2 A manifold $\mathcal{M}$ acted upon by a rotation around some axis. The points that belong to the axis are the only ones left fixed by the rotation, and are therefore the only ones that contribute to the integral of formula (3.54)

3.4.3 Characters and Fixed Points

Formula (3.54) is one of the key results of this chapter. Its two most salient features are (i) the fact that the character of $\mathcal{T}$ is completely specified by that of $\mathcal{S}$ and the space $\mathcal{M}$, and (ii) the fact that it is an integral over the points of $\mathcal{M}$ that are left fixed by f. At first sight the latter observation is a surprise: there is no obvious reason why a sum over all states of the induced representation would collapse to an integral over fixed points of f, though in practice this is due to the scalar products $\langle \Psi_k | \Psi_{f \cdot k} \rangle$ in the trace (3.55). This collapse is an instance of *localization*: an integral localizes to a small subset of points in $\mathcal{M}$ (Fig. 3.2), so that the evaluation of (3.54) becomes child's play. We shall encounter this situation with the BMS_3 group in Sect. 10.3.

Remark The relation between characters of group representations and fixed point theorems is much deeper and more general than the superficial description given here. Indeed one can show [22] (see also [19]) that (3.54) coincides with the Lefschetz number of $\mathcal{T}[f]$ when the latter is seen as an endomorphism acting on a space of $\mathcal{E}$-valued sections on $\mathcal{M}$. In turn, the fact that $\mathcal{T}[f]$ is derived by (3.32) from a diffeomorphism action of f on $\mathcal{M}$ turns out to imply that its Lefschetz number is given by the Atiyah–Bott fixed point theorem [23].

3.5 Systems of Imprimitivity*

This technical section is for advanced reading: other than for a key corollary that implies the exhaustivity of induced representations for semi-direct products, it is inconsequential to the remainder of the thesis and may be skipped in a first reading.

We saw in Eq. (3.46) that the identity operator $\mathbb{I}$ can be written as an integral of projectors $\Psi_k \langle \Psi_k | \cdot \rangle = |\Psi_k\rangle\langle\Psi_k|$. This leads to a seemingly random idea: why not combine these projectors into more general operators? For example, if U is any Borel subset of $\mathcal{M}$, we can associate with it a projection operator

$$P_U \equiv \int_U d\mu(k) \sum_{\ell=1}^{N} \langle \Psi_{k,\ell} | \cdot \rangle \, \Psi_{k,\ell} \,. \tag{3.56}$$

In that language the identity operator is $\mathbb{I} = P_{\mathcal{M}}$. As it turns out this idea is one of the key properties of induced representations, and sparked the whole development of the theory by Mackey in the fifties [24–26]. In particular it leads to the *imprimitivity theorem*, which roughly states that *any* representation that admits a suitable family of projectors (3.56) is necessarily induced. An important corollary of that result is the fact that all irreducible unitary representations of semi-direct products are induced.

The plan of this section is the following. We first define the notion of systems of imprimitivity as suitable families of projection operators, and show that any induced representation admits such a family. We then state (without proof) the imprimitivity theorem, which we eventually use to define a restricted notion of equivalence for induced representations. The presentation is based on [1, 2], but our approach will be heuristic at times; we refer to [27, 28] for a mathematically rigorous presentation.

3.5.1 Projections and Imprimitivity

Here we describe the operators (3.56) in the framework of projection-valued measures and show that they form a system of imprimitivity.

Projection-Valued Measures

Let us put (3.56) in a more general context. Observe that, given the Borel set U, the projector P_U acts on wavefunctions by setting them to zero everywhere outside of U:

$$\big(P_U \cdot \Psi\big)(q) = \begin{cases} \Psi(q) & \text{if } q \in U, \\ 0 & \text{otherwise.} \end{cases} \tag{3.57}$$

The construction of such projectors motivates the following definition:

Definition Let $\mathcal{M}$ be a manifold, $\mathscr{H}$ a Hilbert space, End$(\mathscr{H})$ the space of linear operators in $\mathscr{H}$. Then a *projection-valued measure* on $\mathcal{M}$ with respect to $\mathscr{H}$ is a map

$$P : \{\text{Borel subsets of } \mathcal{M}\} \to \text{End}(\mathscr{H}) : U \mapsto P_U \tag{3.58}$$

satisfying the following properties:

- $P_{\mathcal{M}}$ is the identity operator $\mathbb{I}$ in $\mathscr{H}$.
- For any pair of Borel sets U and V, we have $P_{U\cap V} = P_U P_V$; in particular each P_U is a projector.
- The map P is σ-additive in the sense that, if U_1, U_2, etc. are disjoint Borel sets,

$$P_{U_1 \cup U_2 \cup \cdots} = P_{U_1} + P_{U_2} + \cdots. \tag{3.59}$$

The terminology here is inspired by measure theory: the map (3.58) is an operator analogue of (3.1) and property (3.59) corresponds to (3.2). Thus a projection-valued measure measures the "size" of a subset U not by a real number $\mu(U)$, but by the rank of a projector P_U.

It is easy to verify that the projectors (3.56) define a projection-valued measure on $\mathcal{M}$ with respect to $\mathscr{H}$. One can build such a family for any induced representation. In terms of plane waves (3.43), this measure infinitesimally reads

$$dP(k) = d\mu(k) \sum_{\ell=1}^{N} \Psi_{k,\ell} \langle \Psi_{k,\ell} | \cdot \rangle = d\mu(k) \sum_{\ell=1}^{N} |\Psi_{k,\ell}\rangle\langle\Psi_{k,\ell}| \equiv d\mu(k)\mathbb{I}_k \tag{3.60}$$

at any $k \in \mathcal{M}$. Here we have used both our notation and the standard Dirac one; we have also introduced an operator $\mathbb{I}_k = \sum_{\ell=1}^{N} |\Psi_{k,\ell}\rangle\langle\Psi_{k,\ell}|$ such that the identity operator in $\mathscr{H}$ is an integral $\mathbb{I} = \int_{\mathcal{M}} d\mu(k)\mathbb{I}_k$. Analogously to (3.3), the operator P_U is the integral of dP over U. For a particle on the real line, for example, the projection-valued measure in momentum space would read $dP = dk\,|k\rangle\langle k|$ with $k \in \mathbb{R}$.

Systems of Imprimitivity

There is one property that makes the projection-valued measure (3.56) very special. Namely, the transitive action of G on $\mathcal{M}$ gives rise to an action (3.48) on wavefunctions; the latter, in turn, yields an action on the projectors $|\Psi_k\rangle\langle\Psi_k|$. So the fact that (3.56) acts in the space of a representation provides a relation between the geometry of $\mathcal{M}$ and the action of G on the projection-valued measure, which motivates the following definition:

Definition Let $\mathcal{M}$ be a manifold, G a Lie group acting on $\mathcal{M}$. Let $\mathcal{T}$ be a unitary representation of G in a Hilbert space $\mathscr{H}$. Then a *system of imprimitivity* for $\mathcal{T}$ based on $\mathcal{M}$ is a projection-valued measure P on $\mathcal{M}$ with respect to $\mathscr{H}$ such that

$$P_{f \cdot U} = \mathcal{T}[f] \circ P_U \circ \mathcal{T}[f]^{-1} \tag{3.61}$$

for all $f \in G$ and any Borel subset U of $\mathcal{M}$. The system is said to be *transitive* if the action of G on $\mathcal{M}$ is transitive.

In this language the projectors (3.56) imply that any induced representation has a transitive system of imprimitivity:

Proposition Let $\mathcal{S}$ be a unitary representation of a closed subgroup H of G, $\mathcal{T} = \mathrm{Ind}_H^G(\mathcal{S})$ the corresponding induced representation, and $\mathcal{M} \cong G/H$. Then the associated projection-valued measure (3.56) is a transitive system of imprimitivity for $\mathcal{T}$ based on $\mathcal{M}$. In the notation (3.60) this is to say that

$$dP(f \cdot k) = \mathcal{T}[f] \circ dP(k) \circ \mathcal{T}[f]^{-1} \tag{3.62}$$

for all $k \in \mathcal{M}$ and any $f \in G$. We shall refer to (3.56) as the *canonical system of imprimitivity* of the induced representation $\mathcal{T}$.

Proof Transitivity is obvious, so the only subtlety is proving (3.61). Let us pick a group element $f \in G$ and a Borel set $U \subseteq \mathcal{M}$. We start from the definition (3.56) and relate $\mathcal{T}[f] \circ P_U \circ \mathcal{T}[f]^{-1}$ to $P_{f \cdot U}$ using formula (3.48) and the fact that $\mathcal{T}$ is unitary:

$$\mathcal{T}[f] \circ P_U \circ \mathcal{T}[f]^{-1} = \int_U d\mu(k)\rho_f(k) \sum_{\ell=1}^{N} \mathcal{S}[g_{f\cdot k}^{-1}\, f\, g_k]\Psi_{f\cdot k,\ell}\langle \mathcal{S}[g_{f\cdot k}^{-1}\, f\, g_k]\Psi_{f\cdot k,\ell}|\cdot\rangle.$$

Here the sum over ℓ allows us to cancel the two $\mathcal{S}[\cdots]$'s by unitarity. Using also $d\mu(k)\rho_f(k) = d\mu(f \cdot k)$ and renaming the integration variable, the right-hand side boils down to $P_{f \cdot U}$. ■

Remark The word "imprimitive" means "which is not primitive" and was introduced by Galois [29] in the context of permutation groups. The action of a group on a set shuffles the elements of this set, and the action is *imprimitive* if these permutations preserve some (non-trivial) partition of the set. In the present case the group G acts on the Hilbert space $\mathscr{H}$ by the induced representations (3.48), and property (3.62) says that this action preserves the partition of $\mathscr{H}$ into isomorphic subspaces $\mathcal{E}_k \cong \mathcal{E}$ with definite momentum k.

3.5.2 Imprimitivity Theorem

The considerations of the previous pages open the door to a highly non-trivial statement, namely the fact that *any* representation that admits a transitive system of imprimitivity is an induced representation:

Imprimitivity theorem Let G be a finite-dimensional Lie group, H a closed subgroup of G. Let $\mathcal{T}$ be a continuous, unitary representation of G in some Hilbert space $\mathscr{H}$ and let P be a system of imprimitivity for $\mathcal{T}$ on $\mathcal{M} = G/H$. Then there exists a unitary representation $\mathcal{S}$ of H in some Hilbert space $\mathcal{E}$ such that the pair $(\mathcal{T}, P)$ is unitarily equivalent to $\left(\text{Ind}_H^G(\mathcal{S}), P^{\mathcal{S}}\right)$ where $P^{\mathcal{S}}$ is the canonical system of imprimitivity (3.56) associated with $\text{Ind}_H^G(\mathcal{S})$. More precisely, there exists an isometry $\mathcal{U} : L^2(G/H, \mu, \mathcal{E}) \to \mathscr{H}$, where μ is a quasi-invariant measure on G/H, that intertwines the representations $\mathcal{T}$ and $\text{Ind}_H^G(\mathcal{S})$ and that satisfies

$$\mathcal{U} \circ P_U^{\mathcal{S}} \circ \mathcal{U}^{-1} = P_U \tag{3.63}$$

for any Borel set $U \subseteq \mathcal{M}$.

The complete proof of this theorem can be found in [30] and is reproduced in [1]. Given the representation $\mathcal{T}$ and the system of imprimitivity P, the key subtlety is to construct a Hilbert space $\mathcal{E}$ and a representation $\mathcal{S}$ of H in $\mathcal{E}$. We will not dwell on this proof any further, but turn now to some of its applications.

Equivalent Induced Representations

Here we describe a restricted notion of equivalence for induced representations, culminating with the observation that two induced representations $\mathcal{T}_1, \mathcal{T}_2$ are "equivalent" if and only if they are induced from equivalent representations $\mathcal{S}_1, \mathcal{S}_2$. The proofs rely crucially on the imprimitivity theorem, but we omit them; they can be found in [1].

Definition Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be two representations of G induced by some representations $\mathcal{S}_1, \mathcal{S}_2$ (respectively) of a subgroup H. Let their respective carrier spaces be $\mathscr{H}_1, \mathscr{H}_2$, and let P_1, P_2 (respectively) be their canonical systems of imprimitivity. Then a linear map $A : \mathscr{H}_1 \to \mathscr{H}_2$ *intertwines* the pairs $(\mathcal{T}_1, P_1)$ and $(\mathcal{T}_2, P_2)$ if

$$A \circ \mathcal{T}_1[f] = \mathcal{T}_2[f] \circ A \qquad \text{and} \qquad A \circ (P_1)_U = (P_2)_U \circ A \tag{3.64}$$

for any $f \in G$ and any Borel set U in G/H.

Equivalence theorem Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be unitary representations of H in the Hilbert spaces $\mathcal{E}_1$ and $\mathcal{E}_2$ (respectively). Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be the corresponding induced representations, and P_1, P_2 the associated canonical systems of imprimitivity. Then there exists a (continuous) vector space isomorphism between the space of operators intertwining $\mathcal{S}_1$ and $\mathcal{S}_2$ and the space of intertwiners between $(\mathcal{T}_1, P_1)$ and $(\mathcal{T}_2, P_2)$.

As a corollary, the space of intertwiners between $(\mathcal{T}_1, P_1)$ and $(\mathcal{T}_2, P_2)$ contains an isometry if and only if the space of intertwiners between $\mathcal{S}_1$ and $\mathcal{S}_2$ does. In other words, if we declare that the pairs $(\mathcal{T}_1, P_1)$ and $(\mathcal{T}_2, P_2)$ are equivalent once there exists an isometry A satisfying (3.64), then we have

$$(\mathcal{T}_1, P_1) \sim (\mathcal{T}_2, P_2) \qquad \text{if and only if} \qquad \mathcal{S}_1 \sim \mathcal{S}_2. \tag{3.65}$$

In particular, if $\mathcal{S}_1$ and $\mathcal{S}_2$ are equivalent (in the usual sense), then so are the induced representations $\mathcal{T}_1$ and $\mathcal{T}_2$ (regardless of their systems of imprimitivity). The converse is not true, since the "if and only if" of (3.65) also involves the systems of imprimitivity associated with $\mathcal{T}_1$ and $\mathcal{T}_2$. In other words, saying that two induced representations are equivalent *without* saying anything about their systems of imprimitivity is not sufficient to conclude that they are induced from the same spin representation $\mathcal{S}$.

As mentioned below Eq. (3.38), irreducibility of $\mathcal{S}$ does not generally imply irreducibility of the corresponding induced representation. In the next chapter we shall state a stronger result for semi-direct products, but for now we display a theorem that provides a slightly weaker criterion for the irreducibility of induced representations.

Definition Let $\mathcal{T}$ be an induced representation, P the associated canonical system of imprimitivity. We call the pair $(\mathcal{T}, P)$ *irreducible* if the space of operators intertwining it with itself consists of multiples of the identity.

Irreducibility theorem Let $\mathcal{T}$ be induced by $\mathcal{S}$ and let P be its system of imprimitivity. Then the pair $(\mathcal{T}, P)$ is irreducible if and only if $\mathcal{S}$ is irreducible.

The latter theorem shows that a suitable notion of irreducibility is preserved along the induction process, since an irreducible $\mathcal{S}$ will lead to an induced representation $\mathcal{T}$ *and* a system of imprimitivity P which, together, will be considered irreducible in the above sense. But the theorem does *not* say that an induced representation $\mathcal{T}$ on its own is irreducible if it is induced from an irreducible $\mathcal{S}$.

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Chapter 4
Semi-direct Products

In this chapter we introduce semi-direct products such as the Poincaré group, the Galilei group and the Bargmann group. We describe their irreducible unitary representations, which are induced from representations of their translation subgroup combined with a so-called *little group*. We interpret these representations as *particles* propagating in space-time and having definite transformation properties under the corresponding symmetry group. This picture will be instrumental in our study of the BMS_3 group.

The plan is as follows. In Sect. 4.1 we define semi-direct products and introduce the key notions of *momentum orbits*, little groups and particles. We also explain why irreducible unitary representations are always induced, and describe these representations in general terms. The remaining sections are devoted to applications of these considerations. In Sect. 4.2 we describe relativistic particles, i.e. unitary representations of the Poincaré group, with a particular emphasis in Sect. 4.3 on the three-dimensional setting (which will be useful when dealing with BMS_3). Section 4.4 is devoted to non-relativistic particles, i.e. unitary representations of Bargmann groups. Useful references include [1, 2] for the general theory, and [3–5] for its application to Poincaré.

4.1 Representations and Particles

In short, a semi-direct product group consists of two pieces: a non-Abelian group G of transformations that can be interpreted as "rotations" or "boosts", and another group A that consists of transformations analogous to translations that are acted upon by rotations and boosts. This structure is denoted $G \ltimes A$ and is common to the Poincaré groups (1.2) as well as the BMS groups (1.1)–(1.9). In this section we define such groups in abstract terms, define the associated notion of "momenta" and describe their irreducible unitary representations, which we interpret as particles.

B. Oblak, *BMS Particles in Three Dimensions*, Springer Theses,
DOI 10.1007/978-3-319-61878-4_4

4.1.1 Semi-direct Products

Definition Let G and A be Lie groups; we denote elements of G as f, g, etc. and those of A as α, β, etc. Let $\sigma : G \times A \to A : (f, \alpha) \mapsto \sigma_f(\alpha)$ be a smooth action of G on A where each σ_f is an automorphism of A. Then the *semi-direct product* of G and A with respect to σ is the group denoted

$$G \ltimes_\sigma A \qquad \text{or} \qquad G \ltimes A \tag{4.1}$$

whose elements are pairs (f, α) where $f \in G$ and $\alpha \in A$, with a group operation

$$(f, \alpha) \cdot (g, \beta) = \big(f \cdot g, \alpha \cdot \sigma_f(\beta)\big). \tag{4.2}$$

This definition implies for instance that the inverse of (f, α) is

$$(f, \alpha)^{-1} = \big(f^{-1}, [\sigma_{f^{-1}}(\alpha)]^{-1}\big). \tag{4.3}$$

It follows that A is a normal subgroup of $G \ltimes A$: identifying A with the set of elements $(e, \alpha) \in G \ltimes A$ (where e is the identity in G), one finds

$$(g, \beta) \cdot (e, \alpha) \cdot (g, \beta)^{-1} \overset{(4.2)}{=} \big(e, \beta \cdot \sigma_g(\alpha) \cdot \beta^{-1}\big) \in A. \tag{4.4}$$

It is equally easy to verify that G is a subgroup of $G \ltimes A$, though it is generally *not* a normal subgroup. Indeed, upon identifying G with the subgroup of $G \ltimes A$ consisting of elements (f, e_A) (where e_A is the identity in A), we find

$$(g, \beta) \cdot (f, e_A) \cdot (g, \beta)^{-1} \overset{(4.2)}{=} \big(gfg^{-1}, \beta \cdot \sigma_{gfg^{-1}}(\beta^{-1})\big). \tag{4.5}$$

For this to be an element of G, we must require that $\beta \cdot \sigma_{gfg^{-1}}(\beta^{-1})$ coincides with e_A, which is the statement that $\sigma_f(\alpha) = \alpha$ for any $\alpha \in A$. Thus G is a normal subgroup of A if and only if its action σ is trivial, in which case $G \ltimes A$ is isomorphic to the direct product $G \times A$. From now on we always take the action σ to be *non*-trivial.

Rotations and Translations

A case of great interest, both for the general theory and for our specific purposes, occurs when A is a *vector group*. By this we mean a vector space endowed with the Abelian group operation given by the addition of vectors: $\alpha \cdot \beta \equiv \alpha + \beta$. In that case the identity in A is the vanishing vector $e_A = 0$.

Definition Let G be a Lie group, A a vector space, σ a representation of G in A, and consider the semi-direct product $G \ltimes_\sigma A$ whose elements are pairs (f, α) with group operation

$$\boxed{(f, \alpha) \cdot (g, \beta) = \big(f \cdot g, \alpha + \sigma_f(\beta)\big).} \tag{4.6}$$

Elements of G are then called *rotations* or *boosts* while elements of A are *translations*.

Note that, since A is a vector group, the inverse (4.3) of (f, α) is $(f, \alpha)^{-1} = \big(f^{-1}, -\sigma_{f^{-1}}(\alpha)\big)$. Relation (4.4) also simplifies to $(g, \beta) \cdot (e, \alpha) \cdot (g, \beta)^{-1} = (e, \sigma_g \alpha)$. From now on the words "semi-direct product" and the notation $G \ltimes A$ will always refer to a group (4.1) with A a vector group. (This is why the second factor in (4.1) was denoted "A" in the first place.)

The terminology of "rotations" and "translations" is justified by the semi-direct products commonly encountered in physics:

- The *Euclidean group* in n space dimensions takes the form (4.1) where rotations span the group $G = \mathrm{O}(n)$ while translations belong to $A = \mathbb{R}^n$, with the action σ of rotations on translations given by the vector representation of $\mathrm{O}(n)$.
- The *Poincaré group* in D space-time dimensions takes the form (4.1) where rotations and boosts span the Lorentz group $\mathrm{O}(D-1, 1)$ while space-time translations span $A = \mathbb{R}^D$ (which is sometimes written $\mathbb{R}^{D-1,1}$); the action σ is the vector representation of the Lorentz group.
- The BMS groups (1.1)–(1.9) all take the form (4.1) with G a specific non-Abelian group and A an Abelian vector group of so-called "supertranslations". A similar structure will hold in three space-time dimensions.

Note that the definition of $G \ltimes A$ singles out the normal subgroup A, so G and A live on unequal footings. In particular the Lie algebra of $G \ltimes A$ contains a non-trivial Abelian ideal and is *not* semi-simple. This implies that, in contrast to simple Lie groups, the representations of $G \ltimes A$ must somehow distinguish the roles of G and A by making them act on the carrier space in radically different ways. We will illustrate this in the pages that follow (see e.g. formula (4.23)).

4.1.2 Momenta

Suppose we wish to build unitary representations of a semi-direct product $G \ltimes_\sigma A$. Where should we start? A simple approach is to note that the restriction to A of any unitary representation of $G \ltimes A$ is a (reducible) unitary representation of A. So instead of directly looking for representations of $G \ltimes A$, let us consider the simpler problem of building unitary representations of the group of translations, A.

We denote by A^* the vector space dual to A. It consists of linear forms $p : A \to \mathbb{R} : \alpha \mapsto \langle p, \alpha \rangle$, which motivates the definition of a bilinear pairing

$$\langle \cdot, \cdot \rangle : A^* \times A \to \mathbb{R} : (p, \alpha) \mapsto \langle p, \alpha \rangle. \tag{4.7}$$

Since A is Abelian, any one of its irreducible unitary representations is one-dimensional and takes the form

$$\mathcal{R} : A \to \mathbb{C} : \alpha \mapsto e^{i\langle p, \alpha \rangle} \tag{4.8}$$

for some fixed element p of A^*. Indeed, any representation $\mathcal{R}$ of A is such that

$$\mathcal{R}[\alpha + \beta] = \mathcal{R}[\alpha]\,\mathcal{R}[\beta] \tag{4.9}$$

where the right-hand side is a composition of linear operators. Assuming that A has a countable basis so that α and β have components α^i, β^i, the derivative of (4.9) with respect to β^i yields

$$\partial_j \mathcal{R}[\alpha] = i\big(- i\partial_j \mathcal{R}[0]\big)\mathcal{R}[\alpha] \tag{4.10}$$

where each $(-i\partial_j\mathcal{R}[0])$ is Hermitian by unitarity. Hence $\mathcal{R}[\alpha] = \exp[i(-i\partial_j \mathcal{R}[0])\alpha^j]$, which can be diagonalized into a direct sum of multiplicative operators (4.8).

For example, for the Euclidean group in n dimensions, $\alpha = (\alpha^1, \ldots, \alpha^n)$ is an n-component vector and $\langle p, \alpha\rangle = p_i\alpha^i$ where $p = (p_1, \ldots, p_n)$ is a "covector". For the Poincaré group in D space-time dimensions, $\alpha = (\alpha^0, \ldots, \alpha^{D-1})$ is a D-vector and $\langle p, \alpha\rangle = p_\mu\alpha^\mu$ for some energy-momentum covector $(p_0, \ldots, p_{D-1})$. When interpreting the corresponding unitary representations as "particles", the quantity p represents the particle's *momentum* vector.[1] Accordingly, from now on the dual space A^* will be called the *space of momenta*, and its elements will be denoted as p, q or k. In the BMS_3 groups, translations and momenta are vectors with infinitely many components. Note that two irreducible representations of the form (4.8) are equivalent if and only if their momenta coincide.

Remark In proving that all irreducible unitary representations of A takes the form (4.8), we relied crucially on Eq. (4.9). The latter assumes that $\mathcal{R}$ is an exact representation of A, which is not a restrictive assumption as long as there exists no central extension of $G \ltimes A$ that turns A into a non-Abelian group. The Poincaré groups, the Bargmann groups and the BMS_3 group all satisfy this property, so one may safely assume that A is Abelian even upon switching on central extensions. By contrast, the symmetry group of warped conformal field theories [6] is a semi-direct product whose central extension makes translations non-Abelian [7].

4.1.3 Orbits and Little Groups

We now ask how irreducible, unitary representations of the Abelian group A are embedded in unitary representations of the larger group $G \ltimes_\sigma A$. Let $\mathcal{T}$ be a unitary representation of the latter; then its restriction to A is, in general, reducible. It is typically a direct sum, or rather a direct integral, of irreducible representations (4.8)[2]:

[1] More precisely the momentum *vector* is obtained by raising the indices of the *covector* p thanks to some metric on A left invariant by G, but we will keep referring to p as the "momentum vector".

[2] Our notation here is not mathematically precise; we refer to [2] for a more rigorous treatment.

$$\mathcal{T}[(e,\alpha)] = \int_{\mathcal{O}} d\mu(q)\, e^{i\langle q,\alpha\rangle}\, \mathbb{I}_q \qquad \forall\, \alpha \in A\,. \tag{4.11}$$

Here $\mathcal{O}$ is a certain subset of A^*, μ is some measure on $\mathcal{O}$, and each $\mathbb{I}_q$ is an identity operator acting in a suitable Hilbert space at momentum q. The question then is:

What is the minimal set $\mathcal{O}$ of momenta appearing in the decomposition (4.11)? (4.12)

The answer will lead to the notion of *orbits*; hence the notation "$\mathcal{O}$" in (4.11).

Call $\mathscr{H}$ the Hilbert space of the representation $\mathcal{T}$. Suppose there exists a subspace $\mathcal{E}$ of $\mathscr{H}$ where translations are represented by multiplicative operators (4.8) with a certain momentum p:

$$\mathcal{T}[(e,\alpha)]\Big|_{\mathcal{E}} = e^{i\langle p,\alpha\rangle}\, \mathbb{I}_{\mathcal{E}} \qquad \forall\, \alpha \in A\,, \tag{4.13}$$

where $\mathbb{I}_{\mathcal{E}}$ is the identity operator in $\mathcal{E}$. We shall refer to this property by saying that the representation $\mathcal{T}$ "contains the momentum p". Now pick some group element $f \in G$. By virtue (4.6) and since $\mathcal{T}$ is a representation, one has

$$\mathcal{T}[(e,\alpha)] \cdot \mathcal{T}[(f,0)] = \mathcal{T}[(f,0)] \cdot \mathcal{T}[(e,\sigma_{f^{-1}}\alpha)]. \tag{4.14}$$

One can then act with both sides of this equation on the space $\mathcal{E}$; the last term on the right-hand side produces a multiplicative operator (4.13) with α replaced by $\sigma_{f^{-1}}\alpha$. This operator is a c-number and therefore commutes with $\mathcal{T}[(f,0)]$. We conclude that, on the space $\mathcal{T}[(f,0)] \cdot \mathcal{E} \equiv \mathcal{E}'$, all translations are again represented by multiplicative operators, but now with an additional insertion of $\sigma_{f^{-1}}$ in the phase $\langle p,\alpha\rangle$:

$$\mathcal{T}[(e,\alpha)]\Big|_{\mathcal{E}'} = e^{i\langle p,\sigma_{f^{-1}}\alpha\rangle}\, \mathbb{I}_{\mathcal{E}'} \qquad \forall\, \alpha \in A\,. \tag{4.15}$$

This motivates the following definition for the action of boosts on momenta:

Definition For any momentum $p \in A^*$ and any $f \in G$, we write

$$\sigma_f^*(p) \equiv p \circ \sigma_{f^{-1}}\,, \tag{4.16}$$

i.e. $\langle \sigma_f^*(p), \alpha\rangle \equiv \langle p, \sigma_{f^{-1}}\alpha\rangle$ for all translations α. This defines a representation σ^* of G in the space of momenta, known as the *dual representation* corresponding to σ. To reduce clutter, we will often denote it by

$$\sigma_f^*(p) \equiv f \cdot p\,. \tag{4.17}$$

In terms of the dual representation (4.17) we can rewrite (4.15) as $\mathcal{T}[(e,\alpha)]\big|_{\mathcal{E}'} = e^{i\langle f\cdot p,\alpha\rangle}\, \mathbb{I}_{\mathcal{E}'}$, where $\mathcal{E}' = \mathcal{T}[(f,0)] \cdot \mathcal{E}$. Thus, whenever the representation $\mathcal{T}$ contains a momentum p, compatibility with the structure of $G \ltimes A$ implies that it also contains the boosted momentum $f \cdot p$, where f is any element of G. This is the answer to the question (4.12): if there exists a momentum p such that (4.13) holds, then the representation also contains all momenta that belong to the *orbit* (3.13) of p under G,

$$\boxed{\mathcal{O}_p \equiv \{f \cdot p \,|\, f \in G\}\,.} \tag{4.18}$$

This orbit is the minimal set of momenta needed to cook up a representation of $G \ltimes A$; we will see below that it is also sufficient. In fact, the whole classification of irreducible unitary representations of $G \ltimes A$ will be provided by a partition of the space of momenta into G-orbits. Note that this partition is scale-invariant in the following sense: since the action of boosts on momenta is linear, the orbits $\mathcal{O}_p$ and $\mathcal{O}_{\lambda p}$ are diffeomorphic for any real number $\lambda \neq 0$.

Each orbit $\mathcal{O}_p$ is a homogeneous space for the action (4.16) of G. Accordingly we define the *little group* of a momentum p as the set of rotations that leave it fixed,

$$G_p \equiv \{f \in G \,|\, f \cdot p = p\}\,. \tag{4.19}$$

It is the stabilizer (3.14) for the action of G on the homogeneous space $\mathcal{O}_p$. As in (3.15) there is a diffeomorphism $\mathcal{O}_p \cong G/G_p$. Note that the little group of the vanishing momentum $p = 0$ is the whole group G.

The notion of orbits is perhaps the one most important concept needed to understand representations of semi-direct products. We will encounter it repeatedly later on. Orbits hint at a geometrization of representation theory analogous to the one mentioned at the end of Sect. 3.3, and therefore suggest that representations of $G \ltimes A$ are closely related to induced representations. In the next pages we will confirm this intuition by showing how to associate representations of $G \ltimes A$ with a given orbit.

Note that the action (4.16) of G on the space of momenta leaves the pairing (4.7) invariant in the sense that $\langle f \cdot p, \sigma_f \alpha\rangle = \langle p, \alpha\rangle$. This has an important implication: when A is finite-dimensional it is isomorphic to its dual, so (4.7) defines a non-degenerate bilinear form on A and the action σ^* of G on momenta is equivalent to σ. We shall see illustrations of this in the Poincaré groups. By contrast, when A is *infinite*-dimensional, σ^* may not be equivalent to σ despite the property $\langle f \cdot p, \sigma_f \alpha\rangle = \langle p, \alpha\rangle$. This observation will be relevant to the BMS_3 group in part III.

4.1.4 Particles

We now explain how to build irreducible unitary representations of $G \ltimes A$ starting from a momentum orbit $\mathcal{O}_p$. Inspired by the Poincaré group, we refer to such representations as *particles*. We start by describing scalar particles and identify them with induced representations of $G \ltimes A$. This identification will then allow us to introduce spin.

Scalar Particles

Let $p \in A^*$ be a momentum with orbit (4.18). The latter is a homogeneous space and therefore admits a quasi-invariant measure μ. Let then $\mathscr{H} = L^2(\mathcal{O}_p, \mu, \mathbb{C})$ be the Hilbert space of square-integrable wavefunctions in momentum space,

$$\Psi : \mathcal{O}_p \to \mathbb{C} : q \mapsto \Psi(q). \tag{4.20}$$

The scalar product of wavefunctions is (3.7), with $(\Phi(q)|\Psi(q)) = \Phi^*(q)\Psi(q)$.

Now let us endow $\mathscr{H}$ with a unitary action $\mathcal{T}$ of $G \ltimes A$. In other words, if $\Psi \in \mathscr{H}$ is a wavefunction, we wish to define the object

$$\mathcal{T}[(f, \alpha)] \cdot \Psi \tag{4.21}$$

where (f, α) belongs to $G \ltimes A$ and where $\mathcal{T}[(f, \alpha)]$ is some unitary operator. Linearity implies that the result should be proportional to Ψ, so

$$\big(\mathcal{T}[(f, \alpha)] \cdot \Psi\big)(q) = (\text{some number}) \times \Psi(\text{some point on } \mathcal{O}_p)$$

where the unknown quantities may depend on q, f and α. Note that the quantity multiplying $\Psi(\cdots)$ on the right-hand side must be a *number*, as opposed to an operator, because Ψ takes its values in $\mathbb{C}$ (this will change upon adding spin). Now recall that the reason for introducing orbits in the first place was to represent translations by multiplicative operators (4.8). Accordingly the translation α in (4.21) should produce a momentum-dependent phase factor:

$$\big(\mathcal{T}[(f, \alpha)] \cdot \Psi\big)(q) = e^{i\langle q, \alpha\rangle} \times \Psi(\text{some point on } \mathcal{O}_p)\,.$$

Finally, since Ψ is a wavefunction in momentum space, its argument on the right-hand side should represent the fact that a boost f maps a particle with momentum k on a particle with momentum $f \cdot k$. This is exactly the situation encountered in the quasi-regular representation (3.22) so we can borrow that construction:

$$\big(\mathcal{T}[(f, \alpha)] \cdot \Psi\big)(q) = e^{i\langle q, \alpha\rangle}\, \Psi(f^{-1} \cdot q)\,. \tag{4.22}$$

In particular, the intuition depicted in Fig. 3.1 remains valid.

Formula (4.22) defines a representation $\mathcal{T}$ of $G \ltimes A$, as can be verified by following the same steps as for the quasi-regular representation (3.22). It is also irreducible by virtue of the fact that the orbit $\mathcal{O}_p$ is a homogeneous space. Finally, it is unitary if the measure μ in (3.7) is invariant under G. If the measure has a non-trivial Radon-Nikodym derivative (3.19), the representation (4.22) can be made unitary by inserting a compensating term in front of the exponential, as in (3.24):

$$\big(\mathcal{T}[(f, \alpha)] \cdot \Psi\big)(q) = \sqrt{\rho_{f^{-1}}(q)}\; e^{i\langle q, \alpha\rangle}\, \Psi(f^{-1} \cdot q). \tag{4.23}$$

We call this representation a *scalar particle* with momentum orbit $\mathcal{O}_p$. Note how translations and rotations have radically different roles: translations multiply wavefunctions by momentum-dependent phase factors, while boosts move them around on the orbit by changing their argument. In particular, pure translations act as

$$\mathcal{T}[(e,\alpha)]\cdot\Psi(q)=e^{i\langle q,\alpha\rangle}\,\Psi(q). \tag{4.24}$$

Thinking of wavefunctions as sections of a complex line bundle over $\mathcal{O}_p$ with fibres $\mathcal{E}_q\cong\mathbb{C}$, formula (4.24) can be rewritten symbolically as

$$\mathcal{T}[(e,\alpha)]=\int_{\mathcal{O}_p}d\mu(q)\,e^{i\langle q,\alpha\rangle}\,\mathbb{I}_q \tag{4.25}$$

where $\mathbb{I}_q$ is the identity operator in the fibre at q. This is precisely the anticipated expression (4.11) (Fig. 4.1).

Particles Are Induced Representations

Formula (4.23) is almost identical to the quasi-regular representation (3.24), and more generally to the induced representation (3.32). To investigate this relation, let G_p be the little group (4.19) of p and consider the subgroup $G_p\ltimes A$ of $G\ltimes A$. Define a map

$$\mathcal{S}:G_p\ltimes A\to\mathbb{C}:(f,\alpha)\mapsto\mathcal{S}[(f,\alpha)]\equiv e^{i\langle p,\alpha\rangle}, \tag{4.26}$$

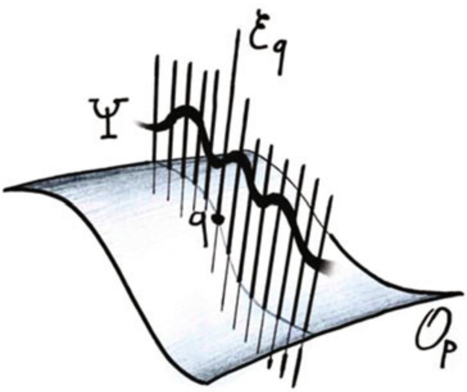

Fig. 4.1 A momentum orbit $\mathcal{O}_p$ crossed by one-dimensional fibres isomorphic to $\mathbb{C}$. (For simplicity the fibres are depicted as if they were real rather than complex.) The fibre at $q\in\mathcal{O}_p$ is denoted $\mathcal{E}_q$ and the disjoint union of such fibres is a (complex) line bundle over $\mathcal{O}_p$. A wavefunction Ψ is a section of that bundle. Translations (4.25) act by complex multiplication $z\mapsto e^{i\langle q,\alpha\rangle}z$ in each fibre $\mathcal{E}_q$

which is a one-dimensional representation of $G_p \ltimes A$. Indeed, for all (f, α) and (g, β) belonging to $G_p \ltimes A$, $\mathcal{S}$ preserves the group structure in the sense that

$$\mathcal{S}[(f,\alpha)] \cdot \mathcal{S}[(g,\beta)] \overset{(4.26)}{=} e^{i\langle p,\alpha\rangle + i\langle p,\beta\rangle} \overset{f\in G_p}{=} e^{i\langle p,\alpha\rangle + i\langle f^{-1}\cdot p,\beta\rangle}$$
$$\overset{(4.16)}{=} e^{i\langle p,(\alpha+\sigma_f\beta)\rangle} \overset{(4.26)}{=} \mathcal{S}[(fg, \alpha+\sigma_f\beta)] \overset{(4.6)}{=} \mathcal{S}[(f,\alpha)\cdot(g,\beta)].$$

Furthermore, (4.26) is unitary so we can use it to induce a unitary representation

$$\mathcal{T} = \mathrm{Ind}_{G_p \ltimes A}^{G\ltimes A}(\mathcal{S}) \tag{4.27}$$

of $G \ltimes A$. Using the general formula (3.32) and the diffeomorphisms $\mathcal{O}_p \cong G/G_p \cong (G \ltimes A)/(G_p \ltimes A)$, we see that the induced representation (4.27) acts on wavefunctions exactly in the way displayed in Eq. (4.23). Note that the little group G_p is represented trivially in the "spin" representation (4.26). This is why we say that the particle (4.23) is *scalar*: its states are essentially unaffected by the rotations that span G_p. The picture (4.27) suggests a simple generalization of this behaviour, as we now explain.

Spinning Particles

To generalize (4.23), let $\mathcal{R}$ be an irreducible, unitary representation of G_p in some space $\mathcal{E}$ and consider the spin representation

$$\mathcal{S} : G_p \ltimes A \to \mathrm{GL}(\mathcal{E}) : (f,\alpha) \mapsto e^{i\langle p,\alpha\rangle}\,\mathcal{R}[f]\,. \tag{4.28}$$

This reduces to (4.26) when $\mathcal{R}$ is trivial, and the corresponding induced representation of $G \ltimes A$ is

$$\mathcal{T} = \mathrm{Ind}_{G_p\ltimes A}^{G\ltimes A}(\mathcal{S}) = \mathrm{Ind}_{G_p\ltimes A}^{G\ltimes A}\big(e^{i\langle p,\cdot\rangle}\,\mathcal{R}\big)\,. \tag{4.29}$$

Its action on wavefunctions is analogous to (3.32) and generalizes (4.23):

$$\boxed{\left(\mathcal{T}[(f,\alpha)]\cdot\Psi\right)(q) = \sqrt{\rho_{f^{-1}}(q)}\;\, e^{i\langle q,\alpha\rangle}\,\mathcal{R}[g_q^{-1}\,f\,g_{f^{-1}\cdot q}]\cdot\Psi(f^{-1}\cdot q)\,,} \tag{4.30}$$

where the map $g : \mathcal{O}_p \to G : q \mapsto g_q$ is a continuous family of standard boosts (3.31). In contrast to (4.23), Ψ now takes its values in $\mathcal{E}$ rather than $\mathbb{C}$.

We call the representation (4.30) a *spinning particle* with spin $\mathcal{R}$ and momenta belonging to $\mathcal{O}_p$. It is an irreducible unitary representation of $G \ltimes A$ acting on the Hilbert space $\mathscr{H} = L^2(\mathcal{O}_p, \mu, \mathcal{E})$. The operator

$$\mathcal{R}[g_q^{-1}\,f\,g_{f^{-1}\cdot q}] \equiv W_q[f] \tag{4.31}$$

is the *Wigner rotation* (3.35) associated with f and q. It is the transformation that corresponds to f in the space of internal degrees of freedom $\mathcal{E}$ at q and it entangles momentum and spin degrees of freedom. The decomposition (4.25) still holds in the spinning case, with $\mathbb{I}_q$ the identity operator in the fibre $\mathcal{E}_q \cong \mathcal{E}$ at q.

From this point on, the whole machinery of induced representations applies to unitary representations (4.30) of $G \ltimes A$. In particular they are independent of the choice of the quasi-invariant measure μ and of the family of standard boosts g_q. The plane waves (3.43) provide a basis of the Hilbert space and represent one-particle states with definite momentum and definite spin. They transform under $G \ltimes A$ according to

$$\mathcal{T}[(f,\alpha)] \cdot \Psi_{k,\ell} = \sqrt{\rho_f(k)}\ e^{i\langle f\cdot k,\alpha\rangle}\mathcal{R}\left[g_{f\cdot k}^{-1} \cdot f \cdot g_k\right] \cdot \Psi_{f\cdot k,\ell}\,. \tag{4.32}$$

This is just formula (3.48) applied to (4.29); it is the plane wave analogue of (4.30). Using this, one can go on and evaluate characters along the lines that led to the Frobenius formula (3.54). One finds

$$\chi[(f,\alpha)] = \text{Tr}\big(\mathcal{T}[(f,\alpha)]\big) = \int_{\mathcal{O}_p} d\mu(k)\,\delta(k, f\cdot k)\,e^{i\langle k,\alpha\rangle}\chi_{\mathcal{R}}[g_k^{-1} f g_k]\,, \tag{4.33}$$

where $\chi_{\mathcal{R}}$ is the character of the representation $\mathcal{R}$ of G_p. As before one can check that this formula defines a class function and that $\chi[(f,\alpha)]$ vanishes when f is not conjugate to an element of the little group. The delta function localizes the integral to the momenta that are left fixed by the action of f on $\mathcal{O}_p$.

4.1.5 Exhaustivity Theorem

Equation (4.30) is an irreducible unitary representations of $G \ltimes A$. As it turns out, *all* irreducible representations of $G \ltimes A$ take this form for some momentum orbit $\mathcal{O}_p$ and some spin $\mathcal{R}$. We refer to this property as the *exhaustivity theorem* for induced representations.

This theorem has enormous practical value: it provides the classification of all irreducible unitary representations of a semi-direct product $G \ltimes_\sigma A$ when A is a vector group. This classification can be performed thanks to the following algorithm:

1. Consider the space of momenta, A^*. For each $p \in A^*$, determine the orbit $\mathcal{O}_p$ given by (4.18). This foliates A^* into disjoint momentum orbits, and each point of A^* belongs to exactly one orbit.
2. We call *set of orbit representatives* a set of momenta that exhaust all orbits in a non-redundant way, in the sense that (i) each orbit contains one of the representatives, and (ii) different representatives belong to different orbits. Find a set of orbit representatives, compute the little group of each representative, and find standard boosts connecting each representative to the points of its orbit.

3. For each representative p with little group G_p, classify all irreducible unitary representations of G_p. Given such a representation $\mathcal{R}$, the associated induced representation of $G \ltimes A$ is (4.30).

We will illustrate this classification for the Poincaré groups in Sect. 4.2 and for the Bargmann groups in Sect. 4.4, and of course for BMS_3 in Chap. 10.

The proof of the exhaustivity theorem is essentially an upgraded version of our arguments in Sect. 4.1.3 and relies on two crucial ingredients: the first is the commutativity of the vector group A, and the second is the imprimitivity theorem of Sect. 3.5. Thanks to commutativity, any unitary representation of A can be written as a direct integral (4.11) of irreducible representations specified by certain momenta $q \in A^*$. (This is known as the SNAG theorem.) This implies that any unitary representation $\mathcal{T}$ of $G \ltimes A$ is imprimitive. Indeed, relation (4.14) can be rewritten as

$$\mathcal{T}[(f,0)] \cdot \mathcal{T}[(e,\alpha)] \cdot \mathcal{T}[(f,0)]^{-1} = \mathcal{T}[(e,\sigma_f \alpha)]$$

whereupon the direct integral representation (4.11) yields

$$\mathcal{T}[(f,0)] \cdot d\mu(q)\mathbb{I}_q \cdot \mathcal{T}[(f,0)]^{-1} = d\mu(f \cdot q)\mathbb{I}_{f \cdot q}\,, \tag{4.34}$$

which is precisely the statement (3.62) that the projection-valued measure $d\mu(q)\mathbb{I}_q$ is a system of imprimitivity for $\mathcal{T}$ on A^*. The imprimitivity theorem then implies that the representation $\mathcal{T}$ is induced. The last step of the proof consists in showing that, if $\mathcal{T}$ is irreducible, then the measure μ in (4.34) localizes to a single momentum orbit. We refer to [2] for details.

Remark The exhaustivity theorem relies on an extra technical assumption that we haven't mentioned so far. Namely, one says that $G \ltimes A$ is *regular* if the space of momenta A^* and the action (4.16) of G are such that A^* contains a countable family of Borel sets, each a union of momentum orbits, such that each orbit is the limit of a decreasing sequence of such sets. As it turns out regularity is necessary for the measure μ in (4.34) to be localized on a momentum orbit. All semi-direct products treated in part I of this thesis are regular. As for the BMS_3 group of part III, the issue of regularity will be discussed briefly in Sect. 10.1.

4.2 Poincaré Particles

In this section and the next ones we study examples of semi-direct products to illustrate induced representations. Here we deal with the Poincaré group — the isometry group of Minkowski space — whose representations describe relativistic particles. Following the algorithm of page XXXX we will find that these particles are classified by two parameters: their mass and their spin. In view of treating the BMS_3 group in part III, we relegate the detailed description of relativistic particles in three dimensions to Sect. 4.3.

The plan is the following. First we define the Poincaré group as a semi-direct product of the Lorentz group with the group of space-time translations. Then we turn to the classification of its momentum orbits and describe the corresponding particles. We also compute their characters and end with the observation that Lorentz transformations generally entangle momentum and spin degrees of freedom.

The classification of relativistic particles was first performed by Wigner [8], and their relation to wave equations was worked out in [9]. These results are among the foundations of quantum mechanics and field theory; see e.g. [2–5].

4.2.1 Poincaré Groups

Lorentz Transformations

We consider the vector space $\mathbb{R}^D$; its elements are column vectors α, β, etc. with components α^μ, β^μ where $\mu = 0, 1, \ldots, D-1$. Here $\mathbb{R}^D$ is to be interpreted as a space-time manifold with dimension $D \geq 2$. We endow $\mathbb{R}^D$ with a non-degenerate bilinear form given by the *Minkowski metric*,

$$(\alpha, \beta) \equiv \eta_{\mu\nu}\alpha^\mu\beta^\nu, \qquad (\eta_{\mu\nu}) = \begin{pmatrix} -1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}. \tag{4.35}$$

We write $(\alpha, \alpha) \equiv \alpha^2$ for any α. The sign of α^2 determines whether α is time-like, null or space-like, corresponding respectively to $\alpha^2 < 0$, $\alpha^2 = 0$ or $\alpha^2 > 0$.

Definition The *Lorentz group* $\mathrm{O}(D-1, 1)$ in D dimensions is the group of linear transformations $\mathbb{R}^D \to \mathbb{R}^D : \alpha \mapsto f \cdot \alpha$ that preserve (4.35) in the sense that

$$(f \cdot \alpha, f \cdot \beta) = (\alpha, \beta). \tag{4.36}$$

It consists of $D \times D$ matrices $f = (f^\mu{}_\nu)$ such that

$$f^t \cdot \eta \cdot f = \eta, \qquad \text{i.e.} \qquad f^\lambda{}_\mu \eta_{\lambda\rho} f^\rho{}_\nu = \eta_{\mu\nu} \tag{4.37}$$

where the dot denotes matrix multiplication. In particular, the Minkowskian norm is left invariant by Lorentz transformations.

Topology of Lorentz Groups

The Lorentz group $\mathrm{O}(D-1, 1)$ is disconnected. Indeed, any Lorentz matrix f has determinant $\det(f) = \pm 1$. This cuts the group in two pieces consisting of matrices with positive and negative determinant, corresponding to transformations that

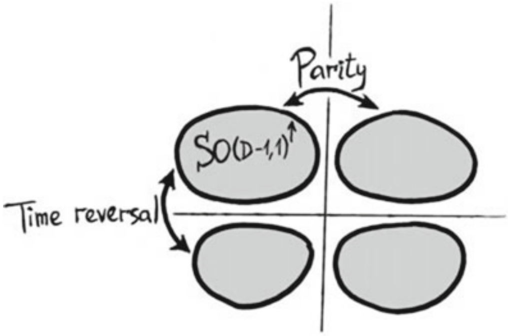

Fig. 4.2 The four connected components of the Lorentz group. The *upper left* component is the proper orthochronous Lorentz group $\mathrm{SO}(D-1,1)^{\uparrow}$. It can be mapped on the other components using parity and time reversal. In particular the proper Lorentz group is generated by $\mathrm{SO}(D-1,1)^{\uparrow}$ together with time reversal, while the orthochronous Lorentz group is generated by $\mathrm{SO}(D-1,1)^{\uparrow}$ together with parity

preserve or break (respectively) the orientation of the spatial coordinates. The subgroup of $\mathrm{O}(D-1,1)$ consisting of Lorentz matrices with positive determinant is the *proper* Lorentz group, $\mathrm{SO}(D-1,1)$. Any improper Lorentz matrix is the product of a proper Lorentz transformation with parity. In addition, one can show that any Lorentz matrix f satisfies $|f^0{}_0| \geq 1$, which again cuts the Lorentz group in two pieces: matrices with positive or negative $f^0{}_0$, corresponding to transformations that preserve or invert (respectively) the orientation of the arrow of time. The subgroup consisting of Lorentz transformations with positive $f^0{}_0$ is the *orthochronous* Lorentz group $\mathrm{O}(D-1,1)^{\uparrow}$. Any Lorentz matrix that reverts the arrow of time is the product of an orthochronous Lorentz matrix with time reversal. The situation is depicted in Fig. 4.2.

In this section we focus on the connected Lorentz group, i.e. the proper orthochronous Lorentz group $\mathrm{SO}(D-1,1)^{\uparrow}$. The latter satisfies an important property known as *standard decomposition*: any proper, orthochronous Lorentz transformation is a product $f = R_1 \cdot \Lambda \cdot R_2$, where R_1 and R_2 are spatial rotations and Λ is a pure boost [10]. In what follows we often refer to $\mathrm{SO}(D-1,1)^{\uparrow}$ simply as "the Lorentz group".

The Lorentz group is not simply connected: in space-time dimension $D \geq 4$, its fundamental group is isomorphic to $\mathbb{Z}_2$. The universal cover of the connected Lorentz group is then called the *spin group*, so that

$$\mathrm{SO}(D-1,1)^{\uparrow} \cong \mathrm{Spin}(D-1,1)/\mathbb{Z}_2 \tag{4.38}$$

where the $\mathbb{Z}_2$ subgroup of $\mathrm{Spin}(D-1,1)$ consists of the identity matrix and its opposite. In four dimensions, $\mathrm{Spin}(3,1) = \mathrm{SL}(2,\mathbb{C})$. In $D = 3$ dimensions the situation is a bit different; we will return to it in the next section. In any case the Lorentz group is always multiply connected, and therefore admits topological projective representations; this will be important for representations of Poincaré.

Poincaré Groups

Definition The *Poincaré group* or *inhomogeneous Lorentz group* in D space-time dimensions is the semi-direct product

$$\mathrm{IO}(D-1,1) \equiv \mathrm{O}(D-1,1) \ltimes \mathbb{R}^D \tag{4.39}$$

whose elements are pairs (f, α) where f is a Lorentz transformation, α a space-time translation. The group operation is $(f, \alpha) \cdot (g, \beta) = (f \cdot g, \alpha + f \cdot \beta)$ where the dots on the right-hand side denote matrix multiplication and the action of matrices on column vectors. The *connected Poincaré group* is the largest connected subgroup of (4.39),

$$\mathrm{ISO}(D-1,1)^{\uparrow} \equiv \mathrm{SO}(D-1,1)^{\uparrow} \ltimes \mathbb{R}^D\,, \tag{4.40}$$

and its universal cover is

$$\mathrm{Spin}(D-1,1)^{\uparrow} \ltimes \mathbb{R}^D \tag{4.41}$$

where spin transformations act on $\mathbb{R}^D$ according to the composition of the homomorphism given by (4.38) with the vector representation of the Lorentz group.

The Poincaré group turns out to have no algebraic central extensions, so its only non-trivial projective transformations are of topological origin. The Poincaré Lie algebra is generated by $D(D-1)/2$ Lorentz generators and D translation generators; we will not display their brackets here.

4.2.2 Orbits and Little Groups

From now on we focus on the connected Poincaré group (4.40), to which we refer simply as "the Poincaré group".

Momenta and Orbits

The space of Poincaré momenta is $(\mathbb{R}^D)^* = \mathbb{R}^D$; its elements are D-dimensional covectors $p = (p_0, p_1, \ldots, p_{D-1})$, where p_0 is to be interpreted as the energy of a relativistic particle, while $\mathbf{p} = (p_1, \ldots, p_{D-1})$ is its spatial momentum.[3] Given a momentum p and a space-time translation α, the pairing (4.7) is $\langle p, \alpha \rangle = p_\mu \alpha^\mu$.

The Minkowski metric (4.35) provides a Lorentz-invariant pairing between translation vectors and can be used to define an isomorphism

$$\mathcal{I} : \mathbb{R}^D \to (\mathbb{R}^D)^* : \alpha \mapsto (\alpha, \cdot) \tag{4.42}$$

[3]Strictly speaking the energy of the particle is $p^0 = -p_0$, but this detail will not affect our discussion so we neglect it for simplicity.

where the components of $(\alpha, \cdot)$ are those of α lowered with the Minkowski metric. Using $\mathcal{I}$, one verifies that the action σ^* of Lorentz transformations on momenta is equivalent to their action σ on translations:

$$\sigma^*_f = \mathcal{I} \circ \sigma_f \circ \mathcal{I}^{-1}. \tag{4.43}$$

As a consequence, momentum orbits coincide with orbits of translations under Lorentz transformations, and consist of momenta q with constant Minkowskian norm squared q^2. We thus conclude that

the orbits of momenta of relativistic particles are connected hyperboloids specified by an equation of the form $q_0^2 - \mathbf{q}^2 = const.$ in $\mathbb{R}^D$.

The only exception to this rule is the trivial orbit of the vanishing momentum $p = 0$, which contains only one point. The word "connected" appears here because we are dealing with the connected Poincaré group (4.40). By contrast the momentum orbits of (4.39) are generally disconnected.

The connected Poincaré group has six distinct families of momentum orbits, which we now describe. Further details can be found e.g. in [4, 5].

- Let $p = 0$ be the vanishing momentum. Its orbit $\mathcal{O}_0 = \{0\}$ contains a single point. Its little group is the whole Lorentz group.
- Let p be a timelike momentum with positive energy, $p_0 > 0$. Its orbit $\mathcal{O}_p$ is *massive with positive energy* and consists of momenta q satisfying

$$q_0^2 - \mathbf{q}^2 = M^2 > 0, \qquad q_0 > 0, \tag{4.44}$$

where we have introduced the *mass squared* $M^2 \equiv -p^2$. We can choose as orbit representative the rest frame momentum

$$p = (M, 0, \ldots, 0), \qquad M > 0. \tag{4.45}$$

The little group of (4.45) is the group of spatial rotations

$$G_p = \mathrm{SO}(D-1) \tag{4.46}$$

consisting of proper Lorentz transformations that leave the time coordinate fixed. In particular, the orbit is diffeomorphic to the quotient

$$\mathcal{O}_p \cong \mathrm{SO}(D-1,1)^{\uparrow}/\mathrm{SO}(D-1) \cong \mathbb{R}^{D-1} \tag{4.47}$$

and its points can be labelled by the spatial components of momentum (since the zeroth component is then determined by Eq. (4.44)). Massive orbits with different masses are disjoint.

- Let p be a time-like momentum with negative energy, $p_0 < 0$. Its orbit is *massive with negative energy* and consists of momenta q satisfying (4.44) with $q_0 < 0$. A typical orbit representative is (4.45) with $M < 0$, and orbits with different masses are disjoint. The little group is a again $\mathrm{SO}(D-1)$.
- Let p be a null momentum with positive energy, $p_0 > 0$. Its orbit $\mathcal{O}_p$ is *massless with positive energy*. It consists of momenta q satisfying (4.44) with $M^2 = 0$. A typical orbit representative is

$$p = (E, E, 0, \ldots, 0) \tag{4.48}$$

where the energy E is positive; different values of E yield the same orbit. Note that there is no rest frame for massless particles. The little group of (4.48) is isomorphic to the Euclidean group

$$G_p \cong \mathrm{SO}(D-2) \ltimes \mathbb{R}^{D-2} = \mathrm{ISO}(D-2). \tag{4.49}$$

In particular, the orbit is diffeomorphic to the quotient

$$\mathcal{O}_p \cong \mathrm{SO}(D-1,1)^{\uparrow}/\mathrm{ISO}(D-2) \cong \mathbb{R} \times S^{D-2} \tag{4.50}$$

and its points can be labelled by the spatial components of momentum (since the zeroth component is then determined by $q^2 = 0$).
- Let p be a null energy-momentum vector with negative energy. Its orbit is *massless with negative energy* and consists of null momenta q with $q_0 < 0$. A typical orbit representative is (4.48) with negative E. The little group is (4.49) and the orbit can be represented as a quotient (4.50).
- Let p be a space-like momentum. Its orbit is *tachyonic* and consists of momenta q satisfying (4.44) with $M^2 < 0$. A typical orbit representative is

$$p = \left(0, 0, \ldots, 0, \sqrt{-M^2}\right). \tag{4.51}$$

The little group is the lower-dimensional Lorentz group $\mathrm{SO}(D-2,1)^{\uparrow}$ consisting of transformations that leave the spatial coordinate x^{D-1} fixed. In particular the orbit is diffeomorphic to the quotient

$$\mathcal{O} \cong \mathrm{SO}(D-1,1)^{\uparrow}/\mathrm{SO}(D-2,1)^{\uparrow} \cong \mathbb{R} \times S^{D-2}. \tag{4.52}$$

Tachyonic orbits with different negative values of M^2 are disjoint. Note that rotations always allow us to map p on $-p$, which is why any tachyonic orbit representative can be written as (4.51).

This enumeration exhausts all Poincaré momentum orbits. Among the six families of orbits, three contain only one orbit: the trivial orbit and the two massless orbits. The remaining three families all contain infinitely many orbits labelled by a

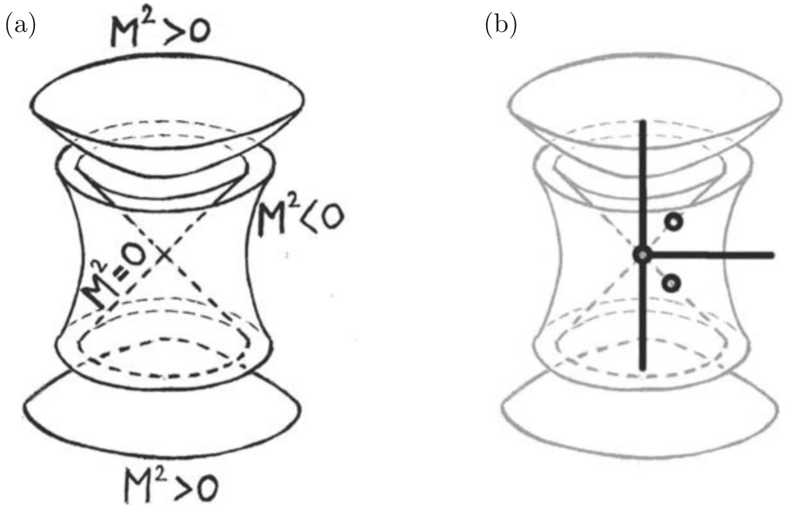

Fig. 4.3 On the *left*, **a** represents a few momentum orbits of the Poincaré group in three dimensions, embedded in $\mathbb{R}^3$ with the vertical axis corresponding to p_0 and the two horizontal axes (not represented in the figure) corresponding to spatial components of momentum. Orbits can be massive, massless or tachyonic depending on whether M^2 is positive, vanishing or negative, respectively. The cross in the middle is the trivial orbit of $p = 0$, consisting of a single point. On the *right*, **b** is a schematic representation of momentum orbits: each point of the diagram corresponds to an orbit representative, where massive orbits are represented by a *vertical line*, tachyonic ones by a *horizontal line*, and discrete orbits (the two massless ones and the trivial one) by *dots*. This schematic representation will be useful in parts II and III for the interpretation of BMS_3 supermomentum orbits

non-vanishing mass squared, corresponding to massive particles and tachyons. These orbits and their representatives are schematically depicted in Fig. 4.3.

To complete the description of orbits we now display standard boosts for massive particles (the other cases are less important for our purposes so we skip them). We take as orbit representative the momentum (4.45) of a particle at rest, and look for a family of boosts g_q such that $g_q \cdot p = (\sqrt{M^2 + \mathbf{q}^2}, \mathbf{q})$ that depend continuously on $\mathbf{q}$. One readily verifies that the matrices [11]

$$g_q = \begin{pmatrix} \sqrt{1 + \mathbf{q}^2/M^2} & q_j/M \\ q_i/M & \delta_{ij} + \frac{q_i q_j}{\mathbf{q}^2}\left(\sqrt{1 + \mathbf{q}^2/M^2} - 1\right) \end{pmatrix} \tag{4.53}$$

satisfy these requirements. Here $i, j = 1, \ldots, D-1$ are spatial indices. Each such matrix is a boost in the direction $\mathbf{q}/|\mathbf{q}|$ with rapidity $\text{arccosh}[\sqrt{1 + \mathbf{q}^2/M^2}]$.

Remark The little groups displayed in (4.46) and (4.49) hold for the connected Poincaré group (4.40). If we replace the latter by its universal cover (4.41), then the little groups are replaced by their double covers (assuming that $D \geq 4$). In particular the little group of massive particles becomes $\text{Spin}(D-1)$ while that of massless particles becomes $\mathrm{Spin}(D-2) \ltimes \mathbb{R}^{D-2}$, with the convention that Spin(2) is the double cover of SO(2). Note that $\mathrm{Spin}(3) = \mathrm{SU}(2)$.

4.2.3 Particles

According to the exhaustivity theorem of Sect. 4.1.5, the momentum orbits in Fig. 4.3 roughly classify relativistic particles. The states of each particle are wavefunctions on its momentum orbit, valued in a spin representation of the little group and transforming under Poincaré transformations according to formula (4.30). Provided we know all irreducible unitary representations of all little groups, we have effectively classified all irreducible unitary representations of the Poincaré group.

Vacuum

Vacuum representations of Poincaré are those whose orbit $\mathcal{O}_0 = \{0\}$ is trivial and is left invariant by the whole Lorentz group. In that case a spin representation is a (projective) irreducible unitary representation $\mathcal{R}$ of $\mathrm{SO}(D-1,1)^{\uparrow}$. The latter is simple but non-compact, so its only finite-dimensional irreducible unitary transformation is the trivial one; the corresponding induced representation of Poincaré is trivial as well. All other irreducible unitary representations of the Lorentz group are infinite-dimensional; the corresponding induced representations of Poincaré are such that translations act trivially, while Lorentz transformations act non-trivially on an infinite-dimensional Hilbert space $\mathcal{E}$ of spin-like degrees of freedom. These representations can be interpreted as "vacua with spin" but are generally discarded as unphysical.

Massive Particles

The momenta of a massive particle with mass M span an orbit (4.44) with little group (4.46). The spin representation $\mathcal{R}$ then is a finite-dimensional, irreducible, generally projective unitary representation of $\mathrm{SO}(D-1)$ specified by some highest weight λ. For example, when $D=4$, $\mathcal{R}$ is a highest-weight representation of $\mathrm{SO}(3) = \mathrm{SU}(2)/\mathbb{Z}_2$ with spin $s \geq 0$; the latter is either an integer or a half-integer. The carrier space of $\mathcal{R}$ has dimension $2s+1$ and is generated by states $|-s\rangle, |-s+1\rangle, \ldots, |s-1\rangle, |s\rangle$ with definite spin projection along a prescribed axis. In that case the highest weight λ coincides with s. The higher-dimensional case is analogous except that the number of coefficients specifying λ is the rank $\lfloor (D-1)/2 \rfloor$ of $\mathrm{SO}(D-1)$. We will illustrate this point in Sect. 11.1 when dealing with partition functions of higher-spin fields in Minkowski space.

Given a spin representation, the remainder of the construction is straightforward: formula (3.4) yields a Lorentz-invariant measure that can be used to define scalar products (3.7) of wavefunctions, and the Poincaré representation acts according to (4.30) with the Radon-Nikodym derivative set to $\rho_f = 1$ thanks to the choice of measure. For example, when $D=4$ and $s=1/2$, any state takes the form (3.9).

Massless Particles

The description of massless particles is analogous to that of massive ones, up to the key difference that the massless little group is the Euclidean group (4.49). It is a semi-direct product (4.1) with an Abelian normal subgroup, so the exhaustivity theorem ensures that its irreducible unitary representations are induced and classified by momentum-like orbits of their own. From the Poincaré viewpoint each induced representation of (4.49) is a spin representation for a massless particle.

In that context one makes the distinction between two types of massless particles: particles with *discrete spin* are those given by spin representations of (4.49) with vanishing Euclidean momentum. These are Euclidean analogues of the "spinning vacua" described earlier, except that they are finite-dimensional. They amount to making the action of $\mathbb{R}^{D-2}$ in (4.49) trivial, and coincide with (projective) irreducible unitary representations of $\mathrm{SO}(D-2)$. Thus massless particles with discrete spin have a finite-dimensional space of spin degrees of freedom. By contrast, massless particles with *continuous* or *infinite spin* are those whose spin representations of (4.49) have non-trivial Euclidean momentum. The space of spin degrees of freedom is infinite-dimensional in that case, since it consists of wavefunctions on a Euclidean momentum orbit $\mathrm{SO}(D-2)/\mathrm{SO}(D-3) \cong S^{D-3}$. Particles with continuous spin are generally discarded on the grounds that they are unphysical, although they have recently been described in a field-theoretic framework [12–14].

Tachyons

Tachyons are particles moving faster than light. Their little group is $\mathrm{SO}(D-2,1)^{\uparrow}$. It is simple and non-compact, so tachyons either have no spin at all, or have continuous spin. They are generally considered as unphysical.

4.2.4 Massive Characters

Having completed the enumeration of relativistic particles, we now evaluate characters of massive irreducible unitary representations of the Poincaré group. Massless characters are relegated to Sect. 4.2.5. These computations are important for our purposes, as we will rely on them in Chap. 11. To our knowledge, Poincaré characters were first studied in [15, 16] before reappearing more recently in [17–19].

Setting the Stage

By virtue of the Frobenius formula (4.33), the character of an induced representation vanishes when evaluated on a transformation f that does not belong to the little group. Since the character is a class function, only the conjugacy class of f matters for the final result. Accordingly, for a massive particle in D dimensions we let f be a rotation

$$f = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & \cos\theta_1 & -\sin\theta_1 & \cdots & 0 & 0 & 0 \\ 0 & \sin\theta_1 & \cos\theta_1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \cos\theta_r & -\sin\theta_r & 0 \\ 0 & 0 & 0 & \cdots & \sin\theta_r & \cos\theta_r & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 1 \end{pmatrix} \tag{4.54}$$

written here for even D with $r = \lfloor (D-1)/2 \rfloor$; if D is odd we erase the last row and the last column. We assume for simplicity that all angles $\theta_1, \ldots, \theta_r$ are non-zero.

Now let μ be a quasi-invariant measure on a momentum orbit with mass M, and let δ be the corresponding delta function. Given an arbitrary space-time translation α, our goal is to evaluate the character of (f, α) using Eq. (4.33). To do this we treat separately odd and even space-time dimensions.

Odd Dimensions

For odd D we erase the last row and column of (4.54). Then the Frobenius formula (4.33) localizes to the unique rotation-invariant point of the momentum orbit, namely the momentum at rest $p = (M, 0, 0, \ldots, 0)$. This allows us to simplify (4.33) by setting $k = p$ in the exponential and the little group character, and pulling them out of the integral. Denoting by λ the spin of the particle (it is a highest weight for $\mathrm{SO}(D-1)$), we find

$$\chi[(f, \alpha)] = e^{iM\alpha^0} \chi_\lambda^{(D-1)}[f] \int_{\mathcal{O}_p} d\mu(k)\, \delta(k, f \cdot k) \tag{4.55}$$

where the replacement of k by p has projected the translation α on its time component α^0. The little group character $\chi_\lambda^{(D-1)}[f]$ is some function of the angles $\theta_1, \ldots, \theta_r$ that we do not need to write down at this stage (in practice it follows from the Weyl character formula and is displayed in Eq. (11.151) below). To obtain (4.55) it only remains to evaluate the integral of the delta function. As coordinates on the orbit we choose the spatial components of momentum, in terms of which the Lorentz-invariant measure on $\mathcal{O}_p$ is (3.4) and the corresponding delta function is (3.42). We thus get

$$d\mu(k)\, \delta(k, q) = \frac{d^{D-1}\mathbf{k}}{\sqrt{M^2 + \mathbf{k}^2}} \sqrt{M^2 + \mathbf{k}^2}\, \delta^{(D-1)}(\mathbf{k} - \mathbf{q}) = d^{D-1}\mathbf{k}\; \delta^{(D-1)}(\mathbf{k} - \mathbf{q})\,, \tag{4.56}$$

where the multiplicative factors of the measure and its delta function cancel out. Note that the same cancellation would have taken place for *any* measure μ proportional to $d^{D-1}\mathbf{k}$, in accordance with the fact that induced representations are insensitive to the choice of measure. Applied to (4.55), the cancellation (4.56) allows us to write

$$\chi[(f, \alpha)] = e^{iM\alpha^0} \chi_\lambda^{(D-1)}[f] \int_{\mathbb{R}^{D-1}} d^{D-1}\mathbf{k}\; \delta^{(D-1)}(\mathbf{k}, f \cdot \mathbf{k}) \tag{4.57}$$

where $f \cdot \mathbf{k}$ denotes the action of the spatial submatrix of (4.54) on $\mathbf{k}$. The integral can be written as

$$\int_{\mathbb{R}^D} d^{D-1}\mathbf{k}\ \delta^{(D-1)}\big((\mathbb{I}-f)\cdot\mathbf{k}\big) = \frac{1}{\det(\mathbb{I}-f)} \tag{4.58}$$

where $\mathbb{I}$ is the $(D-1)$-dimensional identity matrix. In terms of angles θ_i we find

$$\det(\mathbb{I}-f) = \prod_{j=1}^{r} \begin{vmatrix} 1-\cos\theta_j & \sin\theta_j \\ -\sin\theta_j & 1-\cos\theta_j \end{vmatrix} = \prod_{j=1}^{r} 4\sin^2\theta_j = \prod_{j=1}^{r} |1-e^{i\theta_j}|^2. \tag{4.59}$$

Plugging this into (4.58), the character (4.57) finally becomes

$$\chi[(f,\alpha)] = e^{iM\alpha^0}\chi_\lambda^{(D-1)}[f]\prod_{j=1}^{r}\frac{1}{|1-e^{i\theta_j}|^2}\,. \tag{4.60}$$

Note that for a Euclidean time translation $\alpha^0 = i\beta$, this quantity may be seen as the partition function of a relativistic particle in a rotating frame (albeit with purely imaginary angular velocity).

Remark The localization effect (4.58) is a restatement of the Atiyah-Bott fixed point theorem. In that context the term

$$\frac{1}{\det(\mathbb{I}-f)} \tag{4.61}$$

is the Lefschetz number of the operator $\mathcal{T}[(f,\alpha)]$. If f was a number $e^{-\beta\omega}$, (4.61) would coincide with the partition function of a harmonic oscillator with frequency ω at temperature $1/\beta$.

Even Dimensions

For even D the rotation f is exactly given by (4.54). Then the situation is more complicated because the integral (4.33) localizes to a line rather than a point, as in Fig. 3.2. To make things simple we take $\alpha = (\alpha^0, 0, \ldots, 0)$ to be a pure time translation. Formula (4.55) is then replaced by

$$\chi[(f,\alpha)] = \chi_\lambda^{(D-1)}[f]\int_{\mathbb{R}^{D-1}} d^{D-1}\mathbf{k}\ e^{i\alpha^0\sqrt{M^2+\mathbf{k}^2}}\ \delta^{(D-1)}(\mathbf{k}-f\cdot\mathbf{k}) \tag{4.62}$$

where we have already implemented the simplification (4.56). The SO$(D-1)$ character $\chi_\lambda^{(D-1)}$ has been pulled out of the integral because, for D even, boosts along the direction k_{D-1} commute with rotations (4.54). It remains once more to integrate the delta function in (4.62). As far as the first $D-2$ components of $\mathbf{k}$ are concerned, the computation is the same as in the odd-dimensional case and results in a factor (4.58) given by (4.59). But the last component of **k** is untouched by (4.54), so (4.62) becomes

$$\chi[(f,\alpha)] = \chi_\lambda^{(D-1)}[f]\prod_{j=1}^{r}\frac{1}{|1-e^{i\theta_j}|^2}\int_{-\infty}^{+\infty}dk\, e^{i\alpha^0\sqrt{M^2+k^2}}\,\delta^{(1)}(k-k)\,, \quad (4.63)$$

where $k \equiv k_{D-1}$. Here the last term is an infrared-divergent factor

$$\delta(k-k) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}dz \equiv \frac{L}{2\pi} \quad (4.64)$$

where the length scale L is an infrared regulator. The integral in (4.63) then gives

$$\int_{-\infty}^{+\infty}dk\, e^{i\alpha^0\sqrt{M^2+k^2}} = 2M\, K_1(-iM\alpha^0)$$

where K_1 is the first modified Bessel function of the second kind. In conclusion we get

$$\chi[(f,\alpha)] = \frac{ML}{\pi}\,K_1(-iM\alpha^0)\,\chi_\lambda^{(D-1)}[f]\prod_{j=1}^{r}\frac{1}{|1-e^{i\theta_j}|^2}\,, \quad (4.65)$$

whose Wick-rotated version can now be seen as the rotating partition function of a particle trapped in a box of height L.

Time Translations

All characters written above diverge when one of the angles θ_j goes to zero. These divergences are infrared since they are due to delta functions evaluated at zero in momentum space, and can be regularized as in (4.64). A case of particular interest is the character of a pure time translation, whose Wick rotation is a canonical partition function (3.50). Using once more the Frobenius formula (4.33) and the cancellation (4.56), and letting $\alpha = (\alpha^0, 0, \dots, 0)$ be a pure time translation, we find

$$\chi[(e,\alpha)] = N\int_{\mathbb{R}^{D-1}} d^{D-1}\mathbf{k}\, e^{i\alpha^0\sqrt{M^2+\mathbf{k}^2}}\,\delta^{(D-1)}(0) \quad (4.66)$$

where $N \equiv \dim(\mathcal{E})$ is the number of spin degrees of freedom of the particle. The infrared-divergent delta function can be seen as the spatial volume of the system,

$$\delta^{(D-1)}(0) = \frac{1}{(2\pi)^{D-1}}\int_{\mathbb{R}^{D-1}} d^{D-1}\mathbf{x} = \frac{V}{(2\pi)^{D-1}}\,.$$

Using spherical coordinates we can then rewrite (4.66) as

$$\chi[(e,\alpha)] = \frac{NV}{(2\pi)^{D-1}} \frac{2\pi^{(D-1)/2}}{\Gamma((D-1)/2)} \int_0^{+\infty} k^{D-2}\, dk\, e^{i\alpha^0\sqrt{M^2+k^2}}$$
$$= \frac{2NV}{(2\pi)^{D-1}} \left(\frac{2\pi M}{-i\alpha^0}\right)^{(D-2)/2} M K_{D/2}(-iM\alpha^0)\,, \qquad (4.67)$$

where $K_{D/2}$ denotes once more a modified Bessel function of the second kind. This is the character of a pure time translation in a massive Poincaré representation. For $\alpha^0 = i\beta$ purely imaginary, it becomes the canonical partition function of a massive relativistic particle,

$$\text{Tr}\left(e^{-\beta H}\right)_{\text{massive particle}} = \frac{2NV}{(2\pi)^{D-1}} \left(\frac{2\pi M}{\beta}\right)^{(D-2)/2} M\, K_{D/2}(\beta M). \qquad (4.68)$$

4.2.5 Massless Characters

Characters of massless Poincaré representations can be evaluated along the same lines as massive ones, but there are subtleties due to the little group (4.49). The latter admits both finite- and infinite-dimensional irreducible unitary representations, corresponding to massless particles with discrete or continuous spin, respectively. Here we focus on the discrete case. As in the massive case we treat separately even and odd dimensions, this time starting with the former. For simplicity we take α to be a pure time translation.

Even Dimensions

For even D the Lorentz transformation (4.54) belongs to the little group of a massless particle since it leaves invariant the momentum vector $(E, 0, \ldots, 0, E)$. The character computation then is the same as in the even-dimensional massive case; formula (4.63) still holds with $M = 0$ and $\chi^{(D-1)}$ replaced by the character $\chi^{(D-2)}$ of a representation of SO$(D-2)$ instead of SO$(D-1)$. Note that for even D these two groups have the same rank $r = \lfloor (D-1)/2 \rfloor$, so there is no restriction on the values of the angles $\theta_1, \ldots, \theta_r$ (this will change for odd D). Using the regulator (4.64) and the fact that

$$\int_{-\infty}^{+\infty} dk\, e^{i|k|(\alpha^0+i\varepsilon)} = -\frac{2}{i\alpha^0}\,,$$

one finds the character

$$\chi[(f,\alpha)] = \frac{iL}{\pi\alpha^0}\, \chi_\lambda^{(D-2)}[f] \prod_{j=1}^r \frac{1}{|1-e^{i\theta_j}|^2}\,. \qquad (4.69)$$

Up to the replacement of $D-1$ by $D-2$, this is the limit $M \to 0$ of the massive character (4.65).

Odd Dimensions

For odd D the transformation (4.54) (with the last row and column suppressed) is no longer an element of the little group of $(E, 0, \ldots, 0, E)$ so its character vanishes if $\theta_r \neq 0$. This is consistent with the fact that $\mathrm{SO}(D-2)$ has lower rank than $\mathrm{SO}(D-1)$ when D is odd. Accordingly we now take $\theta_r = 0$ in (4.54), being understood that the last row and column are suppressed. From there on the character computation is identical to the cases treated above, except that the infrared divergence of the integral becomes worse and requires two regulators L, L':

$$\int_{\mathbb{R}^2} dk dq\, e^{i\alpha^0\sqrt{k^2+q^2}}\, \delta^{(1)}(k-k)\delta^{(1)}(q-q) = -\frac{LL'}{2\pi(\alpha^0)^2}\,. \tag{4.70}$$

Massless characters in odd dimension D thus read

$$\chi[(f,\alpha)] = \chi_\lambda^{(D-2)}[f]\left(-\frac{LL'}{2\pi(\alpha^0)^2}\right)\prod_{j=1}^{r-1}\frac{1}{|1-e^{i\theta_j}|^2} \tag{4.71}$$

where it is understood that $\theta_r = 0$ in (4.54) and $\chi_\lambda^{(D-2)}$ is a character of $\mathrm{SO}(D-2)$. Note that this expression is *not* the massless limit of (4.60) because in general $\theta_r \neq 0$ in the latter formula. However, upon setting $\theta_r = 0$ in (4.57) and regulating the resulting double infrared divergence as in (4.70), the limit $M \to 0$ does produce an expression of the form (4.71), albeit with a reducible representation of $\mathrm{SO}(D-2)$. We shall return to this in Sect. 11.1. Characters of time translations can be treated as in the massive case and coincide, up to spin multiplicity, with the massless limit of (4.67).

*4.2.6 Wigner Rotations and Entanglement**

We now analyse the Wigner rotation (4.31) and show that, for generic spinning particles, it entangles momentum and spin degrees of freedom. This phenomenon was first investigated in [20] (see also [21] and the related considerations in [22, 23]).

Wigner Rotations

Consider a particle with mass M and spin representation $\mathcal{R}$. We wish to understand the action of the Wigner rotation (4.31) for an arbitrary momentum q belonging to its orbit, and for a boost

$$f = \begin{pmatrix} \cosh\gamma & -\sinh\gamma & 0 & \cdots & 0 \\ -\sinh\gamma & \cosh\gamma & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix} \tag{4.72}$$

with rapidity γ in the direction x^1. Since the g_q's are standard boosts (4.53), the combination $g_q^{-1} f\, g_{f^{-1}\cdot q}$ is a sequence of three pure boosts in a plane, so we may safely take $D = 3$ without affecting the outcome of the computation. The momentum q then reads

$$q = \begin{pmatrix} \sqrt{M^2 + Q^2} \\ Q \cos\varphi \\ Q \sin\varphi \end{pmatrix} \tag{4.73}$$

for some angle φ and some positive number Q. After a mildly cumbersome but straightforward computation, one finds a Wigner rotation matrix

$$g_q^{-1} f\, g_{f^{-1}\cdot q} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix} \tag{4.74}$$

whose entries are given by

$$\begin{aligned} \sin\theta &= -\frac{M \sin(\varphi) \sinh(\gamma)}{\sqrt{Q^2 \sin^2(\varphi) + (Q \cos(\varphi) \cosh(\gamma) + \sqrt{M^2 + Q^2} \sinh(\gamma))^2}}\,, \\ \cos\theta &= \frac{Q \cosh(\gamma) + \sqrt{M^2 + Q^2} \cos(\varphi) \sinh(\gamma)}{\sqrt{Q^2 \sin^2(\varphi) + (Q \cos(\varphi) \cosh(\gamma) + \sqrt{M^2 + Q^2} \sinh(\gamma))^2}}\,. \end{aligned} \tag{4.75}$$

This is a pure rotation, as it should. It represents the fact that a boost acting on a particle with non-zero momentum is seen, from the rest frame of the particle, as a boost combined with a rotation (4.74) rather than a pure boost. The rotation only affects spin degrees of freedom; scalar particles are insensitive to it.

The Wigner rotation (4.75) is responsible for the phenomenon of *Thomas precession* [24] (see also [11], Sect. 11.8). The latter is visible in atomic physics, where the spin of an electron orbiting around a nucleus undergoes a slow precession due to the fact that the electron's acceleration is a sequence of boosts directed towards the nucleus.

Momentum/spin Entanglement

In (3.8) we saw that a space of $\mathcal{E}$-valued wavefunctions on $\mathcal{O}_p$ is a tensor product of $\mathcal{E}$ with the scalar space $L^2(\mathcal{O}_p, \mu, \mathbb{C})$. For a relativistic particle, the former consists of spin degrees of freedom while the latter accounts for momenta (or positions after Fourier transformation). For example, any state of a massive particle with spin $1/2$ takes the form (3.9) and describes the separate propagation of the two spin states $|+\rangle$ and $|-\rangle$. (For simplicity we use the Dirac notation until the end of this section.)

Hilbert space factorizations such as (3.8) are seldom preserved by unitary maps. Indeed, if $\mathscr{H} = A \otimes B$ and $|\Psi\rangle \in \mathscr{H}$ is a state with unit norm, the reduced density matrix associated with $|\Psi\rangle$ and acting in B is $\rho \equiv \mathrm{Tr}_A |\Psi\rangle\langle\Psi|$. When U is a unitary operator in $\mathscr{H}$, it is generally *not* true that the reduced density matrix of $U \cdot |\Psi\rangle$ is unitarily equivalent to ρ. In particular, U does not preserve the degree of entanglement between A and B. Accordingly one may ask [20] whether Poincaré representations spoil the splitting (3.8). To answer this, consider for definiteness a massive spin $1/2$ particle in four dimensions. We start from a normalized $\mathcal{E}$-valued wavefunction

$$\Psi(q) = \psi(q)|+\rangle, \qquad \text{i.e.} \qquad |\Psi\rangle = |\psi\rangle \otimes |+\rangle \tag{4.76}$$

where ψ is some complex-valued wavefunction while $|+\rangle$ is one of the two members of an orthonormal basis $|+\rangle, |-\rangle$ of $\mathcal{E}$. This state represents a particle with spin up (say along the x^3 axis) propagating with a momentum probability distribution $d\mu(q)|\psi(q)|^2$. (For definiteness we take the measure μ to be the Lorentz-invariant expression (3.4).) The corresponding reduced density matrix obtained by tracing over spin degrees of freedom acts on the scalar Hilbert space $L^2(\mathcal{O}_p, \mu, \mathbb{C})$ and reads

$$\rho = \langle +|\Big(|\psi\rangle|+\rangle\langle\psi|\langle+|\Big)|+\rangle + \langle -|\Big(|\psi\rangle|+\rangle\langle\psi|\langle+|\Big)|-\rangle = |\psi\rangle\langle\psi|\,,$$

which is a pure state. Now let us act on (4.76) with a Lorentz transformation f. According to (4.30), and writing $U \equiv \mathcal{T}[(f, 0)]$, the resulting wavefunction is

$$(U \cdot \Psi)(q) = W_q[f] \cdot \Psi(f^{-1} \cdot q) \overset{(4.76)}{=} \psi(f^{-1} \cdot q)\, W_q[f]|+\rangle \tag{4.77}$$

where $W_q[f]$ is the Wigner rotation (4.31). Denoting $\phi(q) \equiv \psi(f^{-1} \cdot q)$ we now find that the entries of the reduced density matrix of (4.77) are

$$\tilde{\rho}(q, q') = \big(\phi(q)\chi_+(q)\big)\big(\phi(q')\chi_+(q')\big)^* + \big(\phi(q)\chi_-(q)\big)\big(\phi(q')\chi_-(q')\big)^* \tag{4.78}$$

where we have defined $\chi_\pm(q) \equiv \langle\pm|W_q[f]|+\rangle$. In general expression (4.78) is *not* equal to a product $\tilde{\psi}(q)\tilde{\psi}^*(q')$ (for some complex wavefunction $\tilde{\psi}$), so the state (4.78) is not pure! In particular the boosted state (4.77) is generally entangled with respect to the splitting (3.8), even though the original state (4.76) was not. The reason for this is that the Wigner rotation (4.74) generally has non-vanishing $+-$ entries. Note that the functions $\chi_\pm$ satisfy $|\chi_+|^2 + |\chi_-|^2 = 1$ by virtue of the fact that Wigner rotations are unitary, so formula (4.78) indeed defines a density matrix.

These arguments can be generalized to any unitary representation of a semi-direct product (4.1). The only exceptions arise (i) if the spin representation $\mathcal{R}$ is one-dimensional so that $\mathcal{E} = \mathbb{C}$ and the tensor product (3.8) is trivial, or (ii) if f is such that $W_q[f]$ does not depend on q. In both situations the splitting (3.8) is robust against symmetry transformations. An example of momentum-independent Wigner rotations will be provided by the Bargmann group below. Thus the entanglement of spin and momentum due to Wigner rotations is a purely relativistic effect.

4.3 Poincaré Particles in Three Dimensions

Here we apply the considerations of the previous section to the Poincaré group in $D = 3$ space-time dimensions. This exercise will be a helpful guide for the description of BMS_3 particles in part III. To our knowledge, representations of Poincaré in three dimensions have previously been studied in [25, 26].

4.3.1 Poincaré Group in Three Dimensions

Prelude: The Group SL$(2, \mathbb{R})$

Many properties of the Poincaré group in three dimensions rely on the group $\text{SL}(2, \mathbb{R})$, so we start by describing the latter. $\text{SL}(2, \mathbb{R})$ is the group of linear transformations of the plane $\mathbb{R}^2$ that preserve volume and orientation. It consists of real 2×2 matrices with unit determinant:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad ad - bc = 1. \tag{4.79}$$

The centre of $\text{SL}(2, \mathbb{R})$ consists of the identity matrix and its opposite, thus spanning a group $\mathbb{Z}_2$. Furthermore:

Lemma The group $\text{SL}(2, \mathbb{R})$ is connected, but not simply connected. It is homotopic to a circle and its fundamental group is isomorphic to the group of integers $\mathbb{Z}$:

$$\pi_1(\text{SL}(2, \mathbb{R})) \cong \mathbb{Z}. \tag{4.80}$$

Proof Since the determinant of (4.79) is non-zero, the vectors (a, b) and (c, d) in $\mathbb{R}^2$ are linearly independent. We can thus find linear combinations of these vectors that span an orthonormal basis of $\mathbb{R}^2$. In other words there exists a real matrix

$$\bar{K} = \begin{pmatrix} \bar{\alpha} & 0 \\ \bar{\beta} & \bar{\gamma} \end{pmatrix}$$

such that, for any $\text{SL}(2, \mathbb{R})$ matrix S of the form (4.79), the product

$$\mathcal{O} \equiv \bar{K} S = \begin{pmatrix} \bar{\alpha} a & \bar{\alpha} b \\ \bar{\beta} a + \bar{\gamma} c & \bar{\beta} b + \bar{\beta} d \end{pmatrix}$$

belongs to the orthogonal group O(2). We can make $\bar{\alpha}$ positive by setting $\bar{\alpha}^{-1} = \sqrt{a^2 + b^2}$ and we can set $\bar{\gamma} = 1/\bar{\alpha}$ so that $\mathcal{O} \in \text{SO}(2)$. Any matrix $S \in \text{SL}(2, \mathbb{R})$ can therefore be decomposed uniquely as

$$S = \bar{K}^{-1}\mathcal{O} \equiv K\mathcal{O}, \qquad \text{with } \mathcal{O} \in \text{SO}(2) \text{ and } K = \begin{pmatrix} x & 0 \\ y & 1/x \end{pmatrix} \tag{4.81}$$

for some $y \in \mathbb{R}$ and $x \in \mathbb{R}$ strictly positive.[4] This shows that SL(2, $\mathbb{R}$) is connected and homotopic to its maximal compact subgroup consisting of rotations

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \qquad \theta \in \mathbb{R}. \tag{4.82}$$

In particular, the fundamental group of SL(2, $\mathbb{R}$) is isomorphic to $\mathbb{Z}$. ■

Lorentz Transformations in Three Dimensions

The definitions of Sect. 4.2.1 remain valid in three dimensions. In particular the Lorentz group O(2, 1) still has four connected components as in Fig. 4.2, and it is still multiply connected. However, in contrast to the higher-dimensional case, the Lorentz group is now homotopic to a circle and therefore has a fundamental group isomorphic to $\mathbb{Z}$. This is a consequence of the following result:

Proposition There is an isomorphism

$$\text{SO}(2, 1)^{\uparrow} \cong \text{SL}(2, \mathbb{R})/\mathbb{Z}_2 \equiv \text{PSL}(2, \mathbb{R}) \tag{4.83}$$

where the $\mathbb{Z}_2$ subgroup of SL(2, $\mathbb{R}$) consists of the identity matrix and its opposite. In particular, the fundamental group of the connected Lorentz group in three dimensions is isomorphic to $\mathbb{Z}$.

Proof Our goal is to build a homomorphism

$$\phi : \text{SL}(2, \mathbb{R}) \to \text{O}(2, 1) : f \mapsto \phi[f] \tag{4.84}$$

and then use the property

$$\text{Im}(\phi) \cong \text{SL}(2, \mathbb{R})/\text{Ker}(\phi). \tag{4.85}$$

Let A be the Lie algebra $\mathfrak{sl}(2, \mathbb{R})$. Each matrix $\alpha \in \mathfrak{sl}(2, \mathbb{R})$ can be written as a linear combination

$$\alpha = \alpha^{\mu} t_{\mu} \tag{4.86}$$

where the α^{μ}'s are real coefficients and the matrices

$$t_0 \equiv \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad t_1 \equiv \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad t_2 \equiv \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{4.87}$$

form a basis of $\mathfrak{sl}(2, \mathbb{R})$. With these conventions,

[4]This is a rewriting of the Iwasawa decomposition.

$$\det(\alpha) = -\frac{1}{4}\eta_{\mu\nu}\alpha^\mu\alpha^\nu = -\frac{1}{4}\alpha^2. \tag{4.88}$$

Now $\mathrm{SL}(2,\mathbb{R})$ naturally acts on A according to the adjoint representation,

$$A \to A : \alpha \mapsto f\alpha f^{-1}. \tag{4.89}$$

This action preserves the determinant since $\det(f) = 1$, so according to (4.88) it may be seen (for each f) as a Lorentz transformation. This motivates the definition of a homomorphism (4.84) given by

$$f\, t_\mu\, f^{-1} = t_\nu\, \phi[f]^\nu{}_\mu \qquad \forall\, \mu = 0, 1, 2. \tag{4.90}$$

The entries of $\phi[f]$ are quadratic combinations of those of f, so ϕ is a continuous map. Since $\mathrm{SL}(2,\mathbb{R})$ is connected, the image $\mathrm{Im}(\phi)$ is contained in the connected Lorentz group $\mathrm{SO}(2,1)^\uparrow$. In fact one has $\mathrm{Im}(\phi) = \mathrm{SO}(2,1)^\uparrow$, which follows from the standard decomposition theorem for Lorentz transformations (see e.g. [10, 27]). The kernel of ϕ coincides with the centre of $\mathrm{SL}(2,\mathbb{R})$, i.e. $\mathrm{Ker}(\phi) = \{\mathbb{I}, -\mathbb{I}\}$. The isomorphism (4.83) follows upon using (4.85). ■

Remark For future reference note that the homomorphism (4.90) explicitly reads

$$\phi\left[\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right] = \begin{pmatrix} \frac{1}{2}(a^2+b^2+c^2+d^2) & \frac{1}{2}(a^2-b^2+c^2-d^2) & -ab-cd \\ \frac{1}{2}(a^2+b^2-c^2-d^2) & \frac{1}{2}(a^2-b^2-c^2+d^2) & -ab+cd \\ -ac-bd & bd-ac & ad+bc \end{pmatrix}, \tag{4.91}$$

where the argument of ϕ is an $\mathrm{SL}(2,\mathbb{R})$ matrix. This will be useful in Sect. 9.1.

Poincaré Group in Three Dimensions

The Poincaré group for $D = 3$ is defined as in (4.39) and its connected subgroup is (4.40). Owing to the isomorphism (4.83), its double cover can be written as

$$\mathrm{SL}(2,\mathbb{R}) \ltimes \mathbb{R}^3 \tag{4.92}$$

where the action of $\mathrm{SL}(2,\mathbb{R})$ on $\mathbb{R}^3$ is given by (4.89). The latter is in fact the adjoint representation so we can also rewrite (4.92) as

$$\text{double cover of } \mathrm{ISO}(2,1)^\uparrow \;=\; \mathrm{SL}(2,\mathbb{R}) \ltimes_{\mathrm{Ad}} \mathfrak{sl}(2,\mathbb{R})_{\mathrm{Ab}} \tag{4.93}$$

where $\mathfrak{sl}(2,\mathbb{R})_{\mathrm{Ab}}$ is the Lie algebra of $\mathrm{SL}(2,\mathbb{R})$, seen as an Abelian vector group. This observation will turn out to be crucial in part III of this thesis. We stress that (4.93) is *not* the universal cover of the Poincaré group in three dimensions, since $\mathrm{SL}(2,\mathbb{R})$ is homotopic to a circle. This implies that (4.93) admits topological projective representations (which are equivalent to exact representations of its universal cover). There are no algebraic central extensions.

4.3.2 *Particles in Three Dimensions*

Here we describe projective irreducible unitary representations of the connected Poincaré group in three dimensions and point out a few differences with respect to the higher-dimensional case described in Sect. 4.2.

Orbits and Little Groups

The classification of Poincaré momentum orbits in three dimensions is the same as in Sect. 4.2.2 and is summarized in Fig. 4.3. Considering the double cover (4.93) for definiteness, the little groups are as follows.

Let us prove that these are the correct little groups. We shall use the fact that the action of Lorentz transformations on momenta is equivalent to its action on translations, which in turn is equivalent to the adjoint representation of $\mathrm{SL}(2, \mathbb{R})$ according to the definition (4.93). From that point of view a momentum (p_0, p_1, p_2) is represented by a matrix $\eta^{\mu\nu} p_\mu t_\nu$ where $\eta_{\mu\nu}$ is the Minkowski metric in $D = 3$ dimensions and the t_μ's are given by (4.87). Explicitly the matrix is

$$p = \frac{1}{2} \begin{pmatrix} p_2 & -p_0 + p_1 \\ p_0 + p_1 & -p_2 \end{pmatrix}. \tag{4.94}$$

In that language the little group of p is the set of $\mathrm{SL}(2, \mathbb{R})$ matrices that commute with (4.94). It immediately follows that the little group of $p = 0$ is $\mathrm{SL}(2, \mathbb{R})$. For massive orbits we move to a rest frame where $p_0 = M$ and $p_1 = p_2 = 0$; the only matrices leaving p fixed then are rotations (4.82). For tachyons we take $p_0 = p_1 = 0$, $p_2 \neq 0$ and find that the little group consists of matrices of the form

$$\pm \begin{pmatrix} e^x & 0 \\ 0 & e^{-x} \end{pmatrix}, \qquad x \in \mathbb{R}, \tag{4.95}$$

spanning a group $\mathbb{R} \times \mathbb{Z}_2$. Finally, for massless particles we take $p_0 = -p_1 \neq 0$ and $p_2 = 0$; the resulting little group $\mathbb{R} \times \mathbb{Z}_2$ is spanned by matrices of the type

$$\pm \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, \qquad x \in \mathbb{R}. \tag{4.96}$$

This reproduces all little groups listed in Table 4.1.

Table 4.1 Orbits and little groups of the Poincaré group in three dimensions

Orbit	Little group
Trivial	$\mathrm{SL}(2, \mathbb{R})$
Massive	$\mathrm{U}(1) \cong \mathrm{SO}(2)$
Massless	$\mathbb{R} \times \mathbb{Z}_2$
Tachyonic	$\mathbb{R} \times \mathbb{Z}_2$

Note that the little groups listed in Table 4.1 are sensitive to the cover chosen in (4.93). Had we chosen the standard connected Poincaré group $\mathrm{SO}(2,1)^{\uparrow} \ltimes \mathbb{R}^3$, the little groups for massless particles and tachyons would be quotiented by $\mathbb{Z}_2$ and would reduce to $\mathbb{R}$. For the universal cover of the Poincaré group, the little groups would instead get decompactified, so e.g. U(1) would be replaced by $\mathbb{R}$. This has important implications for the spin of relativistic particles in three dimensions.

Massive Particles

The properties of massive particles in three dimensions are the same as in Sect. 4.2.3. In particular their momentum orbits take the form

$$\mathcal{O}_p \cong \mathrm{SO}(2,1)^{\uparrow}/\mathrm{U}(1) \cong \mathrm{SL}(2,\mathbb{R})/S^1. \tag{4.97}$$

The only subtlety is that the group of spatial rotations now is $\mathrm{U}(1) \cong \mathrm{SO}(2)$, so the spin of a massive particle is a one-dimensional irreducible unitary representation of the form (2.13) labelled by some number s. If the double cover (4.93) was the universal cover of the Poincaré group, that number would be restricted to integer or half-integer values. However the fact that (4.93) is homotopic to a circle implies that s may take any real value. Thus massive particles in three dimensions can be *anyons* [28, 29]. The same phenomenon will occur with massive BMS_3 particles.

Remark Wigner rotations do occur in three dimensions, but they do not lead to momentum/spin entanglement when the space of spin degrees of freedom is one-dimensional.

Massless Particles

The spin properties of massless particles in three dimensions are also somewhat peculiar compared to those of their higher-dimensional cousins. Their little group $\mathbb{R} \times \mathbb{Z}_2$ can be seen as a Euclidean group in one dimension, where $\mathbb{Z}_2$ plays the role of rotations while $\mathbb{R}$ is spanned by Euclidean translations. If the latter is represented non-trivially, one obtains an analogue of "continuous spin" particles in three dimensions, although in the present case the space of spin degrees of freedom is actually finite-dimensional. By contrast, when $\mathbb{R}$ is represented trivially, the spin representation boils down to an irreducible unitary representation of $\mathbb{Z}_2$. The latter has exactly two irreducible unitary representations (the trivial one and the fundamental one), so we conclude that "discrete spin" massless particles in three dimensions can only be distinguished by their statistics (bosonic or fermionic); they have no genuine spin. This is consistent with the fact that massless field theories in three dimensions either have no local degrees of freedom at all (such as in gravity or Chern-Simons theory), or have only scalar or Weyl fermion degrees of freedom.

4.3.3 Characters

For future reference, we now list characters of irreducible unitary representations of the Poincaré group in three dimensions. The results of Sects. 4.2.4 and 4.2.5 apply, so the character of a rotation by θ combined with an arbitrary translation α in a Poincaré representation with mass M and spin s is given by formula (4.60),

$$\chi[(\text{rot}_\theta, \alpha)] = e^{iM\alpha^0 + is\theta} \frac{1}{|1 - e^{i\theta}|^2} = e^{iM\alpha^0 + is\theta} \frac{1}{4\sin^2(\theta/2)}\,, \tag{4.98}$$

where we have replaced the little group character by $\chi_\lambda[f] = e^{is\theta}$. In part III we shall encounter the BMS_3 generalization of this expression. Similarly the character (4.68) of Euclidean time translations becomes

$$\text{Tr}\left(e^{-\beta H}\right)_{\text{massive particle}} = \frac{V}{2\pi\beta^2}(1 + \beta M)e^{-\beta M}. \tag{4.99}$$

Characters of massless particles with discrete spin are given by formula (4.71) with $D = 3$, $r = 1$ and $\chi_\lambda = \pm 1$.

4.4 Galilean Particles*

In this section we classify irreducible unitary representations of the Bargmann groups, i.e. *non-relativistic* or *Galilean particles*. This example will be useful as a comparison to the relativistic case, and will also involve a dimensionful central charge that makes it similar to the centrally extended BMS_3 group of part III. This being said, the material exposed in this section is not crucial for our later considerations, so it may be skipped in a first reading. The plan is similar to that of Sect. 4.2: after defining Bargmann groups, we classify their orbits and litte groups, describe non-relativistic particles and compute their characters. We refer to [30, 31] for further reading on the Bargmann groups and to [32] for their representations.

4.4.1 Bargmann Groups

Galilei Groups

Definition The *Galilei group* in D space-time dimensions is a nested semi-direct product

$$\left(\text{O}(D-1) \ltimes \mathbb{R}^{D-1}\right) \ltimes \left(\mathbb{R}^{D-1} \times \mathbb{R}\right) \tag{4.100}$$

whose elements are quadruples $(f, \mathbf{v}, \boldsymbol{\alpha}, t)$ where $f \in \mathrm{O}(D-1)$ is a rotation, $\mathbf{v}$ is a boost belonging to the first $\mathbb{R}^{D-1}$, $\boldsymbol{\alpha}$ is a spatial translation belonging to the second $\mathbb{R}^{D-1}$, and $t \in \mathbb{R}$ is a time translation. The group operation is

$$(f, \mathbf{v}, \boldsymbol{\alpha}, s) \cdot (g, \mathbf{w}, \boldsymbol{\beta}, t) = \big(f \cdot g, \mathbf{v} + f \cdot \mathbf{w}, \boldsymbol{\alpha} + f \cdot \boldsymbol{\beta} + \mathbf{v}t, s + t\big) \tag{4.101}$$

where the dots on the right-hand side denote either matrix multiplication, or the action of a matrix on a column vector. The largest connected subgroup of (4.100) is obtained upon replacing $\mathrm{O}(D-1)$ by $\mathrm{SO}(D-1)$; its universal cover is obtained by replacing $\mathrm{SO}(D-1)$ by its universal cover, $\mathrm{Spin}(D-1)$.

The intricate structure (4.100) translates the fact that space and time live on different footings in Galilean relativity. Thus the analogue of a Lorentz transformation now is a pair $(f, \mathbf{v})$, while space-time translations are pairs $(\boldsymbol{\alpha}, t)$. Boosts and rotations span a group $\mathrm{SO}(D-1) \ltimes \mathbb{R}^{D-1}$ while space-time translations span an Abelian group $\mathbb{R}^D$. In particular each boost is a velocity vector $\mathbf{v}$ acted upon by rotations according to the matrix representation of $\mathrm{O}(D-1)$. Since time is absolute in Galilean relativity, the last entry on the right-hand side of (4.101) is a sum $s + t$ without influence of boosts. The term $\mathbf{v}t$ of the third entry is a time-dependent translation at velocity $\mathbf{v}$. Finally, there is an Abelian subgroup $\mathbb{R}^{2D}$ consisting of pairs

$$(e, \mathbf{v}, \boldsymbol{\alpha}, 0) \tag{4.102}$$

where e is the identity in $\mathrm{O}(D-1)$.

The Lie algebra of the Galilei group is generated by $(D-1)(D-2)/2$ rotation generators, $(D-1)$ boost generators, $(D-1)$ spatial translation generators, and one generator of time translations. We will not display their Lie brackets here.

Bargmann Groups

The Galilei group turns out to admit a non-trivial algebraic central extension:

Definition The *Bargmann group* in D space-time dimensions is a centrally extended semi-direct product

$$\mathrm{Bargmann}(D) \equiv \big(\mathrm{O}(D-1) \ltimes \mathbb{R}^{D-1}\big) \ltimes \big(\mathbb{R}^{D-1} \times \mathbb{R}\big) \times \mathbb{R}\,, \tag{4.103}$$

whose elements are 5-tuples $(f, \mathbf{v}, \boldsymbol{\alpha}, t, \lambda)$ where $(f, \mathbf{v}, \boldsymbol{\alpha}, t)$ belongs to the Galilei group (4.100) while λ is a real number. The group operation is

$$(f, \mathbf{v}, \boldsymbol{\alpha}, s, \lambda) \cdot (g, \mathbf{w}, \boldsymbol{\beta}, t, \mu) = \Big((f, \mathbf{v}, \boldsymbol{\alpha}, s) \cdot (g, \mathbf{w}, \boldsymbol{\beta}, t), \lambda + \mu + \mathbf{v} \cdot f \cdot \boldsymbol{\beta} + \frac{1}{2}\mathbf{v}^2 t\Big) \tag{4.104}$$

where the first entry on the right-hand side is given by (4.101) while $\mathbf{v} \cdot \boldsymbol{\beta} \equiv v^i \beta^i$ is the Euclidean scalar product of $\mathbf{v}$ and $\boldsymbol{\beta}$; in particular, $\mathbf{v}^2 \equiv v^i v^i$.

This central extension says that the Abelian subgroup of boosts and translations (4.102) gets extended into a Heisenberg group (2.37):

$$(e, \mathbf{v}, \boldsymbol{\alpha}, 0, \lambda) \cdot (e, \mathbf{w}, \boldsymbol{\beta}, 0, \mu) = \big(e, \mathbf{v}+\mathbf{w}, \boldsymbol{\alpha}+\boldsymbol{\beta}, 0, \lambda+\mu+\mathbf{v}\cdot\boldsymbol{\beta}\big).$$

In other words, in quantum mechanics, spatial translations and boosts do not commute. Note that, even in the Bargmann group, the normal subgroup of (centrally extended) space-time translations

$$(e, 0, \boldsymbol{\alpha}, t, \lambda) \tag{4.105}$$

remains Abelian. Hence the exhaustivity theorem of Sect. 4.1.5 applies to the Bargmann group: all Galilean particles are induced representations.

For $D \geq 4$ space-time dimensions, Eq. (4.104) is the only algebraic central extension of the Galilei group. But for $D = 3$, the Galilei group admits three non-trivial differentiable central extensions [31], one of which is the one displayed in (4.104). We will not take these extra central extensions into account. As regards topological central extensions, the Galilean situation is identical to that of the Poincaré group. Thus Bargmann(3) has a fundamental group $\mathbb{Z}$ and admits infinitely many topological projective representations, while for $D \geq 4$ the fundamental group of Bargmann(D) is $\mathbb{Z}_2$, leading either to exact representations or to representations up to a sign.

Remark The Bargmann group is a limit of the Poincaré group as the speed of light goes to infinity (see e.g. [3]), known more accurately as an *Inönü-Wigner contraction* [33]. We will not describe this procedure here, although we will encounter a very similar one in part III when showing that the BMS_3 group is an ultrarelativistic limit of two Virasoro groups.

4.4.2 Orbits and Little Groups

We now classify the orbits and little groups of the Bargmann group (4.103). We follow the same strategy as in Sect. 4.2.2.

Generalized Momenta

The Abelian normal subgroup of (4.103) consists of centrally extended translations (4.105). Its dual space consists of generalized momenta

$$(\mathbf{p}, E, M) \tag{4.106}$$

paired with translations according to[5]

$$\langle(\mathbf{p}, E, M), (\boldsymbol{\alpha}, t, \lambda)\rangle = \langle\mathbf{p}, \boldsymbol{\alpha}\rangle - Et - M\lambda \tag{4.107}$$

where $\langle\mathbf{p}, \boldsymbol{\alpha}\rangle \equiv p_i\alpha^i, i = 1, \ldots, D-1$. Accordingly, $\mathbf{p}$ is dual to spatial translations and represents the actual momentum of a particle; E is dual to time translations and

[5]The minus signs are conventional, and included for later convenience.

represents the particle's energy; finally M is a *central charge* dual to the central entries λ in (4.105). Working in units such that $\hbar = 1$, Eq. (4.104) says that λ has dimensions [distance] $\times$ [velocity] so the pairing (4.107) implies that M is a mass scale:

$$[M] = \frac{[\text{energy}] \times [\text{time}]}{[\text{distance}] \times [\text{velocity}]} = [\text{mass}]. \tag{4.108}$$

In fact we will see below that M is the mass of a non-relativistic particle. Note that M lives on a different footing than $\mathbf{p}$ and E, which is why the relation $E = Mc^2$ is invisible in Galilean relativity.

According to the structure (4.103), the group G acting on translations is the Euclidean group spanned by rotations and boosts. Its action is given by

$$\sigma_{(f,\mathbf{v})}(\boldsymbol{\alpha}, t, \lambda) = \Big(f \cdot \boldsymbol{\alpha} + \mathbf{v}t, t, \lambda + \mathbf{v} \cdot f \cdot \boldsymbol{\alpha} + \frac{1}{2}\mathbf{v}^2 t\Big) \tag{4.109}$$

by virtue of (4.104). The pairing (4.107) then yields the action σ^* of boosts and rotations on generalized momenta:

$$\begin{aligned}
&\big\langle \sigma^*_{(f,\mathbf{v})}(\mathbf{p}, E, M), (\boldsymbol{\alpha}, t, \lambda)\big\rangle = \\
\overset{(4.16)}{=}\ &\big\langle (\mathbf{p}, E, M), \sigma_{(f^{-1}, -f^{-1}\cdot\mathbf{v})}(\boldsymbol{\alpha}, t, \lambda)\big\rangle \\
=\ &\Big\langle (\mathbf{p}, E, M), \Big(f^{-1} \cdot \boldsymbol{\alpha} - f^{-1} \cdot \mathbf{v}t, t, \lambda - \mathbf{v} \cdot \boldsymbol{\alpha} + \frac{1}{2}\mathbf{v}^2 t\Big)\Big\rangle \\
\overset{(4.107)}{=}\ &\big\langle \mathbf{p}, f^{-1} \cdot \boldsymbol{\alpha} - f^{-1} \cdot \mathbf{v}t\big\rangle - Et - M\Big(\lambda - \mathbf{v} \cdot \boldsymbol{\alpha} + \frac{1}{2}\mathbf{v}^2 t\Big),
\end{aligned} \tag{4.110}$$

where we have used the fact that rotations preserve Euclidean scalar products. We can then use the Euclidean analogue of the isomorphism (4.42) to identify $(\mathbb{R}^{D-1})^*$ with $\mathbb{R}^{D-1}$ and rewrite the pairing $\langle \mathbf{p}, \boldsymbol{\alpha} \rangle = p_i \alpha^i$ as a scalar product $\mathbf{p} \cdot \boldsymbol{\alpha} = p^i \alpha^i = p_i \alpha_i$, where indices are raised and lowered thanks to the Euclidean metric. This allows us to rewrite $\mathbf{v} \cdot \boldsymbol{\alpha}$ as $\langle \mathbf{v}, \boldsymbol{\alpha} \rangle$ in (4.110), and leads to

$$\sigma^*_{(f,\mathbf{v})}(\mathbf{p}, E, M) = \Big(f \cdot \mathbf{p} + M\mathbf{v}, E + \mathbf{v} \cdot f \cdot \mathbf{p} + \frac{1}{2}M\mathbf{v}^2, M\Big). \tag{4.111}$$

One may recognize here the non-relativistic transformations laws of momentum and energy under rotations and boosts. The mass M is left unchanged, as was to be expected for a central charge.

Orbits

Let us classify orbits of generalized momenta under the transformations (4.111). Since the mass M is invariant, it is a constant quantity specifying each orbit; orbits with different masses are disjoint. In particular, the orbits differ greatly depending on whether M vanishes or not.

A *massless non-relativistic particle* is one for which $M = 0$, whereupon (4.111) simplifies to

$$\sigma^*_{(f,\mathbf{v})}(\mathbf{p}, E, 0) = \Big(f \cdot \mathbf{p}, E + \mathbf{v} \cdot f \cdot \mathbf{p}, 0\Big). \tag{4.112}$$

This implies that the norm of the momentum $\mathbf{p}$ of a massless particle is invariant under rotations and boosts. If $\mathbf{p} = 0$ the particle is static (in all reference frames) and the momentum orbit is trivial. If on the other hand $\mathbf{p} \neq 0$, then the particle moves (in all references frames); its momentum orbit is

$$\mathcal{O}_{(\mathbf{p},E,0)} = \big\{(f \cdot \mathbf{p}, E + \mathbf{v} \cdot f \cdot \mathbf{p}, 0)\big| f \in \mathrm{SO}(D-1),\ \mathbf{v} \in \mathbb{R}^{D-1}\big\} \cong S^{D-2} \times \mathbb{R}$$

where the sphere S^{D-2} is spanned by all momenta $f \cdot \mathbf{p}$ while $\mathbb{R}$ is spanned by the values of energy. The little group is

$$G_{(\mathbf{p},E,0)} = \mathrm{SO}(D-2) \ltimes \mathbb{R}^{D-2} \tag{4.113}$$

and consists of rotations leaving $\mathbf{p}$ invariant together with boosts that are orthogonal to $\mathbf{p}$. Note that this is the same little group (4.49) as for relativistic massless particles.

A *massive non-relativistic particle* is such that $M \neq 0$. Let $(\mathbf{p}, E)$ be its momentum and energy. Then the boost $\mathbf{v} = -\mathbf{p}/M$ plugged in (4.111) maps $(\mathbf{p}, E, M)$ on

$$\sigma^*_{(e,-\mathbf{p}/M)}(\mathbf{p}, E, M) = \Big(0, E + \frac{\mathbf{p}^2}{2M}, M\Big) \tag{4.114}$$

so any massive particle admits a rest frame. If we call $E_0 \equiv E + \mathbf{p}^2/2M$, the orbit of (4.114) under rotations and boosts is a parabola

$$\mathcal{O}_{(0,E,M)} = \left\{\Big(M\mathbf{v}, E_0 + \frac{M\mathbf{v}^2}{2}, M\Big)\Big|\mathbf{v} \in \mathbb{R}^{D-1}\right\} \subset \mathbb{R}^{D-1} \times \mathbb{R}. \tag{4.115}$$

As orbit representative we can take the generalized momentum in the rest frame,

$$(0, E_0, M) \tag{4.116}$$

where E_0 is an arbitrary real number; at fixed M, representatives with different values of E_0 define distinct orbits. The little group is the group of rotations

$$G_{(0,E_0,M)} = \mathrm{SO}(D-1) \tag{4.117}$$

in accordance with the fact that the orbit (4.115) is diffeomorphic to the quotient space $\big(\mathrm{SO}(D-1) \ltimes \mathbb{R}^{D-1}\big)/\mathrm{SO}(D-1) \cong \mathbb{R}^{D-1}$. Note again that this is exactly the same little group (4.46) as for relativistic massive particles. Finally, pure boosts

$$g_{\mathbf{q}} = (e, \mathbf{q}/M) \tag{4.118}$$

provide a continuous family of standard boosts on the orbit (4.115) of (4.116). Note that energy is bounded from below on the orbit if and only if $M > 0$.

4.4.3 Particles

According to the exhaustivity theorem of Sect. 4.1.5, all irreducible unitary representations of Bargmann groups are induced, and they are classified by momentum orbits. Each such representation consists of wavefunctions on an orbit, representing the quantum states of a non-relativistic particle.

For example, the spin of a massive Galilean particle is an irreducible unitary representation of $\mathrm{SO}(D-1)$. The space of states of the particle then consists of wavefunctions on the orbit (4.115) taking values in the space of the spin representation. Scalar products of wavefunctions are defined as usual by (3.7), where μ is some measure on the orbit. For convenience one can pick the standard Lebesgue measure $d^{D-1}\mathbf{q}$, which is left invariant by both rotations and boosts since (4.111) says that they act on (4.115) as Euclidean transformations $\mathbf{q} \mapsto f \cdot \mathbf{q} + M\mathbf{v}$.

In order to write down formula (4.30) explicitly for a non-relativistic particle, we still need to understand the Wigner rotation (4.31). Let us evaluate it for a pair $(f, \mathbf{v})$ at a point $\mathbf{q}$ belonging to the momentum orbit. We have $(f, \mathbf{v}) \cdot \mathbf{q} = f \cdot \mathbf{q} + M\mathbf{v}$, so the standard boost (4.118) for the momentum $(f, \mathbf{v})^{-1} \cdot \mathbf{q}$ is $g_{(f,\mathbf{v})^{-1}\cdot\mathbf{q}} = \big(e, \frac{f^{-1}\cdot\mathbf{q}}{M} - f^{-1} \cdot \mathbf{v}\big)$. Using the group operation (4.101) we read off the Wigner rotation

$$g_{\mathbf{q}}^{-1} \cdot (f, \mathbf{v}) \cdot g_{(f,\mathbf{v})^{-1}\cdot\mathbf{q}} = (f, 0)\,. \tag{4.119}$$

Surprise: the Wigner rotation is blind to boosts! In fact it is momentum-independent and simply coincides with the rotation f. Thus formula (4.30) for the transformation law of non-relativistic one-particle states becomes

$$\big(\mathcal{T}[(f, \mathbf{v}, \boldsymbol{\alpha}, t, \lambda)] \cdot \Psi\big)(q) = e^{-iM\lambda} e^{i\mathbf{q}\cdot\boldsymbol{\alpha} - i\mathbf{q}^2 t/2M}\, \mathcal{R}[f] \cdot \Psi\left((f, \mathbf{v})^{-1} \cdot \mathbf{q}\right), \tag{4.120}$$

where we have also used the fact that the measure $d^{D-1}\mathbf{q}$ is invariant to cancel its Radon-Nikodym derivative. This result differs from the Poincaré transformations of relativistic particles in two key respects. First, Galilean Wigner rotations (4.119) are momentum-independent, so in contrast to (4.75) they do *not* entangle momentum and spin. In fact, there is no Thomas precession for non-relativistic particles. The second difference is the presence of the mass M: formula (4.120) is an *exact* representation of the Bargmann group (4.104), but because $M \neq 0$ it is a *projective* representation of the centreless Galilei group (4.100). This can be seen by noting that for a pure boost $\mathbf{v}$ and a spatial translation $\boldsymbol{\alpha}$, Eq. (4.120) gives

$$\mathcal{T}[\mathbf{v}] \cdot \mathcal{T}[\boldsymbol{\alpha}] = e^{-iM\mathbf{v}\cdot\boldsymbol{\alpha}}\, \mathcal{T}[\boldsymbol{\alpha}] \cdot \mathcal{T}[\mathbf{v}], \tag{4.121}$$

which says that boosts and spatial translations do not commute. In part III we will encounter a similar phenomenon with the BMS_3 group, whose dimensionful central charge will coincide with the Planck mass.

4.4.4 Characters

We now evaluate characters of massive non-relativistic particles. Let $M > 0$ and choose a spin λ, specifying an irreducible unitary representation of the little group $SO(D-1)$. For definiteness we take the rest frame energy $E_0 = 0$ in (4.115). In order for the character (4.33) to be non-zero we must set $\mathbf{v} = 0$. Equation (4.119) then allows us to pull the little group character $\chi_{\mathcal{R}} = \chi_\lambda^{(D-1)}$ out of the momentum integral:

$$\chi[(f, 0, \boldsymbol{\alpha}, t, \lambda)] = e^{-iM\lambda}\chi_\lambda^{(D-1)}[f]\int_{\mathbb{R}^{D-1}} d^{D-1}\mathbf{k}\,\delta^{(D-1)}(\mathbf{k} - f\cdot\mathbf{k})\,e^{i\mathbf{k}\cdot\boldsymbol{\alpha} - i\mathbf{k}^2 t/2M}. \tag{4.122}$$

For simplicity we set $\lambda = 0$ from now on and neglect writing this entry. We take f to be a rotation (4.54) with the first row and column suppressed and all angles $\theta_1, \ldots, \theta_r$ non-zero, $r = \lfloor (D-1)/2 \rfloor$. If D is odd we also erase the last row and column. We treat separately even and odd dimensions.

If D is odd, then the only fixed point of f on (4.115) is the tip $\mathbf{k} = 0$. The integral of (4.122) localizes and (4.58) yields

$$\chi[(f, 0, \boldsymbol{\alpha}, t)] = \chi_\lambda[f]\prod_{j=1}^{r}\frac{1}{|1 - e^{i\theta_j}|^2}\,. \tag{4.123}$$

Note that translations do not contribute to this result. Up to the normalization of energy, it coincides with the relativistic character (4.60).

If D is even, then f leaves fixed the whole axis k_{D-1} as in Fig. 3.2. Integrating first over the rotated coordinates $k_1, \ldots, k_{D-2}$ in (4.122) and writing $k_{D-1} \equiv k$, we find

$$\chi[(f, 0, \boldsymbol{\alpha}, t)] = \chi_\lambda^{(D-1)}[f]\prod_{j=1}^{r}\frac{1}{|1 - e^{i\theta_j}|^2}\int_{-\infty}^{+\infty} dk\,\delta(0)\,e^{ik\alpha^{D-1} - ik^2 t/2M}. \tag{4.124}$$

Here the term $\delta(0) = \delta(k - k)$ is an infrared divergence that we regularize as in (4.64) with a length scale L. Denoting $\alpha^{D-1} \equiv x$, we are left with the integral

$$\int_{-\infty}^{+\infty} dk\,e^{ikx - ik^2 t/2M} = \left(\frac{2\pi M}{it}\right)^{1/2} e^{iMx^2/2t} \tag{4.125}$$

and thus conclude

$$\chi[(f,0,\boldsymbol{\alpha},t)] = \frac{L}{2\pi}\chi_\lambda^{(D-1)}[f]\prod_{j=1}^{r}\frac{1}{|1-e^{i\theta_j}|^2}\left(\frac{2\pi M}{it}\right)^{1/2} e^{iMx^2/2t}. \qquad (4.126)$$

The only dependence of this expression on $\boldsymbol{\alpha}$ appears through the component $\alpha^{D-1} \equiv x$, because we picked a rotation f leaving fixed the direction k_{D-1}. For a general rotation, the component of $\boldsymbol{\alpha}$ appearing in the character would be its projection on the axis left fixed by f.

Two comments are in order. First note that in $D = 2$ space-time dimensions, (4.126) boils down to the quantum propagator of a free non-relativistic particle at time t and separation x, up to an infrared-divergent factor L. This is because the character of a pure spatial translation in two space-time dimensions is

$$\mathrm{Tr}\big(\mathcal{T}[(e,x,t)]\big) = \int_{-\infty}^{+\infty} dk\,\delta(0)\,e^{ikx-itk^2/2M} = \int_{-\infty}^{+\infty}\frac{dy}{2\pi}\int_{-\infty}^{+\infty} dk e^{ikx-itk^2/2M}$$

which we can interpret as the trace of the operator $e^{iPx-iHt}$ in the Hilbert space of a free massive particle on the real line:

$$\mathrm{Tr}\big(\mathcal{T}[(e,x,t)]\big) = \mathrm{Tr}\left(e^{iPx-iHt}\right) = \int_{-\infty}^{+\infty} dy\langle y+x|e^{-iHt}|y\rangle. \qquad (4.127)$$

The integrand of this expression is the propagator of a free non-relativistic particle evaluated between y and $y+x$ at time t and coincides with (4.125).

The second comment concerns the relation between Bargmann characters and Poincaré characters. For even D, (4.126) is the non-relativistic analogue of (4.65) but the functions appearing in the two results are different. By contrast, for odd D, the Bargmann character (4.123) coincides with its Poincaré analogue (4.60). This may be seen as a consequence of the phenomenon (4.58), whose effect is to localize the computation of the character to the region of momentum space surrounding the momentum at rest, that is, the non-relativistic region. By contrast, when the localization is not complete as is the case for even D, the momenta in the integral (4.125) are arbitrarily large and relativistic effects become important. This produces a difference between Bargmann and Poincaré characters. It is particularly apparent for characters of Euclidean time translations, which in the non-relativistic case are given by

$$\chi[(e,0,0,-i\beta)] = \frac{NV}{(2\pi)^{D-1}}\int_{\mathbb{R}^{D-1}} d^{D-1}\mathbf{k}\,e^{-\beta\mathbf{k}^2/2M} = NV\left(\frac{M}{2\pi\beta}\right)^{(D-1)/2}$$

where N is the dimension of the spin representation. This is the non-relativistic version of (4.68). For $D = 3$ (and $N = 1$) it reduces to $VM/(2\pi\beta)$, which is the non-relativistic limit of (4.99).

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Chapter 5
Coadjoint Orbits and Geometric Quantization

In the previous chapters we have seen how representation theory leads to geometric objects such as orbits. The purpose of this chapter is to describe the opposite phenomenon: starting from a *coadjoint orbit* of a group G, we will obtain a representation by *quantizing* the orbit. This construction will further explain why orbits of momenta classify representations of semi-direct products. In addition it will turn out to be a tool for understanding gravity in parts II and III.

The plan is as follows. We start in Sect. 5.1 with basic reminders on symplectic manifolds with symmetries, including their momentum maps. Along the way we introduce the notion of coadjoint orbits, which will turn out to be crucial for the remainder of this thesis. Section 5.2 is then devoted to the quantization of symplectic manifolds, and describes in particular the relation between representation theory and symplectic geometry. In Sect. 5.3 we reformulate geometric quantization in terms of action principles that describe the propagation of a point particle on a group manifold. The two last sections of the chapter are concerned with applications of these considerations to semi-direct products: in Sect. 5.4 we describe the coadjoint orbits and world line actions of such groups in general, while in Sect. 5.5 we illustrate these results with the Poincaré group and the Bargmann group.

Our language in this chapter will be slightly different than in the previous ones, as we rely on differential-geometric tools that were unnecessary for our earlier considerations. Useful references include [1, 2] for symplectic geometry, [3, 4] for quantization, as well as the (sadly unpublished) Modave lecture notes [5].

Remark The presentation adopted here is self-contained, but fairly dense. We urge the reader who is not acquainted with differential geometry to only read Sects. 5.1.1 and 5.1.2, then go directly to part II of the thesis. In doing so one will miss the symplectic aspects of our later considerations, but the other points of our presentation should remain accessible.

B. Oblak, *BMS Particles in Three Dimensions*, Springer Theses,
DOI 10.1007/978-3-319-61878-4_5

5.1 Symmetric Phase Spaces

In this section we study classical systems with symmetries, that is, homogeneous symplectic manifolds. We start by recalling a few basic facts about Lie groups and we define their adjoint and coadjoint representations. We then describe in general terms Poisson and symplectic structures, and show how such structures arise in the case of coadjoint orbits. Finally we discuss the notion of momentum maps associated with the symmetries of a symplectic manifold. We use the notational conventions of Chap. 3.

5.1.1 Lie Groups

A *Lie group* is a group G which also has a structure of smooth manifold such that multiplication and inversion are smooth maps. In particular the operations of left and right multiplication defined in (3.16) and (3.17) are diffeomorphisms. We denote by e the identity in G, and generic group elements are denoted f, g, etc.

Definition A vector field ξ on G is *left-invariant* if $(L_f)_*\xi = \xi$ for all $f \in G$, i.e. if $(L_f)_{*g}\xi_g = \xi_{fg}$ for all $f, g \in G$.[1]

One can verify that any left-invariant vector field is given by $\xi_g = (L_g)_{*e}X$ for some tangent vector $X \in T_eG$. Thus the space of left-invariant vector fields is isomorphic to the tangent space of G at the identity. We shall denote by ζ_X the left-invariant vector field on G given by $(\zeta_X)_g = (L_g)_{*e}X$.

Definition The *Lie algebra* of G is the vector space $\mathfrak{g} = T_eG$ endowed with the Lie bracket

$$[X, Y] \equiv [\zeta_X, \zeta_Y]_e \tag{5.1}$$

where the bracket on the right-hand side is the usual Lie bracket of vector fields evaluated at the identity.

One can show that the bracket (5.1) is such that $\zeta_{[X,Y]} = [\zeta_X, \zeta_Y]$. As a corollary, any smooth homomorphism of Lie groups $\mathcal{F} : G \to H$ is such that its differential $\mathcal{F}_{*e}$ at the identity is a homomorphism of Lie algebras. When interpreting G as a symmetry group, the elements of its Lie algebra are seen as "infinitesimal" symmetries, i.e. transformations near the identity. In practice the Lie algebra structure of $\mathfrak{g}$ is often displayed in terms of a basis $\{t_a | a = 1, \ldots, \dim \mathfrak{g}\}$ of $\mathfrak{g}$ with Lie brackets

$$[t_a, t_b] = f_{ab}{}^c\, t_c\,. \tag{5.2}$$

In that context the coefficients $f_{ab}{}^c \in \mathbb{R}$ are known as the *structure constants* of $\mathfrak{g}$ in the basis $\{t_a\}$.

[1]Recall that the *differential* of a smooth map $\mathcal{F} : \mathcal{M} \to \mathcal{N}$ at $p \in \mathcal{M}$ is the map $\mathcal{F}_{*p} : T_p\mathcal{M} \to T_{\mathcal{F}(p)}\mathcal{N} : \dot{\gamma}(0) \mapsto \frac{d}{dt}\big[\mathcal{F}(\gamma(t))\big]\big|_{t=0}$, where $\gamma(t)$ is a path in $\mathcal{M}$ such that $\gamma(0) = p$.

Exponential Map

Definition Let $X \in \mathfrak{g}$, and let γ_X be the integral curve[2] of the corresponding left-invariant vector field ζ_X such that $\gamma_X(0) = e$. Then the *exponential map* of G is

$$\exp : \mathfrak{g} \to G : X \mapsto \exp[X] \equiv \gamma_X(1). \tag{5.3}$$

One can verify that, for matrix groups, this definition reduces to the standard Taylor series $\sum_{n\in\mathbb{N}} X^n/n!$. We often denote $\exp[X] \equiv e^X$.

Since the exponential map is defined by a vector flow, it automatically satisfies $\exp[(s+t)X] = \exp[sX]\exp[tX]$ for all $s, t \in \mathbb{R}$. In particular any $X \in \mathfrak{g}$ determines a one-dimensional subgroup of G consisting of elements $\exp[tX]$, $t \in \mathbb{R}$. Note that left-invariant vector fields are complete, which ensures the existence of $\exp[tX]$ for all $t \in \mathbb{R}$. Finally, for any smooth homomorphism $\mathcal{F} : G \to H$, one can show that

$$\mathcal{F} \circ \exp_G = \exp_H \circ \mathcal{F}_{*e}\,. \tag{5.4}$$

5.1.2 Adjoint and Coadjoint Representations

Definition Let G be a Lie group with Lie algebra $\mathfrak{g}$. Then the *adjoint representation* of G is the homomorphism

$$\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g}) : g \mapsto \mathrm{Ad}_g \tag{5.5}$$

where Ad_g is the linear operator that acts on $\mathfrak{g}$ according to

$$\mathrm{Ad}_g(X) = \frac{d}{dt}\left(g\, e^{tX}\, g^{-1}\right)\Big|_{t=0}. \tag{5.6}$$

Here one may freely replace e^{tX} by any path $\gamma(t)$ in G such that $\gamma(0) = e$ and $\dot{\gamma}(0) = X$. For matrix groups, Eq. (5.6) reduces to $\mathrm{Ad}_g(X) = gXg^{-1}$.

One can verify that this is indeed a representation of G. Using (5.4), one also shows that it satisfies the identity

$$e^{\mathrm{Ad}_f X} = f\, e^X\, f^{-1} \tag{5.7}$$

where e^X is the exponential map of G. Note that the adjoint representation of any Abelian Lie group is trivial. Finally, the adjoint representation of the Lie algebra $\mathfrak{g}$ is defined as the differential of (5.5) at the identity:

$$\mathrm{ad}_X(Y) \equiv \frac{d}{dt}\big(\mathrm{Ad}_{e^{tX}}(Y)\big)\Big|_{t=0} = [X, Y]\,. \tag{5.8}$$

[2] An *integral curve* of a vector field ξ on a manifold $\mathcal{M}$ is a path $\gamma(t)$ on $\mathcal{M}$ such that $\dot{\gamma}(t) = \xi_{\gamma(t)}$.

In (4.16) we saw how to define dual representations. Let us apply this to the adjoint representation (5.5): we write the dual space of $\mathfrak{g}$ as $\mathfrak{g}^*$, which consists of linear forms $p : \mathfrak{g} \to \mathbb{R} : X \mapsto \langle p, X \rangle$. When interpreting G as a symmetry group, the elements of the dual of $\mathfrak{g}$ can be seen as "momenta", or more generally conserved vectors, associated with the symmetries. In particular the number $\langle p, X \rangle$ then is the Noether charge associated with the symmetry generator X when the system has "momentum" p.

Definition Let G be a Lie group with Lie algebra $\mathfrak{g}$. Then the *coadjoint representation* of G is the homomorphism

$$\text{Ad}^* : G \to \text{GL}(\mathfrak{g}^*) : f \mapsto \text{Ad}^*_f \tag{5.9}$$

which is dual to the adjoint representation in the sense that

$$\text{Ad}^*_f(p) \equiv p \circ (\text{Ad}_f)^{-1}, \tag{5.10}$$

i.e. $\langle \text{Ad}^*_f(p), X \rangle \equiv \langle p, \text{Ad}_{f^{-1}}(X) \rangle$ for all $p \in \mathfrak{g}^*$ and any $X \in \mathfrak{g}$. From now on we refer to elements of $\mathfrak{g}$ and $\mathfrak{g}^*$ as *adjoint* and *coadjoint vectors*, respectively.

The coadjoint representation is a linear action of G on $\mathfrak{g}^*$. In particular one can foliate the space $\mathfrak{g}^*$ into disjoint G-orbits. We call the set

$$\mathcal{W}_p \equiv \left\{ \text{Ad}^*_g(p) \middle| g \in G \right\}$$

the *coadjoint orbit* of p. It is a homogeneous space for the coadjoint action of G. Note that the coadjoint representation of any Abelian group is trivial, so its coadjoint orbits are single points. By contrast, coadjoint orbits of non-Abelian groups are generally non-trivial (except if $p = 0$). We will see in Sect. 5.4.3 that the coadjoint orbits of semi-direct products contain their momentum orbits.

The dual of the infinitesimal adjoint representation (5.8) is the differential of (5.9) at the identity, i.e. the coadjoint representation of the Lie algebra $\mathfrak{g}$:

$$\text{ad}^*_X(p) \equiv \frac{d}{dt} \left(\text{Ad}^*_{e^{tX}}(p) \right) \Big|_{t=0} \overset{(5.10)}{=} -p \circ \text{ad}_X = -p \circ [X, \cdot]. \tag{5.11}$$

Remark The adjoint and coadjoint representations of a group G are generally *inequivalent*. In fact they are equivalent if and only if $\mathfrak{g}$ admits a non-degenerate bilinear form (which is the case e.g. for semi-simple Lie groups).

5.1.3 Poisson Structures

The *phase space* of a system is the set of its classical states. In the previous pages we have reviewed some basic concepts of group theory, and our goal is to eventually apply them to phase spaces with symmetries. Accordingly we now investigate Poisson structures and symplectic structures in more detail.

Definition Let $\mathcal{M}$ be a manifold. A *Poisson structure* on $\mathcal{M}$ is an antisymmetric bilinear map[3]

$$\{\cdot,\cdot\}: C^\infty(\mathcal{M})\times C^\infty(\mathcal{M})\to C^\infty(\mathcal{M}): \mathcal{F},\mathcal{G}\mapsto\{\mathcal{F},\mathcal{G}\}$$

which satisfies the Jacobi identity and the Leibniz rule:

$$\begin{aligned}\{\mathcal{F},\{\mathcal{G},\mathcal{H}\}\}+\{\mathcal{G},\{\mathcal{H},\mathcal{F}\}\}+\{\mathcal{H},\{\mathcal{F},\mathcal{G}\}\}=0 \quad &\text{(Jacobi)},\\ \{\mathcal{F},\mathcal{G}\mathcal{H}\}=\{\mathcal{F},\mathcal{G}\}\mathcal{H}+\mathcal{G}\{\mathcal{F},\mathcal{H}\} \quad &\text{(Leibniz)}.\end{aligned}$$

This map is called the *Poisson bracket* on $\mathcal{M}$, and the pair $\big(\mathcal{M},\{\cdot,\cdot\}\big)$ is a *Poisson manifold* or a *phase space*.

The Poisson bracket endows the space of functions $C^\infty(\mathcal{M})$ with a structure of Lie algebra; the Leibniz identity implies in addition that the map

$$\{\mathcal{F},\cdot\}: C^\infty(\mathcal{M})\to C^\infty(\mathcal{M}): \mathcal{G}\mapsto\{\mathcal{F},\mathcal{G}\}$$

is a derivation for any function $\mathcal{F}\in C^\infty(\mathcal{M})$.[4] These properties together endow the space $C^\infty(\mathcal{M})$ with the structure of a *Poisson algebra*. Note that the existence of a Poisson structure sets no restrictions on the dimension of $\mathcal{M}$. In particular, odd-dimensional manifolds admit Poisson structures, e.g. $\mathcal{M}=\mathbb{R}^3$ with the bracket $\{\mathcal{F},\mathcal{G}\}=\partial_x\mathcal{F}\partial_y\mathcal{G}-\partial_y\mathcal{F}\partial_x\mathcal{G}$. This will change once we turn to symplectic structures.

Definition Let $\big(\mathcal{M},\{\cdot,\cdot\}\big)$ be a Poisson manifold; let $\mathcal{H}\in C^\infty(\mathcal{M})$. We call *Hamiltonian vector field* associated with $\mathcal{H}$ the (unique) vector field $\xi_\mathcal{H}$ on $\mathcal{M}$ such that

$$\xi_\mathcal{H}=-\{\mathcal{H},\cdot\}. \tag{5.12}$$

The existence of $\xi_\mathcal{H}$ is ensured by the one-to-one correspondence between derivations of $C^\infty(\mathcal{M})$ and vector fields on $\mathcal{M}$.

The Hamiltonian vector field associated with a function $\mathcal{H}$ is a differential operator acting on functions on $\mathcal{M}$. Its integral curves are the paths $\gamma(t)$ in $\mathcal{M}$ that satisfy $\dot\gamma(t)=(\xi_\mathcal{H})_{\gamma(t)}$, which in local coordinates on $\mathcal{M}$ corresponds to a set of $\dim(\mathcal{M})$ first-order differential equations $\dot x^i(t)=\xi_\mathcal{H}(x(t))$. These are the equations of motion associated with the Hamiltonian $\mathcal{H}$. The definition (5.12) ensures that $\{\mathcal{H},\mathcal{G}\}=-\xi_\mathcal{H}(\mathcal{G})$, which implies that the equations of motion can be written locally as $\dot x^i=\{x^i,\mathcal{H}\}$ in terms of the Poisson bracket. In particular one has $\{\mathcal{H},\mathcal{G}\}=0$ if and only if $\mathcal{G}$ is constant along integral curves of $\xi_\mathcal{H}$. Note also that

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[3]From now on, real functions on $\mathcal{M}$ are denoted as $\mathcal{F},\mathcal{G},\mathcal{H}$, etc.

[4]A *derivation* of an algebra $\mathcal{A}$ is a linear map $D:\mathcal{A}\to\mathcal{A}: a\mapsto D(a)$ that satisfies the Leibniz rule $D(a\cdot b)=D(a)\cdot b+a\cdot D(b)$.

$$[\xi_{\mathcal{F}}, \xi_{\mathcal{G}}] = -\xi_{\{\mathcal{F},\mathcal{G}\}}\,, \tag{5.13}$$

so the Lie brackets of Hamiltonian vector fields are Hamiltonian.

Now consider the set of all Hamiltonian vector fields on $\mathcal{M}$; at a point $p \in \mathcal{M}$, they span a subspace of the tangent space $T_p\mathcal{M}$. By taking this span for all $p \in \mathcal{M}$, one obtains a subbundle of the tangent bundle $T\mathcal{M}$ (i.e. a distribution on $\mathcal{M}$). Because brackets of Hamiltonian vector fields are Hamiltonian, Frobenius' theorem implies that Hamiltonian vector fields yield a foliation of $\mathcal{M}$ into so-called *symplectic leaves*. Two points belong to the same leaf if they can be joined by the integral curve of a Hamiltonian vector field. In the example of $\mathbb{R}^3$ mentioned above, symplectic leaves are planes $z = \text{const}$. This leads to the definition of symplectic manifolds.

5.1.4 Symplectic Structures

Definition Let $\mathcal{M}$ be a manifold. A *symplectic form* on $\mathcal{M}$ is a closed, non-degenerate two-form ω on $\mathcal{M}$.[5] The pair $(\mathcal{M}, \omega)$ is a *symplectic manifold*.

Non-degeneracy means that, in local coordinates, the components ω_{ij} of ω form an invertible antisymmetric matrix. This implies that all symplectic manifolds are even-dimensional. Note that any symplectic manifold admits a *Liouville volume form*

$$\mu \equiv \underbrace{\omega \wedge \ldots \wedge \omega}_{\dim(\mathcal{M})/2 \text{ times}}\,. \tag{5.14}$$

The symplectic leaves described above are prime examples of symplectic manifolds: they are endowed with a symplectic form ω such that $\omega(\xi_{\mathcal{F}}, \xi_{\mathcal{G}}) \equiv \{\mathcal{F}, \mathcal{G}\}$; this condition determines ω unambiguously because symplectic leaves are, by definition, spanned by the integral curves of Hamiltonian vector fields. Another common example is the phase space $\mathcal{M} = \mathbb{R}^{2n}$ of a non-relativistic particle in $\mathbb{R}^n$, with coordinates $(q^1, \ldots, q^n, p_1, \ldots, p_n)$ and symplectic form

$$\omega = dq^i \wedge dp_i \qquad \text{(implicit sum over } i = 1, \ldots, n). \tag{5.15}$$

Canonical Symplectic Form

The symplectic structure (5.15) has an important generalization: consider a manifold $\mathcal{Q}$ describing the configuration space of a classical system (so $\dim\mathcal{Q}$ is the number of Lagrange variables). The corresponding phase space is the *cotangent bundle* $T^*\mathcal{Q}$, which consists of pairs (q, α) where $q \in \mathcal{Q}$ and $\alpha \in T^*_q\mathcal{Q}$. These pairs are generally interpreted as describing a "position" q and a "momentum" α, but we will see below that the interpretation stemming from semi-direct products is different: q will in fact

[5] Closedness means $d\omega = 0$, where d is the exterior derivative. Non-degeneracy means that for all $p \in \mathcal{M}$, any vector $v \in T_p\mathcal{M}$ such that $\omega_p(v, w) = 0$ for all $w \in T_p\mathcal{M}$ necessarily vanishes.

be a momentum (with $\mathcal{Q}$ a momentum orbit), while α will essentially be a position (or rather a translation vector). The symplectic form on $T^*\mathcal{Q}$ is defined as follows. We let

$$\pi : T^*\mathcal{Q} \to \mathcal{Q} : (q, \alpha) \mapsto q \tag{5.16}$$

be the natural projection and define the *Liouville one-form* θ on $T^*\mathcal{Q}$ by

$$\langle \theta_{(q,\alpha)}, \mathcal{V} \rangle \equiv \langle \alpha, \pi_{*(q,\alpha)} \mathcal{V} \rangle \tag{5.17}$$

for any vector $\mathcal{V} \in T_{(q,\alpha)}T^*\mathcal{Q}$. Then $\omega \equiv -d\theta$ is the *canonical symplectic form* on $T^*\mathcal{Q}$. In the example (5.15), $\mathcal{Q} = \mathbb{R}^n$.

Let us verify that $\omega = -d\theta$ is indeed symplectic. We choose local coordinates $(q^1, \ldots, q^n)$ on some open set $U \subset \mathcal{Q}$ and denote by (q^i, p_j) the corresponding local coordinates on $\pi^{-1}(U)$, so that the form $\alpha \in T^*_q\mathcal{Q}$ reads $\alpha = p_j(dq^j)_q$. Given a vector $\mathcal{V} \in T_{(q,\alpha)}T^*\mathcal{Q}$, one can write

$$\mathcal{V} = a^i \frac{\partial}{\partial q^i} + b_j \frac{\partial}{\partial p_j} \qquad \Rightarrow \qquad \pi_{*(q,\alpha)}\mathcal{V} = a^i \frac{\partial}{\partial q^i} \, .$$

Thus the differential of the projection (5.16) projects $\mathcal{V}$ on its part tangent to $\mathcal{Q}$. The definition (5.17) then implies that $\theta = p_i dq^i$, so

$$\omega = -d\theta = dq^i \wedge dp_i \tag{5.18}$$

is definitely a closed, non-degenerate two-form. It coincides locally with (5.15).

Remark The *Darboux theorem* states that any point of a symplectic manifold has a neighbourhood with local coordinates (q^i, p_j) such that the symplectic form reads (5.18). Thus any symplectic manifold is locally equivalent to a cotangent bundle.

Hamiltonian Vector Fields Revisited

Any symplectic manifold can be endowed with a Poisson structure by mimicking the symplectic leaves described at the end of Sect. 5.1.3. This relies on a new definition of Hamiltonian vector fields:

Definition Let $(\mathcal{M}, \omega)$ be a symplectic manifold, $\mathcal{F} \in C^\infty(\mathcal{M})$. The *Hamiltonian vector field* $\xi_\mathcal{F}$ associated with $\mathcal{F}$ is defined by

$$i_{\xi_\mathcal{F}}\omega = \omega(\xi_\mathcal{F}, \cdot) \stackrel{!}{=} d\mathcal{F}. \tag{5.19}$$

Conversely, a vector field ζ is *Hamiltonian* if there exists a function $\mathcal{F}$ such that $\zeta = \xi_\mathcal{F}$.

Hamiltonian vector fields can be used to define Poisson brackets in the same way as on symplectic leaves of Poisson manifolds: for any two functions $\mathcal{F}, \mathcal{G}$ we write

$$\{\mathcal{F}, \mathcal{G}\} \equiv \omega(\xi_\mathcal{F}, \xi_\mathcal{G}). \tag{5.20}$$

In terms of this bracket the definition (5.19) is equivalent to our earlier definition of Hamiltonian vector fields in (5.12). In local coordinates the definition (5.19) reads

$$\omega_{ij}\xi^i_{\mathcal{F}} = \partial_j\mathcal{F} \qquad \Leftrightarrow \qquad \xi^i_{\mathcal{F}} = \partial_j\mathcal{F}\omega^{ji} \tag{5.21}$$

where ω_{ij} are the components of ω and ω^{ij} is the matrix inverse of ω_{ij}. Accordingly, the bracket (5.20) can be written as $\{\mathcal{F},\mathcal{G}\} = -\omega^{ij}\partial_i\mathcal{F}\partial_j\mathcal{G}$.

Note that symplectic manifolds only contain kinematical data: they tell us the available combinations of "positions" and "momenta" — those combinations are classical states. Classical observables then are real-valued functions on phase space. Once we declare that a certain observable $\mathcal{H}$ is the Hamiltonian, time evolution is given locally by the equations of motion $\dot{x} = \{x, \mathcal{H}\}$.

Symplectomorphisms

Definition Let $(\mathcal{M},\omega)$ and $(\mathcal{N},\Omega)$ be symplectic manifolds. A *symplectomorphism* (or *canonical transformation*) from $\mathcal{M}$ to $\mathcal{N}$ is a diffeomorphism $\phi : \mathcal{M} \to \mathcal{N}$ that preserves the symplectic structure in the sense that[6]

$$\phi^*\Omega = \omega. \tag{5.22}$$

Then $(\mathcal{M},\omega)$ and $(\mathcal{N},\Omega)$ are said to be *symplectomorphic*.

When $\phi : \mathcal{M} \to \mathcal{N}$ is a symplectomorphism, it preserves Poisson brackets in the sense that $\{\phi^*\mathcal{F}, \phi^*\mathcal{G}\} = \phi^*\{\mathcal{F},\mathcal{G}\}$ for all functions $\mathcal{F}, \mathcal{G}$ on $\mathcal{N}$, where the brackets on the left and on the right-hand side are those of $\mathcal{M}$ and $\mathcal{N}$, respectively. Note that the flow of any Hamiltonian vector field on $\mathcal{M}$ defines a one-parameter family of symplectomorphisms of $\mathcal{M}$.

5.1.5 Kirillov–Kostant Structures

We now describe phase spaces whose structure is entirely determined by group theory. They are prototypes for all phase spaces with symmetries.

Kirillov–Kostant Poisson Bracket

Definition Let G be a Lie group with Lie algebra $\mathfrak{g}$. Then the *Kirillov–Kostant Poisson bracket* on $\mathfrak{g}^*$ is defined as

$$\{\mathcal{F},\mathcal{G}\}(p) \equiv \langle p, [\mathcal{F}_{*p}, \mathcal{G}_{*p}]\rangle \tag{5.23}$$

where $\mathcal{F},\mathcal{G} \in C^\infty(\mathfrak{g}^*,\mathbb{R})$ and $\mathcal{F}_{*p}$ denotes the differential of $\mathcal{F}$ at $p \in \mathfrak{g}^*$.[7]

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[6]Recall that the *pullback* of a tensor field T of rank k on a manifold $\mathcal{N}$ by a map $\phi : \mathcal{M} \to \mathcal{N}$ is defined by $(\phi^*T)_p(v_1,\ldots,v_k) \equiv T_{\phi(p)}(\phi_{*p}v_1,\ldots,\phi_{*p}v_k)$ for any $p \in \mathcal{M}$ and all $v_1,\ldots,v_k \in T_p\mathcal{M}$.

[7]$\mathcal{F}_{*p}$ is a linear map from $T_p\mathfrak{g}^* \cong \mathfrak{g}^*$ to $T_{\mathcal{F}(p)}\mathbb{R} \cong \mathbb{R}$ and therefore belongs to $(\mathfrak{g}^*)^* \cong \mathfrak{g}$.

One can associate a Hamiltonian vector field (5.12) with any function $\mathcal{F}$ on a Poisson manifold. In the present case one has the following result:

Proposition Let $\mathcal{F}$ be a real function on $\mathfrak{g}^*$, $\xi_{\mathcal{F}}$ the corresponding Hamiltonian vector field. The associated evolution equation is the *Euler–Poisson equation* on $\mathfrak{g}^*$,

$$\dot{\gamma}(t) = (\xi_{\mathcal{F}})_{\gamma(t)} = \mathrm{ad}^*_{\mathcal{F}_{*\gamma(t)}}\big(\gamma(t)\big)\,. \tag{5.24}$$

Proof Let $\mathcal{G}$ be a function on $\mathfrak{g}^*$ and take $p \in \mathfrak{g}^*$. We are going to compute $\xi_{\mathcal{F}}(\mathcal{G})$ at p in two different ways. First, $(\xi_{\mathcal{F}})_p$ is a vector tangent to $\mathfrak{g}^*$ at p and may therefore be seen as an element of $\mathfrak{g}^*$ (since $\mathfrak{g}^*$ is a vector space). But $(\xi_{\mathcal{F}})_p(\mathcal{G})$ only depends on the differential of $\mathcal{G}$ at p, so we may write

$$(\xi_{\mathcal{F}})_p(\mathcal{G}) = \big\langle(\xi_{\mathcal{F}})_p, \mathcal{G}_{*p}\big\rangle. \tag{5.25}$$

Secondly, using (5.12) and the bracket (5.23), one has

$$(\xi_{\mathcal{F}})_p(\mathcal{G}) = -\{\mathcal{F}, \mathcal{G}\} = -\big\langle p, [\mathcal{F}_{*p}, \mathcal{G}_{*p}]\big\rangle = -\big\langle p, \mathrm{ad}_{\mathcal{F}_{*p}}(\mathcal{G}_{*p})\big\rangle \overset{(5.11)}{=} \big\langle \mathrm{ad}^*_{\mathcal{F}_{*p}}(p), \mathcal{G}_{*p}\big\rangle\,.$$

Comparing this with (5.25), Eq. (5.24) follows. ∎

Corollary The symplectic leaves of the Kirillov–Kostant bracket are the coadjoint orbits of G. In particular, all (finite-dimensional) coadjoint orbits have even dimension.

Proof By the above proposition, the Hamiltonian vector field $\xi_{\mathcal{F}}$ associated with a function $\mathcal{F}$ and evaluated at a point $p \in \mathfrak{g}^*$ is

$$(\xi_{\mathcal{F}})_p = \mathrm{ad}^*_{\mathcal{F}_{*p}}(p). \tag{5.26}$$

Now, given an adjoint vector $X \in \mathfrak{g}$, we can always find a real function $\mathcal{F}$ on $\mathfrak{g}^*$ such that $\mathcal{F}_{*p} = X$. Accordingly Eq. (5.26) implies that the integral curves of all possible Hamiltonian vector fields going through p span the coadjoint orbit of p. ∎

For future reference it is useful to rewrite the Kirillov–Kostant bracket in terms of a basis $\{t_a\}$ of $\mathfrak{g}$ with Lie brackets (5.2). Any adjoint vector can then be written as $X = X^a t_a$. If $\{(t^a)^* | a = 1, \ldots, n\}$ denotes the corresponding dual basis of $\mathfrak{g}^*$, so that $\langle (t^a)^*, t_b\rangle = \delta^a_b$, any coadjoint vector can be written as $p = p_a (t^a)^*$ with real components p_a. This defines global coordinates $\{p_a | a = 1, \ldots n\}$ on $\mathfrak{g}^*$, where each p_a is a real function on $\mathfrak{g}^*$ that associates with a coadjoint vector p its component p_a. To apply (5.23) we note that the differential $(p_a)_*$ of p_a acts on the basis vectors $\frac{\partial}{\partial p_c}$ according to

$$(p_a)_*\Big(\frac{\partial}{\partial p_c}\Big) = \frac{\partial p_a}{\partial p_c} = \delta^c_a\,. \tag{5.27}$$

But since $\mathfrak{g}^*$ is a vector space we can identify $T_p\mathfrak{g}^*$ with $\mathfrak{g}^*$ by declaring that $\partial/\partial p_c$ coincides with $(t^c)^*$, so in fact the differential satisfies $(p_a)_*((t^c)^*) = \delta^c_a$. With this identification the differential $(p_a)_*$ belongs to the dual of the dual, $(\mathfrak{g}^*)^* = \mathfrak{g}$, and may be seen as an adjoint vector. Property (5.27) says that this adjoint vector is precisely t_a. The Poisson bracket follows:

$$\{p_a, p_b\} = f_{ab}{}^c\, p_c\,. \tag{5.28}$$

In parts II and III we will see that the Poisson brackets of three-dimensional gravity coincide with Kirillov–Kostant brackets for suitable asymptotic symmetry groups.

Remark The Euler–Poisson equation (5.24) has numerous applications in physics, particularly when there exists an invertible and self-adjoint "inertia operator" $\mathcal{I} : \mathfrak{g} \to \mathfrak{g}^*$.[8] Indeed one can then consider a quadratic Hamiltonian function $\mathcal{F}(p) = \frac{1}{2}\langle p, \mathcal{I}^{-1}p\rangle$ and Eq. (5.24) becomes $\dot{\gamma}(t) = \text{ad}^*_{\mathcal{I}^{-1}\gamma(t)}\gamma(t)$. For $G = \text{SO}(3)$ this coincides with the equations of motion of a free rigid body; for the Virasoro group, it leads to the Korteweg-de Vries equation (see [2] for details).

Kirillov–Kostant Symplectic Form

Since the coadjoint orbits of G are symplectic leaves of the Kirillov–Kostant Poisson bracket, they have a symplectic structure given by (5.20):

Definition Let G be a Lie group, $p \in \mathfrak{g}^*$ a coadjoint vector with orbit $\mathcal{W}_p$. Then the *Kirillov–Kostant(-Souriau) symplectic form* at $q \in \mathcal{W}_p$ is given by

$$\omega_q(\text{ad}^*_X q, \text{ad}^*_Y q) = \langle q, [X, Y]\rangle \tag{5.29}$$

where $X, Y \in \mathfrak{g}$.

Here $\text{ad}^*_X q$ and $\text{ad}^*_Y q$ are "infinitesimal displacements" of q and represent generic tangent vectors of $\mathcal{W}_p$ at q. One can verify that (5.29) is closed and non-degenerate on $\mathcal{W}_p$, so it is indeed a symplectic form. In addition it is invariant under the coadjoint action of G in the sense that $(\text{Ad}^*_f)^*(\omega) = \omega$ for all $f \in G$. Thus each coadjoint orbit of G is a homogeneous space equipped with a G-invariant symplectic structure. In this sense it is a symmetric phase space. We will see below that, for instance, each coadjoint orbit of the Poincaré group coincides with the space of classical states of a relativistic particle with definite mass and (classical) spin.

5.1.6 Momentum Maps

Noether's theorem states that any classical system with a Lie group of symmetries possesses conserved quantities. Here we investigate this statement in the framework

[8]Here self-adjointness means that $\langle\mathcal{I}(X), Y\rangle = \langle\mathcal{I}(Y), X\rangle$ for any two adjoint vectors X, Y.

of symplectic geometry. Until the end of this section $\mathcal{M}$ is understood to be a manifold acted upon by some group G according to $q \mapsto f \cdot q$.

Group Actions and Infinitesimal Generators

Definition Let $G \times \mathcal{M} \to \mathcal{M} : (f, q) \mapsto f \cdot q$ be a smooth action of a Lie group G on a manifold $\mathcal{M}$. Then the *infinitesimal generator* of the action corresponding to $X \in \mathfrak{g}$ is the vector field ξ_X on $\mathcal{M}$ defined by

$$(\xi_X)_q \equiv \frac{d}{dt} \left(e^{tX} \cdot q \right)\Big|_{t=0} . \tag{5.30}$$

One can show (see e.g. [1]) that this definition implies

$$[\xi_X, \xi_Y] = -\xi_{[X,Y]} \tag{5.31}$$

where the Lie bracket on the left-hand side is that of vector fields, while the bracket on the right is that of $\mathfrak{g}$, given by (5.1).

For example, the representations (5.8) and (5.11) are infinitesimal generators of the adjoint and coadjoint representations of G, respectively.[9] In this language the tangent space at q of an orbit (3.13) consists of all vectors of the form $(\xi_X)_q$, where X spans the Lie algebra $\mathfrak{g}$. The flow of ξ_X is $\mathbb{R} \times \mathcal{M} \to \mathcal{M} : (t, q) \mapsto e^{tX} \cdot q$. In what follows we study group actions where the manifold $\mathcal{M}$ is symplectic.

Momentum Maps

Let $(\mathcal{M}, \omega)$ be a symplectic manifold. An action of G on $\mathcal{M}$ is *symplectic* if each map $q \mapsto f \cdot q$ is a symplectomorphism, in which case G is a symmetry group of $\mathcal{M}$. If ξ_X denotes the infinitesimal generator (5.30) of a symplectic action, then $\mathcal{L}_{\xi_X}\omega = 0$.

Definition Let $G \times \mathcal{M} \to \mathcal{M} : (f, q) \mapsto f \cdot q$ be a symplectic group action. A *momentum map* for this action is a smooth map

$$\mathcal{J} : \mathcal{M} \to \mathfrak{g}^* : p \mapsto \mathcal{J}(p) \tag{5.32}$$

such that, for any $X \in \mathfrak{g}$,

$$i_{\xi_X}\omega = d\big\langle \mathcal{J}(\cdot), X \big\rangle \tag{5.33}$$

where ξ_X is the infinitesimal generator (5.30) and $\langle \mathcal{J}(\cdot), X\rangle$ is the real function on $\mathcal{M}$ that associates with $q \in \mathcal{M}$ the value $\langle \mathcal{J}(q), X\rangle$. In the sequel we write $\langle \mathcal{J}(\cdot), X\rangle \equiv \mathcal{J}_X$, to which we also refer as a "momentum map".

The definition (5.33) can be compared to that of Hamiltonian vector fields, Eq. (5.19), and is equivalent to the statement

[9] Property (5.31) does not contradict the fact that the adjoint and coadjoint representations of $\mathfrak{g}$ are actual representations, i.e. for example that $\mathrm{ad}_X \mathrm{ad}_Y - \mathrm{ad}_Y \mathrm{ad}_X = \mathrm{ad}_{[X,Y]}$. Indeed, the vector fields in (5.31) are derivations acting on functions on $\mathcal{M}$, while ad_X and ad^*_X are generally understood as linear operators acting on $\mathfrak{g}$ and $\mathfrak{g}^*$, respectively.

$$\xi_X = \xi_{\mathcal{J}_X} = -\{\mathcal{J}_X, \cdot\}. \tag{5.34}$$

Here ξ_X is the infinitesimal generator (5.30), while $\xi_{\mathcal{J}_X}$ is the Hamiltonian vector field associated with the function $\mathcal{J}_X$ and $\{\cdot, \cdot\}$ is the Poisson bracket (5.20). Thus the function $\mathcal{J}_X$ generates the transformation corresponding to $X \in \mathfrak{g}$ in phase space, in the sense that for any function $\mathcal{F} \in C^\infty(\mathcal{M})$ we have $\{\mathcal{J}_X, \mathcal{F}\} = -\xi_X(\mathcal{F}) = -\delta_X \mathcal{F}$. From this observation we can derive an important corollary: if $X, Y \in \mathfrak{g}$ and if we consider the corresponding functions $\mathcal{J}_X$ and $\mathcal{J}_Y$, their Poisson bracket acts on classical observables according to

$$\begin{aligned}\big\{\{\mathcal{J}_X, \mathcal{J}_Y\}, \mathcal{F}\big\} &= \big\{\mathcal{J}_X, \{\mathcal{J}_Y, \mathcal{F}\}\big\} - \big\{\mathcal{J}_Y, \{\mathcal{J}_X, \mathcal{F}\}\big\} \\ &\overset{(5.34)}{=} [\xi_X, \xi_Y](\mathcal{F}) \overset{(5.31)}{=} -\xi_{[X,Y]}(\mathcal{F}) \overset{(5.34)}{=} \big\{\mathcal{J}_{[X,Y]}, \mathcal{F}\big\}\end{aligned} \tag{5.35}$$

where in the first equality we have used the Jacobi identity. Since this property is true for any function $\mathcal{F}$, it is tempting to remove it from both ends of the equation and conclude that the momentum map provides a representation of the Lie algebra $\mathfrak{g}$. However, this hasty argument overlooks one crucial possibility, namely the fact that brackets of momentum maps may include a *central term* that commutes with all functions on phase space (see [1] or appendix 5 of [6]). Thus we conclude that:

Proposition Provided phase space is connected, the Poisson algebra of momentum maps is a representation of the Lie algebra $\mathfrak{g}$ up to a (classical) central extension:

$$\{\mathcal{J}_X, \mathcal{J}_Y\} = \mathcal{J}_{[X,Y]} + \mathsf{c}(X, Y) \qquad \forall\, X, Y \in \mathfrak{g}\,, \tag{5.36}$$

for some real two-cocycle c on $\mathfrak{g}$. If the phase space has several connected components, there may be several different cocycles (one for each connected component).

This statement is equivalent to saying that momentum maps provide a projective representation of $\mathfrak{g}$, or alternatively an *exact* representation of a central extension of $\mathfrak{g}$. It is the classical analogue of the symmetry representation theorem of Sect. 2.1; it will be crucial once we use geometric quantization to produce unitary representations. In parts II and III we will see that the surface charges generating asymptotic symmetries in gravity provide a shining illustration of this phenomenon.

Remark Not all symplectic group actions have a momentum map, as there may be no function $\mathcal{J}_X$ such that (5.34) holds. If such a function exists for each $X \in \mathfrak{g}$, then the action does admit a momentum map and is said to be *Hamiltonian*. Note that, if $\mathcal{J}$ and $\mathcal{J}'$ are momentum maps for the same group action, then (5.33) implies that they differ by a constant coadjoint vector (provided $\mathcal{M}$ is connected).

Noether's Theorem

The momentum map gives the conserved quantity $\mathcal{J}(p) \in \mathfrak{g}^*$ associated with each classical state $p \in \mathcal{M}$. As anticipated earlier, coadjoint vectors may thus be seen as "conserved vectors" for symmetric phase spaces. This interpretation stems from the following fundamental result:

Noether's theorem Let $q \mapsto f \cdot q$ be a Hamiltonian action of G on $(\mathcal{M}, \omega)$ with momentum map $\mathcal{J}$. Let also $\mathcal{H} \in C^{\infty}(\mathcal{M})$ be a classical observable invariant under G in the sense that $\mathcal{H}(f \cdot q) = \mathcal{H}(q)$ for all $f \in G$ and all $q \in \mathcal{M}$, and let $\xi_{\mathcal{H}}$ be the associated Hamiltonian vector field (5.12). Then, for any integral curve $\gamma(t)$ of $\xi_{\mathcal{H}}$ with initial condition $\gamma(0)$, one has

$$\mathcal{J}\big(\gamma(t)\big) = \mathcal{J}\big(\gamma(0)\big) \tag{5.37}$$

for any time t belonging to the domain of the curve. In other words the $\dim \mathfrak{g}$ components of the coadjoint vector $\mathcal{J}(\gamma(t))$ are *conserved* when the equations of motion $\dot{\gamma} = (\xi_{\mathcal{H}})_{\gamma}$ are satisfied.

Proof The Hamiltonian $\mathcal{H}$ is invariant under G, so $\frac{d}{dt}\mathcal{H}(e^{tX} \cdot p) = 0$ for any $p \in \mathcal{M}$. Since any integral curve of ξ_X takes the form $e^{tX} \cdot p$ for some initial condition p, this is to say that $\xi_X(\mathcal{H}) = \xi_{\mathcal{J}_X}(\mathcal{H}) = 0$, so $\mathcal{H}$ is constant along integral curves of $\mathcal{J}_X$; equivalently, $\mathcal{J}_X$ is constant along integral curves of $\xi_{\mathcal{H}}$. ■

In a translation-invariant system the momentum map associates a momentum vector with any point in phase space. Similarly, in a rotation-invariant system it coincides with angular momentum. Finally, in a two-dimensional conformal field theory, it coincides with the stress tensor of a given field configuration. We will illustrate these statements below. In the remainder of this section we build momentum maps for specific families of symplectic manifolds.

Momentum Maps for Coadjoint Orbits

Let us build a momentum map (5.33) for a coadjoint orbit $\mathcal{W}_p$ of some group G, endowed with the Kirillov–Kostant symplectic form (5.29). First we note that any path $\gamma(t)$ in $\mathcal{W}_p$ can be written as $\gamma(t) = \mathrm{Ad}^*_{f(t)} p$ for some path $f(t)$ in G. If $\gamma(0) = q$ and $\dot{\gamma}(0) = \mathrm{ad}^*_Y q$ for some $Y \in \mathfrak{g}$, then we find

$$\omega_q\big(\mathrm{ad}^*_X q, \dot{\gamma}(0)\big) = \omega_q\big(\mathrm{ad}^*_X q, \mathrm{ad}^*_Y q\big) \overset{(5.29)}{=} \big\langle q, [X, Y]\big\rangle = \big\langle \mathrm{ad}^*_Y q, X\big\rangle$$

for any $X \in \mathfrak{g}$. Since $\mathrm{ad}^*_X q$ is the infinitesimal generator ξ_X of the coadjoint action of G on $\mathcal{W}_p$, the far left-hand side of this equation coincides with $(i_{\xi_X}\omega)_q\big(\dot{\gamma}(0)\big)$. As a consequence the momentum map (5.33) should be such that

$$\big\langle \mathrm{ad}^*_Y q, X\big\rangle = \frac{d}{dt}\Big(\big\langle \mathcal{J}(\gamma(t)), X\big\rangle\Big)\Big|_{t=0} = \big\langle \mathcal{J}_{*q}\, \mathrm{ad}^*_Y q, X\big\rangle$$

for all $X \in \mathfrak{g}$. This implies that the differential $\mathcal{J}_{*q} : T_q\mathcal{W}_p \to \mathfrak{g}^*$ is just the inclusion, and determines $\mathcal{J}$ up to a constant coadjoint vector. In particular:

Proposition The inclusion of the coadjoint orbit $\mathcal{W}_p$ in $\mathfrak{g}^*$,

$$\mathcal{J} : \mathcal{W}_p \hookrightarrow \mathfrak{g}^* : q \mapsto q, \tag{5.38}$$

is a momentum map for the coadjoint action of G on $(\mathcal{W}_p, \omega)$ when ω is the Kirillov–Kostant symplectic form (5.29).

This result implies that the action of a Lie group on its coadjoint orbits is *always* Hamiltonian. In fact one can show that any symplectic manifold endowed with a transitive Hamiltonian action of some group G is a covering space of a coadjoint orbit of G. In this sense coadjoint orbits are "universal" homogeneous phase spaces.

Note that the momentum map (5.38) automatically realizes $\mathfrak{g}$ symmetry without central extensions. Indeed, if X, Y belong to $\mathfrak{g}$ and if $\mathcal{J}_X, \mathcal{J}_Y$ are the corresponding momentum maps, then at a point $p \in \mathfrak{g}^*$ we find

$$\{\mathcal{J}_X, \mathcal{J}_Y\}(p) \overset{(5.23)}{=} \big\langle p, [(\mathcal{J}_X)_{*p}, (\mathcal{J}_Y)_{*p}]\big\rangle \overset{(5.38)}{=} \big\langle p, [X, Y]\big\rangle = \mathcal{J}_{[X,Y]}(p).$$

This is exactly the statement (5.36) with a vanishing cocycle c. However, one should keep in mind that the group G itself may be centrally extended.

Momentum Maps for Cotangent Bundles

Proposition Consider a symplectic action of G on $\mathcal{M}$. Let $\omega = -d\theta$ for some symplectic potential θ. If the group action leaves θ invariant, then

$$\mathcal{J}_X \equiv \langle \theta, \xi_X \rangle \tag{5.39}$$

defines a momentum map (5.32) that satisfies (5.36) with a vanishing cocycle $\mathsf{c} = 0$.

Proof Since the action leaves θ invariant, one has $\mathcal{L}_{\xi_X}\theta = 0$ for any $X \in \mathfrak{g}$. Writing the Lie derivative as $\mathcal{L}_\xi = d \circ i_\xi + i_\xi \circ d$ and using $\omega = -d\theta$, this is equivalent to $d\langle \theta, \xi_X \rangle = -i_{\xi_X} d\theta = i_{\xi_X}\omega$. One may recognize this as the definition (5.33) of a momentum map given by (5.39). In order to prove that (5.36) is satisfied with a vanishing cocycle c, we evaluate the Poisson bracket

$$\{\mathcal{J}_X, \mathcal{J}_Y\} = \frac{1}{2}\left[\xi_Y\langle\theta, \xi_X\rangle - \xi_X\langle\theta, \xi_Y\rangle\right] = \frac{1}{2}\omega(\xi_X, \xi_Y) - \frac{1}{2}\langle\theta, [\xi_X, \xi_Y]\rangle.$$

Here $\omega(\xi_X, \xi_Y) = \{\mathcal{J}_X, \mathcal{J}_Y\}$ by virtue of (5.20) and (5.34), while Eqs. (5.31) and (5.39) imply that $\langle\theta, [\xi_X, \xi_Y]\rangle = -\mathcal{J}_{[X,Y]}$. Equation (5.36) follows with $\mathsf{c} = 0$. ■

Let us now apply this result to the cotangent bundle $T^*\mathcal{Q}$ of a manifold $\mathcal{Q}$. Let $\phi : \mathcal{Q} \to \mathcal{Q}$ be a diffeomorphism. We define the associated *point transformation* as

$$\bar{\phi} : T^*\mathcal{Q} \to T^*\mathcal{Q} : (q, \alpha) \mapsto \left(\phi^{-1}(q), \alpha \circ \phi_{*\phi^{-1}(q)}\right). \tag{5.40}$$

As one can verify, this definition ensures that

$$\phi \circ \pi \circ \bar{\phi} = \pi \tag{5.41}$$

where $\pi : T^*\mathcal{Q} \to \mathcal{Q}$ is the canonical projection (5.16). Thanks to this property, one can show (see e.g. [1]) that point transformations are symplectomorphisms:

Proposition Consider $T^*\mathcal{Q}$ with the symplectic form $\omega = -d\theta$, where θ is the canonical one-form (5.17). Let $\phi : \mathcal{Q} \to \mathcal{Q}$ be a diffeomorphism and let $\bar{\phi}$ be the associated point transformation (5.40). Then

$$(\bar{\phi})^*\theta = \theta\,. \tag{5.42}$$

In particular, point transformations are symmetries of $T^*\mathcal{Q}$.

Suppose now that there is an action $q \mapsto f \cdot q$ of a Lie group G on the manifold $\mathcal{Q}$. For clarity we will also write $f \cdot q \equiv \sigma^*_f(q)$, where the notation is purposely the same as in Eq. (4.17). Then one can define an action of G on $\mathcal{T}^*\mathcal{Q}$ by

$$f \cdot (q, \alpha) \equiv \big(f \cdot q, \alpha \circ (\sigma^*_{f^{-1}})_{*f\cdot q}\big). \tag{5.43}$$

Proposition (5.42) ensures that this action is symplectic and even preserves the Liouville one-form. Accordingly we can apply (5.39) to build its momentum map:

Proposition A momentum map for (5.43) is provided by the prescription

$$\langle \mathcal{J}(q, \alpha), X\rangle \equiv \langle \alpha, (\xi_X)_q\rangle \tag{5.44}$$

where ξ_X is the infinitesimal generator of the action $q \mapsto f \cdot q = \sigma^*_f(q)$.

Proof Applying (5.39) to the case at hand, we find a momentum map $\mathcal{J}$ given by $\langle \mathcal{J}(q, \alpha), X\rangle = \langle \theta_{(q,\alpha)}, (\bar{\xi}_X)_{(q,\alpha)}\rangle$ where $\bar{\xi}_X$ denotes the infinitesimal generator of (5.43). Then (5.41) implies that $\pi_{*(q,\alpha)}(\bar{\xi}_X)_{(q,\alpha)} = (\xi_X)_q$, so (5.44) follows. ■

As an application of these results one can show for instance that the momentum map given by (5.44) for a translation-invariant system is just the standard momentum vector, while for a rotation-invariant system it yields the angular momentum. See [1].

5.2 Geometric Quantization

Given a symmetric phase space $(\mathcal{M}, \omega)$, one would like to understand how "quantizing" that system produces unitary representation of the symmetry group. This section is devoted to an overview of that problem, particularly as applied to coadjoint orbits. In short, the quantum Hilbert space associated with $(\mathcal{M}, \omega)$ will consist of "wavefunctions", or rather sections of suitable line bundle over $\mathcal{M}$, and will indeed provide unitary representations provided certain quantization conditions are satisfied. Our plan is to start by reviewing the technology of line bundles and their connections, before explaining how it applies to the quantization of symplectic manifolds and discussing the realization of unitary group representations by geometric quantization. The presentation is condensed and superficial; we refer to [3–5] for a much more detailed account of the subject.

5.2.1 Line Bundles and Wavefunctions

The basic idea of geometric quantization is to consider wavefunctions on a symplectic manifold $\mathcal{M}$. These wavefunctions are sections of a complex line bundle over $\mathcal{M}$. Recall that a *fibre bundle* is a quadruple $(\mathcal{L}, \mathcal{M}, \mathcal{F}, \pi)$ where $\pi : \mathcal{L} \to \mathcal{M}$ is a projection and $\mathcal{L}$ is locally diffeomorphic to the product $\mathcal{M} \times \mathcal{F}$, where $\mathcal{M}$ is known as the *base space* and $\mathcal{F}$ is known as the *fibre*. A *vector bundle* is a fibre bundle whose fibres are diffeomorphic to a vector space.

Definition A *complex line bundle* $\mathcal{L}$ over $\mathcal{M}$ is a vector bundle $\pi : \mathcal{L} \to \mathcal{M}$ whose fibres are isomorphic to $\mathbb{C}$.

Thus a complex line bundle consists of infinitely many copies of the complex plane $\mathbb{C}$, one at each point of $\mathcal{M}$, glued together in a smooth way (see Fig. 4.1 with $\mathcal{O}_p$ replaced by $\mathcal{M}$). The bundle locally looks like the direct product of $\mathcal{M}$ with $\mathbb{C}$. When this is true globally, i.e. when $\mathcal{L}$ is diffeomorphic to $\mathcal{M} \times \mathbb{C}$, the line bundle is said to be *trivial*.

Definition Let $\pi : \mathcal{L} \to \mathcal{M}$ be a complex line bundle. Then a *section* of $\mathcal{L}$ is a map $\Psi : \mathcal{M} \to \mathcal{L}$ such that $\pi \circ \Psi = \mathrm{Id}_{\mathcal{M}}$. The space of sections is denoted $\Gamma(\mathcal{M}, \mathcal{L})$.

When $\pi : \mathcal{L} \to \mathcal{M}$ is trivial, the space of sections coincides with the space of complex-valued functions on $\mathcal{M}$. For example, when $\mathcal{M} = \mathbb{R}^2$ is interpreted as the phase space of a particle on a line, complex functions $\Psi(x, p)$ on $\mathbb{R}^2$ would provide sections of the trivial line bundle $\mathbb{R}^2 \times \mathbb{C}$.

Remark If $\mathcal{M}$ is a symplectic manifold and $\mathcal{L}$ is a line bundle over $\mathcal{M}$, not all sections of $\mathcal{L}$ are eligible as quantum wavefunctions because they depend on too many arguments. In the example $\mathcal{M} = \mathbb{R}^2$ just given, sections $\Psi(x, p)$ depend both on positions and on momenta, which violates the uncertainty principle. This will be remedied much later by so-called *polarization* (see Sects. 5.2.2 and 5.2.3).

Connections and Curvature

In order to define quantum operators acting on wavefunctions seen as sections of a line bundle, we need a prescription for computing derivatives of sections. (For instance the momentum operator typically reads $P = -i\partial_x$.) This requires a notion of covariant differentiation:

Definition Let $\mathcal{L}$ be a complex line bundle over $\mathcal{M}$, $\mathrm{Vect}_{\mathbb{C}}(\mathcal{M})$ the space of complex vector fields on $\mathcal{M}$. A *connection* for $\mathcal{L}$ is a map

$$\nabla : \mathrm{Vect}_{\mathbb{C}}(\mathcal{M}) \times \Gamma(\mathcal{M}, \mathcal{L}) \to \Gamma(\mathcal{M}, \mathcal{L}) : (\xi, \Psi) \mapsto \nabla_\xi \Psi$$

which is linear on $\Gamma(\mathcal{M}, \mathcal{L})$ and $\mathrm{Vect}_{\mathbb{C}}(\mathcal{M})$, and satisfies the property $\nabla_{\mathcal{F}\xi}\Psi = \mathcal{F}\,\nabla_\xi\Psi$ as well as the Leibniz rule $\nabla_\xi(\mathcal{F}\,\Psi) = \xi(\mathcal{F})\,\Psi + \mathcal{F}\,\nabla_\xi\Psi$ for any $\mathcal{F} \in C^\infty(\mathcal{M}, \mathbb{C})$. The linear operator ∇_X is known as the *covariant derivative* along ξ.

A connection defines a notion of parallel transport and allows one to connect, or identify, fibres at different points. The extent to which these identifications deform the fibres as one moves around on the base manifold is measured by the *curvature*

$$R : \mathrm{Vect}_{\mathbb{C}}(\mathcal{M}) \times \mathrm{Vect}_{\mathbb{C}}(\mathcal{M}) \times \Gamma(\mathcal{M}, \mathcal{L}) \to \Gamma(\mathcal{M}, \mathcal{L})$$

which is a two-form defined for all $\xi, \zeta \in \mathrm{Vect}_{\mathbb{C}}(\mathcal{M})$ and any section Ψ by

$$R(\xi, \zeta)\Psi \equiv \left(\nabla_\xi \nabla_\zeta - \nabla_\zeta \nabla_\xi - \nabla_{[\xi,\zeta]}\right) \Psi. \tag{5.45}$$

When the curvature vanishes the connection is said to be *flat*. Any trivial vector bundle admits a flat connection, but the converse is not true: there exist non-trivial bundles with flat connections.

Hermitian Structures

Since we eventually wish to interpret sections as wavefunctions, we need to define their scalar products.

Definition A *Hermitian structure* on a line bundle $\mathcal{L} \to \mathcal{M}$ is a smooth map

$$\mathcal{M} \times \Gamma(\mathcal{M}, \mathcal{L}) \times \Gamma(\mathcal{M}, \mathcal{L}) \to \mathbb{C} : (q, \Phi, \Psi) \mapsto \big(\Phi(q)|\Psi(q)\big). \tag{5.46}$$

which is linear in Ψ and antilinear in Φ.[10] Provided $\mathcal{M}$ is endowed with a measure μ, the Hermitian structure can be used to define a space of square-integrable sections with scalar product (3.7).

Now let $\mathcal{L}$ be a complex line bundle over $\mathcal{M}$ endowed with a connection ∇ and a Hermitian structure (5.46). We say that ∇ is *Hermitian* if it is compatible with the Hermitian structure in the sense that

$$\xi \cdot (\Phi|\Psi) = (\nabla_\xi \Phi|\Psi) + (\Phi|\nabla_\xi \Psi) \tag{5.47}$$

where $(\Phi|\Psi)$ is the function $\mathcal{M} \to \mathbb{C}$ whose value at q is $\big(\Phi(q)\big|\Psi(q)\big)$. Condition (5.47) is the Hermitian analogue of the condition of metric-compatibility for connections on the tangent bundle. In the realm of quantum mechanics, property (5.47) will allow us to define self-adjoint operators.

5.2.2 *Quantization of Cotangent Bundles*

Having introduced the setup, we now return to our original problem of defining a quantum Hilbert space associated with a symplectic manifold $(\mathcal{M}, \omega)$. In order for this definition to qualify as a consistent quantization prescription, the Hilbert space

[10]The map being "smooth" means that, given any two smooth sections Φ, Ψ, the assignment $q \mapsto (\Phi(q)|\Psi(q))$ is smooth.

$\mathscr{H}$ must be endowed with an operator algebra that is somehow associated with the Poisson algebra of classical observables. This association must be a linear map that sends a function $\mathcal{F} \in C^\infty(\mathcal{M})$ on a linear operator $\hat{\mathcal{F}}$ in $\mathscr{H}$, in such a way that

$$\big[\hat{\mathcal{F}}, \hat{\mathcal{G}}\big] = i\hbar\,\widehat{\{\mathcal{F}, \mathcal{G}\}}\,. \tag{5.48}$$

Furthermore the constant function $\mathcal{F}(p) = 1$ must be mapped on the identity operator, i.e. $\hat{1} = \mathbb{I}$. Thus, the problem is to find a quantum/classical correspondence that fulfills these criteria.

The solution turns out to be given by so-called *geometric quantization* and consists of two steps: prequantization and polarization. Here we describe these steps for the simple case of cotangent bundles endowed with the symplectic form (5.18), so that $\omega = -d\theta$. More general symplectic manifolds are treated in Sect. 5.2.3.

Prequantization

As a first attempt at quantization, let us consider the space of complex wavefunctions on $\mathcal{M}$. Their scalar products are then given by (3.7) where one may choose μ to be the Liouville volume form (5.14). To define a linear correspondence between classical and quantum observables, one can try to use the Hamiltonian vector fields (5.12):

$$\mathcal{F} \overset{?}{\longmapsto} \hat{\mathcal{F}} \overset{?}{=} -i\hbar\,\xi_{\mathcal{F}}\,. \tag{5.49}$$

Here $\hbar$ is an arbitrary (positive) constant, to be identified with Planck's constant. Indeed, using (5.13) one verifies that (5.49) satisfies the basic consistency requirement (5.48), and is thus at first sight a satisfactory quantization prescription. However, the problem with (5.49) is that the trivial observable $\mathcal{F}(p) = 1$ is mapped on the zero operator $\hat{\mathcal{F}} = 0$ instead of the identity. This inconsistency can be remedied by improving (5.49) as

$$\mathcal{F} \overset{?}{\longmapsto} \hat{\mathcal{F}} \overset{?}{=} -i\hbar\,\xi_{\mathcal{F}} + \mathcal{F}\,, \tag{5.50}$$

where the second term on the right-hand side multiplies wavefunctions by $\mathcal{F}$. This modification ensures that $\mathcal{F} = 1$ is represented by the identity operator, but now relation (5.48) no longer holds.

We seem to be stuck: how are we to define $\hat{\mathcal{F}}$ in such a way that both condition (5.48) and the requirement $\hat{1} = \mathbb{I}$ be satisfied? The way out turns out to be the further improvement that consists in adding to (5.50) the momentum map (5.39):

$$\mathcal{F} \longmapsto \hat{\mathcal{F}} = -i\hbar\,\xi_{\mathcal{F}} - \langle\theta, \xi_{\mathcal{F}}\rangle + \mathcal{F}, \tag{5.51}$$

where θ is such that $\omega = -d\theta$. Indeed, using (5.13) one can verify that the commutators of operators (5.51) close according to (5.48), and furthermore the constant observable $\mathcal{F} = 1$ is represented, as it should, by the identity operator $\hat{\mathcal{F}} = \mathbb{I}$. Thus, provided θ exists, the prescription (5.51) is a consistent quantization of the algebra of classical observables on $\mathcal{M}$.

In the present case we are assuming that $\mathcal{M} = T^*\mathcal{Q}$ is a cotangent bundle, so θ is just the Liouville one-form (5.17) and (5.51) is a globally well-defined differential operator that quantizes the classical observable $\mathcal{F}$. One says that cotangent bundles are *quantizable*. For example, on $\mathbb{R}^{2n}$ with the symplectic form (5.15) the position and momentum operators given by prequantization are

$$\hat{q}^j = i\hbar\frac{\partial}{\partial p_j} + q^j, \qquad \hat{p}_j = -i\hbar\frac{\partial}{\partial q^j}\,.$$

Note that (5.51) may be seen as a differential operator

$$\hat{\mathcal{F}} = -i\hbar\nabla_{\xi_{\mathcal{F}}} + \mathcal{F} \tag{5.52}$$

where $\nabla_\xi = \xi - \frac{i}{\hbar}\langle\theta, \xi\rangle$ is a covariant derivative determined by the connection whose connection one-form is θ. From this viewpoint the symplectic potential is seen as an Abelian gauge field on $\mathcal{M} = T^*\mathcal{Q}$, and the corresponding field strength/curvature is the symplectic form $\omega = -d\theta$.

Polarization

Since the symplectic form is exact, the map (5.51) provides a globally well-defined quantization prescription and our job here is almost done. But there is still a problem: the would-be wavefunctions $\Psi : \mathcal{M} \to \mathbb{C}$ depend at this stage on *all* coordinates of $\mathcal{M} = T^*\mathcal{Q}$. For example, on $\mathbb{R}^{2n}$ we would have $\Psi = \Psi(q^i, p_j)$. In particular, in the current situation one could easily devise a wavefunction with arbitrarily accurate values of position and momentum, violating Heisenberg uncertainty. The purpose of *polarization* is to cure this pathology by cutting in half the number of coordinates on which wavefunctions are allowed to depend.

In the case of cotangent bundles it is common to declare that polarized wavefunctions only depend on the coordinates of $\mathcal{Q}$, and not on the transverse coordinates in each fibre $T_q^*\mathcal{Q}$. On $\mathbb{R}^{2n}$ this would correspond to saying that polarized wavefunctions $\Psi(q^i)$ do not depend on the coordinates p_j, which is generally interpreted by saying that wavefunctions are written "in position space" — although we shall see below that the analogue of this polarization for semi-direct products leads instead to the "momentum space" picture of Chap. 4. The scalar product of wavefunctions is obtained by endowing the manifold $\mathcal{Q}$ with a measure, resulting in a Hilbert space of polarized wavefunctions.

Polarization also affects quantum observables since they must preserve the polarization while still satisfying the commutation relations (5.48). As a result, the space of quantizable classical observables is a subset of the full space $C^\infty(\mathcal{M})$. For instance, in $\mathbb{R}^{2n}$ with Darboux coordinates q^i, p_j $(i, j = 1, \ldots, n)$, the classical observables whose quantization preserves the polarization $\partial_p\Psi = 0$ all take the form

$$\mathcal{F}(q, p) = p_j\mathcal{F}^j(q) + \mathcal{G}(q) \tag{5.53}$$

for some functions $\mathcal{F}^j$, $\mathcal{G}$. Observables which are not of this form do not preserve the polarization and are therefore not quantizable in this sense. One should keep in mind, however, that this does *not* mean that all quantum operators acting in the polarized Hilbert space are forced to take the form (5.53). Rather, quantizable classical observables give rise to a vector space of Hermitian quantum operators, and the full algebra of quantum observables is generated by sums and products of these operators. For example, the non-relativistic Hamiltonian $\hat{p}^2$ is obtained by squaring the operator that quantizes the classical observable p, although there exists no quantizable classical observable whose quantization would yield the operator $\hat{p}^2$. In this way one essentially recovers standard quantum mechanics from the quantization of the phase space $T^*\mathcal{Q}$.

5.2.3 Quantization of Arbitrary Symplectic Manifolds*

We now describe geometric quantization *without* assuming that the symplectic form is exact. As it turns out, relaxing that assumption leads to serious complications. Since these subtleties will have very few immediate effects on the remainder of our exposition, we urge the hasty reader to go directly to Sect. 5.2.4.

As before, the requirement that the commutators of quantum observables satisfy (5.48) leads to the quantization prescription (5.51), where the one-form θ is such that $\omega = -d\theta$. However, in contrast to cotangent bundles, there is in general no such one-form on $\mathcal{M}$ because ω need not be exact. Thus the best one can do is to treat (5.51) locally: if $\{U_i | i \in \mathcal{I}\}$ is a contractible open cover of $\mathcal{M}$, the Poincaré lemma ensures that there exist one-forms θ_i such that

$$\omega|_{U_i} = -d\theta_i \qquad \forall\, i \in \mathcal{I} \tag{5.54}$$

since ω is closed. Then, *locally* on each U_i, one can define operators

$$\hat{\mathcal{F}}\Big|_i \equiv -i\hbar\xi_{\mathcal{F}} - \langle\theta_i, \xi_{\mathcal{F}}\rangle + \mathcal{F} \tag{5.55}$$

that provide a linear correspondence between classical and quantum observables. The problem then is to glue together operators defined on different open sets. On any non-empty intersection $U_j \cap U_k$ one has $d\theta_j = d\theta_k$ so there exists a function $\mathcal{G}_{jk}$ on $U_j \cap U_k$ such that

$$\theta_j - \theta_k = d\mathcal{G}_{jk}\,. \tag{5.56}$$

Using (5.55) one can then show that the multiplicative operator

$$\ell_{kj} \equiv e^{i\mathcal{G}_{kj}/\hbar} \tag{5.57}$$

(acting on functions on $U_j \cap U_k$) is such that

$$\hat{\mathcal{F}}\Big|_k = \ell_{kj} \circ \hat{\mathcal{F}}\Big|_j \circ \ell_{kj}^{-1} \qquad \text{on } \; U_j \cap U_k \tag{5.58}$$

for any classical observable $\mathcal{F} \in C^\infty(\mathcal{M})$. This result indicates that the action of $\hat{\mathcal{F}}$ on functions depends on whether one defines it on U_j or on U_k. It is an ambiguity in the definition of the operator corresponding to $\mathcal{F}$, which threatens the consistency of the construction based on (5.55). The way out is think of $\hat{\mathcal{F}}$ as a differential operator acting not on functions, but on *sections* of a complex line bundle over $\mathcal{M}$. Indeed, if the line bundle is chosen properly, one may hope that its transition functions for some local trivialization associated with the open covering $\{U_i\}$ coincide with the multiplication maps (5.57), so that the local formula (5.55) provides globally well-defined differential operators acting on sections.

One is thus led to the problem of determining whether there exists a line bundle whose transition functions take the form (5.57) for the covering $\{U_i | i \in \mathcal{I}\}$, in such a way that the operator (5.55) can be written *globally* as (5.52) for a connection ∇ whose local connection one-forms are the θ_i's. This can be addressed in the framework of Čech cohomology, which we will not describe here. The bottom line is that such a line bundle with such a connection exists *if and only if* the cohomology class of $\omega/2\pi\hbar$ is integral in the cohomology space $\mathcal{H}^2_{\text{de Rham}}(\mathcal{M}, \mathbb{R})$, i.e. if

$$\left[\frac{\omega}{2\pi\hbar}\right] \in \mathcal{H}^2_{\text{de Rham}}(\mathcal{M}, \mathbb{Z}). \tag{5.59}$$

This *quantization condition* is equivalent to demanding that the integral of $\omega/2\pi\hbar$ over any closed two-surface be an integer.[11] The only quantizable symplectic manifolds are those that satisfy this requirement.

The reason why we did not see this condition in the case of cotangent bundles is that their symplectic form is *globally* exact, so that its cohomology class vanishes and the requirement (5.59) is trivially satisfied. In fact one can show that the curvature two-form (5.45) of the connection determined by (5.55) is $R = i\omega/\hbar$, consistently with the fact that the curvature of any line bundle is integral. In particular the connection used to define quantum operators (5.52) for cotangent bundles is flat.

Provided the quantization condition (5.59) is satisfied, one can endow the space of sections of the line bundle with a Hermitian structure and use it to define the scalar product (3.7) thanks to the Liouville volume form (5.14). One can show that the Hermitian structure can always be chosen in a way (5.47) compatible with the connection determined by ω, so that all operators (5.55) are Hermitian.This completes the first step of geometric quantization, i.e. prequantization.

As in the case of cotangent bundles, the Hilbert space of sections produced by prequantization is "too large" in the sense that wavefunctions depend on too many arguments. Polarization corrects this problem by "cutting in half" the number of coordinates on which wavefunctions are allowed to depend. Since this procedure

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[11] Here "closed" means "compact without boundary".

will not be directly visible in our later considerations, we skip its presentation and refer instead to [3, 5] for a much more thorough discussion.

Remark When $\mathcal{M}$ is a coadjoint orbit, there exists a simple reformulation of the integrality condition (5.59). Namely, if $\mathcal{W}_p \cong G/G_p$ is the coadjoint orbit of $p \in \mathfrak{g}^*$ with stabilizer G_p, the Kirillov–Kostant symplectic form (5.29) is integral if and only if there exists a character χ of G_p whose differential (at the identity $e \in G$) satisfies $d\chi_e = \frac{i}{\hbar} j\big|_{\mathfrak{g}_p}$, with $\mathfrak{g}_p$ the Lie algebra of G_p. The textbook example of this phenomenon is provided by coadjoint orbits of SU(2), which are spheres embedded in $\mathfrak{su}(2)^*$: the quantization condition requires that the radius of such a sphere be an integer or half-integer multiple of $\hbar$, corresponding to the statement that highest-weight representations of $\mathfrak{su}(2)$ have integer or half-integer "spin".

5.2.4 Symmetries and Representations

We now combine the results of Sect. 5.1 with the tools of geometric quantization to address the following question: given a symplectic manifold $(\mathcal{M}, \omega)$ acted upon by a group G, does quantization produce a unitary representation of G?

We will assume that the action of G is Hamiltonian, with a momentum map (5.32). We also assume that we have chosen a certain value for Planck's constant $\hbar$ and that $\omega/2\pi\hbar$ is integral in the sense (5.59). Then $(\mathcal{M}, \omega)$ is quantizable and prequantization can be carried out independently of the group action. In particular, for each adjoint vector $X \in \mathfrak{g}$ there is a classical observable $\mathcal{J}_X$ given by (5.33), and the corresponding operator (5.52) is

$$\hat{\mathcal{J}}_X = -i\hbar\nabla_{\xi_X} + \mathcal{J}_X \tag{5.60}$$

where we have used property (5.34) to replace $\xi_{\mathcal{J}_X}$ by the infinitesimal generator (5.30). By virtue of (5.36) the map $X \mapsto \hat{\mathcal{J}}_X$ is a homomorphism, possibly up to a central extension. Thus the assignment (5.60) provides a (generally projective) representation of the Lie algebra $\mathfrak{g}$, acting on a space of sections on $\mathcal{M}$.

The subtlety arises with polarization, since then the wavefunctions of the system satisfy extra conditions which may not be preserved by (5.60). To avoid such pathologies one has to choose a *G-invariant* polarization. In that case each operator (5.60) is a well-defined Hermitian operator acting on polarized wavefunctions, and one obtains a projective, unitary representation of $\mathfrak{g}$. It was shown by Kostant [7] that, when the action of G on $\mathcal{M}$ is transitive, the homomorphism $X \mapsto \hat{\mathcal{J}}_X$ exponentiates to a unitary representation of the group G. This is true in particular when $\mathcal{M}$ is a coadjoint orbit [4]. In addition, when G is semi-simple, compact or solvable, the representations obtained in this way are irreducible. Thus geometric quantization does produce unitary representations of groups, which is the conclusion we were hoping to obtain.

Remark One can discuss semi-classical approximations in symplectic terms, and this applies in particular to coadjoint orbits. Indeed, the Liouville volume form (5.14) measures the "size" of portions of phase space and can be used to compare identical manifolds endowed with different symplectic structures. If ω and $\lambda\omega$ (with $\lambda > 0$) are two symplectic forms on $\mathcal{M}$, then large λ assigns a larger measure to a given portion of $(\mathcal{M}, \lambda\omega)$ than to the same portion in $(\mathcal{M}, \omega)$. In this sense large λ is a semi-classical regime with respect to $(\mathcal{M}, \omega)$, with $1/\lambda$ playing the role of the coupling constant. In the case of coadjoint orbits, by linearity, $\mathcal{W}_p$ is diffeomorphic to $\mathcal{W}_{\lambda p}$ for any $\lambda \neq 0$, but the definition (5.29) ensures that the symplectic form on $\mathcal{W}_{\lambda p}$ is "larger" (for $\lambda > 1$ say) than that on $\mathcal{W}_p$. Thus, for λ large enough the quantization of $\mathcal{W}_{\lambda p}$ can be treated semi-classically. Note that this intuition breaks down if the orbit is invariant under scalings, i.e. $\mathcal{W}_{\lambda p} = \mathcal{W}_p$.

5.3 World Lines on Coadjoint Orbits

In this section we reformulate the observations of the previous pages in terms of action principles and path integrals. In doing so we will develop a group-theoretic world line formalism, which will eventually allow us (in Sect. 5.4) to interpret representations of semi-direct products as actual quantized point particles propagating in space-time.

We will start with general geometric considerations explaining how to associate an action principle with any quantizable symplectic manifold. After a group-theoretic interlude on the Maurer–Cartan form, we will focus on coadjoint orbits and describe their world line actions as gauged non-linear Sigma models. Useful references are [8–10]; see also [11].

5.3.1 World Lines and Quantization Conditions

Our approach here is similar to [12]. Let $(\mathcal{M}, \omega)$ be a symplectic manifold, $p \in \mathcal{M}$. Since ω is closed, there exists a neighbourhood U of p such that $\omega|_U = -d\theta$ for some one-form θ on U. Now let $\gamma : [0, 1] \to U : t \mapsto \gamma(t)$ be a path in phase space contained in U. We can associate with it an action

$$S\big|_U[\gamma] \equiv \int_\gamma \theta = -\int_\gamma d^{-1}\omega\,, \tag{5.61}$$

where the notation $-d^{-1}\omega$ means "whatever one-form θ such that $\omega = -d\theta$". This is a purely kinematical Hamiltonian action associated with the symplectic form ω. For example, when $\mathcal{M} = \mathbb{R}^{2n}$ with $\omega = dq^i \wedge dp_i = -d(p_i dq^i)$, expression (5.61) is globally well-defined and reads

$$S[q^i(t), p_j(t)] = \int_0^1 dt\, p_j(t)\dot{q}^j(t) \tag{5.62}$$

which is the standard reparameterization-invariant kinetic term of any Hamiltonian action. There is no term involving p^2 or any other combination of q's and p's because there is no Hamiltonian at this stage.

For a generic symplectic form ω the definition (5.61) is not enough: one needs an action principle that makes sense for *any* path in $\mathcal{M}$, regardless of exactness. So let $\{U_i | i \in \mathcal{I}\}$ be a contractible open covering of $\mathcal{M}$ such that $\omega|_{U_i} = -d\theta_i$ for each $i \in \mathcal{I}$. We can then write an action (5.61) on each U_i, but we can also attempt to define $S[\gamma]$ for *any* path γ by

$$S[\gamma] \equiv -\int_\gamma d^{-1}\omega\,. \tag{5.63}$$

We refer to this functional as the *geometric action* for $(\mathcal{M}, \omega)$ evaluated on the path γ; its definition follows from the geometry of $\mathcal{M}$. In particular, when a group G acts on $\mathcal{M}$ by symplectomorphisms, the action automatically has global G symmetry. In Sect. 5.5 we will interpret (5.63) as the action of a point particle in space-time.

The action (5.63) can be evaluated as follows. Given a path γ, we can cover its image by open sets U_j, with $j \in \mathcal{J} \subset \mathcal{I}$. If only one U_j suffices we can simply use the original definition (5.61) to evaluate the action. If there are two open sets, say U_1 and U_2, then we call γ_j the portion of the path γ contained in U_j (for $j = 1, 2$) and γ_{12} the portion contained in $U_1 \cap U_2$. We can then define

$$S[\gamma] \equiv \int_{\gamma_1} \theta_1 + \int_{\gamma_2} \theta_2 - \int_{\gamma_{12}} \theta_1 \tag{5.64}$$

where the last term removes the overcounting due to a double integration on $U_1 \cap U_2$. There is a subtlety in this expression: we chose to write $\omega|_{U_1 \cap U_2} = -d\theta_1$ in the last term, but we could equally well have chosen $\omega = -d\theta_2$; this would have given a different compensating term in (5.64), hence a different value for the action! This is a problem at first sight, but one may recall that the action as such need not be a single-valued functional on the space of paths in phase space. The truly important quantity is the complex number

$$e^{iS[\gamma]/\hbar} \tag{5.65}$$

which determines the path integral measure and leads to transition amplitudes in the quantum theory. Thus we are free to have a multivalued action as long as all ambiguities are integer multiples of $2\pi\hbar$. This is in effect a *quantization condition* on the parameters of the action.

A simple reformulation of this condition is obtained by considering a closed path γ (so $\gamma(0) = \gamma(1)$) and evaluating the action along that path. Using Stokes' theorem one can write

$$S[\gamma] = -\oint_{\gamma} d^{-1}\omega = -\int_{\Sigma_\gamma} \omega \tag{5.66}$$

where Σ_γ is a two-surface with boundary γ. As expected this is a multivalued functional of γ. Requiring the exponential $e^{iS[\gamma]}$ to be single-valued then implies that the integral of ω over any closed two-surface must be an integer multiple of $2\pi\hbar$, which is the old Bohr–Sommerfeld quantization condition and coincides with the integrality requirement (5.59) mentioned above. One can also show, more generally, that this condition is sufficient to ensure that (5.65) is single-valued on the space of paths. Thus the quantization condition determined by the action functional (5.63) coincides with the condition that follows from geometric quantization. This applies, in particular, to the coadjoint orbits of any Lie group.

Given an action (5.63) that satisfies the quantization condition, one can choose a Hamiltonian $\mathcal{H} \in C^\infty(\mathcal{M})$ and compute transition amplitudes using path integrals with the action functional

$$S[\gamma] = -\int_{\gamma} d^{-1}\omega - \int_0^T dt\, \mathcal{H}(\gamma(t)). \tag{5.67}$$

Note that this expression is no longer invariant under time reparameterizations for generic choices of the Hamiltonian function.

Remark The geometric actions (5.63) associated with coadjoint orbits of centrally extended loop groups describe certain families of Wess–Zumino–Witten models [10]. In that context the single-valuedness of (5.65) leads to the quantization of the Kac–Moody level [13–15].

5.3.2 *Interlude: The Maurer–Cartan Form*

When the phase space $\mathcal{M}$ is a coadjoint orbit $\mathcal{W}_p$ of a group G, any path γ on $\mathcal{W}_p$ can be written as

$$\gamma(t) = \mathrm{Ad}^*_{f(t)} p \tag{5.68}$$

for some path $f(t)$ in G. Geometric actions such as (5.63) can then be seen as functionals of paths on a group manifold. This reformulation turns out to rely on the Maurer–Cartan form of G, which we now study.

Definition Let G be a Lie group and let $L_f : G \to G : g \mapsto f \cdot g$ denote left multiplication by $f \in G$. Then the (left) *Maurer–Cartan form* on G is

$$\Theta_f \equiv (L_{f^{-1}})_{*f}\,. \tag{5.69}$$

At any point f, the map Θ_f is the differential of left multiplication by f^{-1}.

It follows from (5.69) that the Maurer–Cartan form at f is an isomorphism between the tangent spaces $T_f G$ and $T_e G$, the latter being identified as usual with the Lie algebra of G. Thus Θ is a $\mathfrak{g}$-valued one-form on G and may be seen as a section of the vector bundle $T^*G \otimes \mathfrak{g}$. It is also left-invariant in the sense that

$$L_g^*(\Theta) = \Theta \tag{5.70}$$

for any group element g. When G is a matrix group, any f can be written as a matrix and the entries of f define local coordinates on G. One can then think of df as the matrix whose entries are the differentials of these coordinates, and the left Maurer–Cartan form can be written as

$$\Theta_f = f^{-1} \cdot df\,. \tag{5.71}$$

One can similarly define a right Maurer–Cartan form $(R_{f^{-1}})_{*f}$, where R denotes right multiplication (3.17). Its expression for a matrix group is $df \cdot f^{-1}$.

Proposition The Maurer–Cartan form (5.69) satisfies the *Maurer–Cartan equation*

$$(d\Theta)(\xi, \zeta) + [\Theta(\xi), \Theta(\zeta)] = 0 \tag{5.72}$$

for all vector fields ξ, ζ on G, where $[\cdot, \cdot]$ denotes the Lie bracket (5.1) in $\mathfrak{g}$.

Proof Recall that the exterior derivative of Θ is such that, for all vector fields ξ, ζ,

$$(d\Theta)(\xi, \zeta) \equiv \xi \cdot \Theta(\zeta) - \zeta \cdot \Theta(\xi) - \Theta([\xi, \zeta])\,, \tag{5.73}$$

where $[\cdot, \cdot]$ is the Lie bracket of vector fields. If ξ and ζ are left-invariant, they can be written as $\xi_g = (L_g)_{*e} X$ and $\zeta_g = (L_g)_{*e} Y$ for some adjoint vectors X, Y. Then (5.69) implies that $\Theta(\xi) = X$ is constant on $\mathfrak{g}$, and (5.73) reduces to

$$(d\Theta)(\xi, \zeta) + \Theta([\xi, \zeta]) = 0. \tag{5.74}$$

By left-invariance we may write $\Theta([\xi, \zeta]) = [\Theta(\xi), \Theta(\zeta)]$ where the bracket on the right-hand side now is the Lie bracket (5.1) of $\mathfrak{g}$. Equation (5.74) then takes the form (5.72) save for the fact that ξ and ζ are left-invariant. This condition can be relaxed upon recalling that the span of left-invariant vector fields at a point $g \in G$ is the whole tangent space $T_g G$. ∎

Kirillov–Kostant from Maurer–Cartan

Thanks to (5.68), the Maurer–Cartan form provides a convenient rewriting of the Kirillov–Kostant symplectic form (5.29) in terms of vectors tangent to a group manifold. Indeed, let

$$\pi : G \to \mathcal{W}_p : g \mapsto \mathrm{Ad}_g^*(p) \tag{5.75}$$

be the natural projection. We then define a two-form $\boldsymbol{\omega}$ on G by

$$\boldsymbol{\omega} \equiv \pi^* \omega \tag{5.76}$$

where ω is the Kirillov–Kostant symplectic form (5.29). One may think of $\boldsymbol{\omega}$ as the analogue of (5.29) on the group G.

Lemma Let $g \in G$, and consider tangent vectors $v, w \in T_g G$. Then

$$\boldsymbol{\omega}_g(v, w) = \big\langle p, [\Theta_g(v), \Theta_g(w)]\big\rangle \tag{5.77}$$

where the bracket on the right-hand side is that of $\mathfrak{g}$.

Proof The definition of the two-form (5.76) explicitly reads

$$(\pi^*\omega)_g(v, w) = \omega_{\pi(g)}(\pi_{*g}v, \pi_{*g}w) \overset{!}{=} \boldsymbol{\omega}_g(v, w). \tag{5.78}$$

We can represent the vector v by a path γ in G such that $\dot{\gamma}(0) = v$, so that

$$\pi_{*g}(v) = \frac{d}{dt}\big(\pi(\gamma(t))\big)\Big|_{t=0} = \frac{d}{dt}\left(\text{Ad}^*_{\gamma(t)}(p)\right)\Big|_{t=0}. \tag{5.79}$$

In turn we can write $\gamma = g \cdot \gamma_0(t)$ where $\gamma_0(0) = e$ is the identity. Then $\dot{\gamma}_0(0)$ belongs to the Lie algebra $T_e G = \mathfrak{g}$ of G and is given by

$$\dot{\gamma}_0(0) = \frac{d}{dt}\left(g^{-1} \cdot \gamma(t)\right)\Big|_{t=0} = \frac{d}{dt}\Big(L_{g^{-1}}\big(\gamma(t)\big)\Big)\Big|_{t=0} = (L_{g^{-1}})_{*g}(v) \overset{(5.69)}{=} \Theta_g(v) \tag{5.80}$$

where we used $v = \dot{\gamma}(0)$. We can now use this in (5.79) to obtain

$$\pi_{*g}(v) = \frac{d}{dt}\left(\text{Ad}^*_g(\text{Ad}^*_{\gamma_0(t)}(p))\right)\Big|_{t=0} \overset{(5.11)}{=} \text{Ad}^*_g(\text{ad}^*_{\dot{\gamma}_0(0)}p) \overset{(5.80)}{=} \text{Ad}^*_g(\text{ad}^*_{\Theta(v)}p).$$

Equation (5.77) follows upon plugging this result (and its analogue for w) in (5.78). ■

Formula (5.77) is sometimes rewritten as

$$\boldsymbol{\omega} = \frac{1}{2}\big\langle p, [\Theta \wedge, \Theta]\big\rangle \tag{5.81}$$

where $[\Theta \wedge, \Theta]$ is the $\mathfrak{g}$-valued two-form such that $[\Theta \wedge, \Theta]_g(v, w) \equiv 2[\Theta_g(v), \Theta_g(w)]$ for all tangent vectors $v, w \in T_g G$. In what follows we call $\boldsymbol{\omega}$ the *symplectic form on G* since it is related by (5.76) to the Kirillov–Kostant symplectic form (in particular $d\boldsymbol{\omega} = 0$), but one should keep in mind that this terminology is actually incorrect:

Lemma The two-form $\boldsymbol{\omega}$ is degenerate. Its kernel consists of left-invariant vector fields ζ_X for which X belongs to the Lie algebra of the stabilizer of p.

Proof Let $v, w \in T_g G$; there are two (unique) adjoint vectors $X, Y \in T_e G = \mathfrak{g}$ such that $v = (L_g)_{*e} X$ and $w = (L_g)_{*e} Y$, so that $\Theta_g(v) = X$ and similarly for w. Formula (5.77) can then be rewritten as

$$\boldsymbol{\omega}_g(v, w) = \big\langle p, [X, Y]\big\rangle = -\langle \text{ad}^*_X p, Y\rangle. \tag{5.82}$$

The kernel of $\boldsymbol{\omega}$ consists of vectors $v = (L_g)_{*e} X$ such that (5.82) vanishes for any $Y \in \mathfrak{g}$, which is to say that $\text{ad}^*_X p = 0$. The latter property holds if and only if X belongs to the Lie algebra of the stabilizer of p. ■

This lemma confirms that $\boldsymbol{\omega}$ is not a symplectic form because its components do not form an invertible matrix. The rank of $\boldsymbol{\omega}$ is $\dim(G) - \dim(G_p)$, where G_p is the stabilizer of p. This number coincides (as it should) with the dimension of the coadjoint orbit of p, which proves by the way that the original form (5.29) on $G/G_p \cong \mathcal{W}_p$ is invertible.

5.3.3 *Coadjoint Orbits and Sigma Models*

The "symplectic form" (5.81) is a closed two-form, and is therefore locally exact. As such it can be used to define a kinetic action functional analogous to (5.63),

$$S[f(t)] \equiv -\int_{f(t)} d^{-1}\boldsymbol{\omega}, \tag{5.83}$$

whose argument is a path $f(t)$ in G. Using the Maurer–Cartan equation (5.72) in (5.77), one can write

$$\boldsymbol{\omega}_f = -\big\langle p, d\Theta_f\big\rangle = -d\big(\langle p, \Theta\rangle\big)_f$$

where the exterior derivative goes through the coadjoint vector p by linearity. Thus the action (5.83) becomes

$$S[f(t)] = \int_{f(t)} \big\langle p, \Theta\big\rangle = \int_0^T dt\, \big\langle p, \Theta_{f(t)}\big(\dot{f}(t)\big)\big\rangle. \tag{5.84}$$

It describes the dynamics of paths $f(t) \in G$ and may be seen as the (kinetic piece of the) action of a non-linear Sigma model. When G is a simple matrix group, adjoint and coadjoint vectors can be identified so that $\langle p, \cdot\rangle = \text{Tr}[X\cdot]$ for some $X \in \mathfrak{g}$, and (5.71) allows us to recast the integrand of (5.84) in the form $\text{Tr}\big[X f^{-1}\dot{f}\,\big]$.

Note that the global G symmetry of (5.84) is manifest: if $f(t)$ is a path in G and $g \in G$ is an arbitrary constant group element, then left-invariance of Θ readily implies $S[g \cdot f(t)] = S[f(t)]$. In addition (5.84) is the integral of a one-form and is thus invariant under redefinitions of the time parameter t. As in (5.67) one can include a Hamiltonian in the action, at the cost of breaking time reparameterization invariance.

A key subtlety with (5.84) is that the group variable $f(t)$ is the group element that appears in a coadjoint action $\text{Ad}^*_{f(t)}p$, as in (5.68). The latter coadjoint vector is invariant under multiplication of $f(t)$ from the right by any (generally time-dependent) group element $h(t)$ belonging to the stabilizer of p. This means that (5.84) should be invariant under gauge transformations $f(t) \mapsto f(t) \cdot h(t)$, and therefore describes a *gauged* non-linear Sigma model. Let us check that (5.84) does indeed admit such a symmetry. Using the Leibniz rule we find

$$S[f \cdot h] = \int_0^T \big\langle p, \Theta_{f(t)h(t)}\big((R_h)_{*f(t)}\dot{f}(t)\big)\big\rangle + \int_0^T \big\langle p, \Theta_{f(t)h(t)}\big((L_{f(t)})_{*h(t)}\dot{h}(t)\big)\big\rangle \tag{5.85}$$

where the adjoint vector paired with p in the first term can be rewritten as

$$\Theta_{f(t)h(t)}\big((R_h)_{*f(t)}\dot{f}(t)\big) = \text{Ad}_{h^{-1}}\Theta_{f(t)}\dot{f}(t)$$

thanks to the definitions (5.69) and (5.6). This implies that the first term of (5.85) coincides with the original action (5.84). As for the second term in (5.85), we use left-invariance of Θ to rewrite it as a Sigma model action evaluated on a path wholly contained in the stabilizer G_p:

$$S[h(t)] = \int_0^T dt\big\langle p, \Theta_{h(t)}\dot{h}(t)\big\rangle. \tag{5.86}$$

The counterpart of $h(t)$ in the coadjoint orbit of p is the constant path $\text{Ad}^*_{h(t)}p = p$, but in the Sigma model it carries a generally non-vanishing action (5.86). Thus gauge-invariance of (5.84) may be true, but is not obvious at this stage since the gauge-transformed action (5.85) differs from (5.84) by the extra term (5.86). To reconcile this observation with the much desired gauge-invariance of (5.84), we note that the exterior derivative of the integrand of (5.86) vanishes. Indeed, for all $v, w \in T_hG_p$ the Maurer–Cartan equation (5.72) yields

$$d\big\langle p, \Theta\big\rangle_h(v, w) = -\big\langle p, [\Theta_h(v), \Theta_h(w)]\big\rangle \overset{(5.11)}{=} \big\langle \text{ad}^*_{\Theta_h(v)}p, \Theta_h(w)\big\rangle = 0$$

where the last equality follows from the fact that $\Theta_h(v)$ belongs to the Lie algebra of the stabilizer of p. Thus the integrand of (5.86) is closed, and is therefore locally exact. In particular, for a path $h(t)$ located in a sufficiently small neighbourhood of the identity in H, there exists a function $\mathcal{F}(t)$ such that

$$\big\langle p, \Theta_{h(t)}\dot{h}(t)\big\rangle = \dot{\mathcal{F}}(t) \tag{5.87}$$

for any $t \in [0, T]$. The integral (5.86) of this quantity is a boundary term, so the action functional (5.84) is indeed gauge-invariant, albeit up to boundary terms that can be cancelled by requiring for instance that initial and final configurations be fixed.

We stress that this gauge symmetry is unavoidable if (5.84) is interpreted as the Sigma model version of the action (5.63) on a coadjoint orbit. In particular the inclusion of a Hamiltonian is now subject to a constraint: in order to reproduce (5.67), the Hamiltonian expressed in terms of group variables must be invariant under stabilizer gauge transformations.

*5.3.4 Coadjoint Orbits and Characters of SL(2, $\mathbb{R}$)**

As an application of the above considerations, we now classify the coadjoint orbits of SL(2, $\mathbb{R}$) and quantize some of them, showing along the way that they are equivalent to one-dimensional harmonic oscillators. As an application we evaluate SL(2, $\mathbb{R}$) characters by geometric quantization. We refer e.g. to [16, 17] for more details on the coadjoint orbits of SL(2, $\mathbb{R}$), and to [8, 10] for similar computations in more general cases. This section is not crucial for the remainder of the thesis and may be skipped in a first reading.

Coadjoint Orbits of SL(2, $\mathbb{R}$)

For the basic properties of the group SL(2, $\mathbb{R}$) we refer to Sect. 4.3. Its Lie algebra $\mathfrak{sl}(2, \mathbb{R})$ consists of real, traceless 2×2 matrices. Any such matrix is a real linear combination $X = x^\mu t_\mu$ of basis elements (4.87) whose brackets read

$$[t_\mu, t_\nu] = \epsilon_{\mu\nu}{}^\rho\, t_\rho\,. \tag{5.88}$$

Here $\epsilon_{\mu\nu\rho}$ is the completely antisymmetric tensor such that $\epsilon_{012} = +1$, and indices are raised and lowered using the Minkowski metric $\eta_{\mu\nu} = \text{diag}(-\ +\ +)$. For future reference we also note that, in the complex basis

$$\ell_0 \equiv -t_0\,, \qquad \ell_1 \equiv t_2 - i t_1\,, \qquad \ell_{-1} \equiv t_2 + i t_1\,, \tag{5.89}$$

the Lie brackets (5.88) take the form

$$i[\ell_m, \ell_n] = (m-n)\ell_{m+n} \tag{5.90}$$

for $m, n = -1, 0, 1$. On account of the isomorphism (4.83) this can also be seen as the Lorentz algebra in three dimensions.

The $\mathfrak{sl}(2, \mathbb{R})$ algebra has a non-degenerate bilinear form

$$(X, Y) \equiv 2\,\text{Tr}(XY) = \eta_{\mu\nu} x^\mu x^\nu \tag{5.91}$$

which is left invariant by the adjoint action (4.89) of $\mathrm{SL}(2,\mathbb{R})$. We can then use the isomorphism (4.42) to intertwine the adjoint and coadjoint representations of $\mathrm{SL}(2,\mathbb{R})$ as in (4.43). In particular we may identify coadjoint with adjoint vectors, and coadjoint orbits coincide with adjoint orbits under that identification. Those are exactly the momentum orbits of the Poincaré group in three dimensions, which were described in Sects. 4.2 and 4.3. This provides the classification of coadjoint orbits of $\mathrm{SL}(2,\mathbb{R})$ and an exhaustive family of orbit representatives is depicted schematically in Fig. 4.3.

Note that the fact that coadjoint orbits of $\mathrm{SL}(2,\mathbb{R})$ coincide with Poincaré momentum orbits in three dimensions follows from the structure $G \ltimes_{\mathrm{Ad}} \mathfrak{g}_{\mathrm{Ab}}$ of the double cover (4.93) of the Poincaré group. We will encounter a similar structure in the BMS_3 group, albeit with an infinite-dimensional group G.

Kirillov–Kostant Symplectic Form

We can write any coadjoint vector of $\mathrm{SL}(2,\mathbb{R})$ as $q = q_\mu (t^\mu)^*$ where $(t^\mu)^* = \eta^{\mu\nu}(t_\nu, \cdot)$ is the dual basis corresponding to (4.87). The components q_μ are global coordinates on $\mathfrak{sl}(2,\mathbb{R})^*$ and their Kirillov–Kostant Poisson brackets read

$$\{p_\mu, p_\nu\} = \epsilon_{\mu\nu}{}^\rho\, p_\rho \tag{5.92}$$

on account of (5.88) and the general result (5.28). Now consider a "massive" orbit

$$\mathcal{W}_p = \left\{ q_\mu (t^\mu)^* \middle| q_0 = \sqrt{h^2 + q_1^2 + q_2^2} \right\} \cong \mathrm{SL}(2,\mathbb{R})/\mathrm{U}(1) \tag{5.93}$$

with orbit representative $p = h(t^0)^*$ and stabilizer U(1). We denote the "mass" of the orbit by h rather than M because its quantization will eventually correspond to a representation of $\mathfrak{sl}(2,\mathbb{R})$ with highest weight h (or more precisely $h + 1/2$). The restriction of (5.92) to the orbit gives rise to the Kirillov–Kostant symplectic form (5.29), which we now evaluate.

We can label the points of (5.93) by their "spatial components" (q_1, q_2). In order to write down (5.29) in these coordinates, we need a dictionary between the components x^μ of X and those of the corresponding vector field $\mathrm{ad}^*_X q$ in terms of the coordinates q_1, q_2. We first evaluate $\mathrm{ad}^*_X q$ for $X = x^\mu t_\mu$; using (5.88), for any adjoint vector $Y = y^\mu t_\mu$ we find $\langle \mathrm{ad}^*_X q, Y\rangle = -\langle q, x^\mu y^\nu \epsilon_{\mu\nu}{}^\rho\, t_\rho\rangle$. For q belonging to (5.93) one obtains

$$\begin{aligned}\mathrm{ad}^*_X q = &\Big(- q_1 X^2 + q_2 X^1\Big)(t^0)^* + \Big(- \sqrt{h^2 + q_1^2 + q_2^2}\, X^2 - q_2 X^0\Big)(t^1)^* \\ &+ \Big(\sqrt{h^2 + q_1^2 + q_2^2}\, X^1 + q_1 X^0\Big)(t^2)^*.\end{aligned} \tag{5.94}$$

This is an infinitesimal variation of q tangent to $\mathcal{W}_p$. Any such variation can be expressed in terms of the coordinates (q_1, q_2): for an infinitesimal variation $(\delta q_1, \delta q_2)$ of (q_1, q_2), the variation of q_0 on the orbit (5.93) is

$$\delta q_0 = \frac{q_1 \delta q_1 + q_2 \delta q_2}{\sqrt{h^2 + q_1^2 + q_2^2}} \,. \tag{5.95}$$

The variation of q produced by a vector $v = v_1 \partial_{q_1} + v_2 \partial_{q_2}$ tangent to $\mathcal{W}_p$ takes the same form with δq_i replaced by v_i. Given such a vector v at q one may ask what Lie algebra element X is such that $v = \text{ad}^*_X q$. Owing to (5.94) and (5.95) we may choose

$$x^0 = 0 \,, \qquad x^1 = \frac{V_2}{\sqrt{h^2 + q_1^2 + q_2^2}} \,, \qquad x^2 = \frac{-V_1}{\sqrt{h^2 + q_1^2 + q_2^2}} \,. \tag{5.96}$$

This solution to $v = \text{ad}^*_X q$ is not unique for a given v due to the non-trivial stabilizer U(1), but it is all we need for evaluating the Kirillov–Kostant symplectic form. Indeed, using (5.29) and the fact that the orbit (5.93) is two-dimensional, we find

$$\omega = \frac{dq_2 \wedge dq_1}{\sqrt{h^2 + q_1^2 + q_2^2}} \,, \tag{5.97}$$

which coincides (up to sign) with the Lorentz-invariant volume form (1.6) on the mass shell $\mathcal{W}_p$. One can rewrite it in global Darboux coordinates

$$P \equiv \left(\frac{2\sqrt{h^2 + q_1^2 + q_2^2} - 2h}{q_1^2 + q_2^2} \right)^{1/2} q_1 \,, \qquad Q \equiv \left(\frac{2\sqrt{h^2 + q_1^2 + q_2^2} - 2h}{q_1^2 + q_2^2} \right)^{1/2} q_2 \tag{5.98}$$

such that (5.97) simply becomes

$$\omega = dQ \wedge dP \,. \tag{5.99}$$

Hence the Kirillov–Kostant symplectic form on the orbit (5.93) is globally exact and the quantization condition (5.59) is trivially satisfied for any value of h.

Characters as Path Integrals

We can now quantize the orbit $\mathcal{W}_p$ with the symplectic form (5.99). The associated line bundle is trivial and its sections are just complex-valued functions on $\mathcal{W}_p$; polarized sections can be chosen to depend only on the coordinate Q. One can then evaluate characters of suitable unitary representations of SL$(2, \mathbb{R})$ by computing traces of operators in the resulting Hilbert space, as follows.

The character of a representation is the trace of the exponential of a certain Lie algebra generator. When interpreting the latter as a Hamiltonian, the character may be seen as a partition function. Here we take the Hamiltonian to be the generator of rotations, corresponding to the basis element t_0 in (4.87). As a function on phase space the Hamiltonian maps the point $q_\mu(t^\mu)^*$ on its component q_0, so on the orbit (5.93) we have

$$\mathcal{H} = \frac{1}{\ell}\sqrt{h^2 + q_1^2 + q_2^2} \tag{5.100}$$

where we have included a prefactor[12] $\hbar c/\ell \equiv 1/\ell$ to ensure that $\mathcal{H}$ has dimensions of energy (we think of h, q_μ as being dimensionless). In Darboux coordinates Q, P, we find the Hamiltonian of a harmonic oscillator:

$$\mathcal{H} = \frac{h}{\ell} + \frac{1}{2\ell}(P^2 + Q^2). \tag{5.101}$$

Thus the quantization of the orbit $\mathcal{W}_p$ with the Hamiltonian (5.100) is a quantum harmonic oscillator on the line!

This tremendous simplification allows us to evaluate characters. In principle we could use the path integral formalism, but the operator approach is much simpler since we know the spectrum of the Hamiltonian. Its eigenvalues are

$$\frac{h+1/2}{\ell}, \frac{h+3/2}{\ell}, \frac{h+5/2}{\ell}, \ldots, \frac{h+1/2+n}{\ell}, \ldots$$

each with unit multiplicity. In particular the partition function at temperature $1/\beta$ is that of a harmonic oscillator, $e^{-\beta h/\ell}/(2\sinh[\beta/\ell])$. For future reference we rewrite it as follows: we call L_0 the operator that generates rotations so that $\hat{\mathcal{H}} = \frac{1}{\ell}L_0$, and we write $e^{-\beta/\ell} \equiv q$. We also allow β to be complex as long as its real part is positive. Then the partition function can be written as

$$\mathrm{Tr}\left(q^{L_0}\right) = \frac{q^{h+1/2}}{1-q}\,. \tag{5.102}$$

In Sect. 8.4.1 we will show that this is the character of a unitary representation of the $\mathfrak{sl}(2,\mathbb{R})$ Lie algebra with highest weight $h+1/2$. In the present case one can think of the “1/2” as a quantum correction to the classical weight h.

5.4 Coadjoint Orbits of Semi-direct Products

We now apply the considerations of the previous sections to the semi-direct products[13] described in Chap. 4. In particular we explain how the induced representations of Sect. 4.1 emerge from geometric quantization of coadjoint orbits. The plan is as follows. We first work out general expressions for the adjoint representation, the

[12] We denote by c the speed of light in the vacuum.

[13] Here the words “semi-direct product” refer to a group (4.1) with an *Abelian vector group A*.

Lie bracket and the coadjoint representation of any semi-direct product.[14] Then we expose a general classification of coadjoint orbits, seen as fibre bundles over cotangent bundles of momentum orbits. Finally we turn to geometric quantization and describe the world line actions associated with coadjoint orbits. The considerations of this section can be found e.g. in [18], and also in more recent works [19–21]. The textbooks [22, 23] contain detailed computations and examples.

5.4.1 Adjoint Representation of $G \ltimes A$

We consider a semi-direct product (4.1) with A a vector group. Then the Lie algebra of $G \ltimes A$ is a *semi-direct sum*

$$\mathfrak{g} \in\!\!\!\!+_{\Sigma} A, \tag{5.103}$$

where $\mathfrak{g}$ is the Lie algebra of G and A is its own Lie algebra since it is a vector group. The symbol Σ denotes the differential of the action σ at the identity, $\Sigma : \mathfrak{g} \rightarrow \text{End}(A) : X \mapsto \Sigma_X$, where Σ_X is the infinitesimal generator (5.30) associated with X:

$$\Sigma_X : A \rightarrow A : \alpha \mapsto \Sigma_X \alpha \equiv \frac{d}{dt} \left(\sigma_{e^{tX}} \alpha\right)\big|_{t=0} \,. \tag{5.104}$$

We will denote elements of (5.103) as pairs (X, α) where $X \in \mathfrak{g}$ and $\alpha \in A$; in the terminology of (4.6), X is an infinitesimal rotation/boost while α is a translation.

The adjoint representation of $G \ltimes A$ is given by (5.6), which yields

$$\begin{aligned} \text{Ad}_{(f,\alpha)}(X, \beta) &\overset{(4.6)}{=} \frac{d}{dt}\Big(f e^{tX} f^{-1}, \alpha + t\sigma_f \beta - \sigma_{f e^{tX} f^{-1}} \alpha\Big)\Big|_{t=0} \\ &= \big(\text{Ad}_f X, \sigma_f \beta - \Sigma_{\text{Ad}_f X} \alpha\big) \end{aligned} \tag{5.105}$$

where the symbol "Ad" on the right-hand side denotes the adjoint representation of G. (More generally, in case of ambiguous notations, the argument of a group action determines which group it refers to.) In particular, rotation generators transform according to the adjoint representation of G, while translations are subject to mixed transformations involving both the finite action σ and its differential Σ.

From (5.105) one can read off the Lie bracket in $\mathfrak{g} \in\!\!\!\!+ A$ upon using (5.8):

$$\big[(X, \alpha), (Y, \beta)\big] = \big([X, Y], \Sigma_X \beta - \Sigma_Y \alpha\big). \tag{5.106}$$

The presence of Σ on the right-hand side justifies calling $\mathfrak{g} \in\!\!\!\!+ A$ a *semi*-direct sum. Note that, if A was non-Abelian, the second entry on the right-hand side would include a bracket of generators of A.

[14]The sequence "group $\rightsquigarrow$ adjoint $\rightsquigarrow$ coadjoint" will be ubiquitous in this thesis.

The structure of the algebra (5.106) can be made more transparent by choosing a basis. Let t_a be a basis of $\mathfrak{g}$ satisfying the brackets (5.2), and let α_i be a basis of A (here $a = 1, \ldots, \dim \mathfrak{g}$ and $i = 1, \ldots, \dim A$). Introducing the basis elements

$$j_a \equiv (t_a, 0), \qquad p_i \equiv (0, \alpha_i)$$

that generate the semi-direct sum $\mathfrak{g} \Subset A$, the Lie bracket (5.106) yields

$$[j_a, j_b] = f_{ab}{}^c\, j_c\,, \qquad [j_a, p_i] = g_{ai}{}^k\, p_k\,, \qquad [p_i, p_j] = 0 \tag{5.107}$$

where $g_{ai}{}^k\, p_k \equiv \Sigma_{t_a} p_i$ so that the coefficients $(g_a)_i{}^k$ are the entries of the matrix representing the linear operator $\Sigma_{t_a} : A \to A$ in the basis α_i. The brackets (5.107) make the semi-direct structure manifest since the bracket $[j, p]$ gives p's while the bracket $[p, p]$ vanishes on account of the fact that A is Abelian. This structure will appear repeatedly in this thesis.

5.4.2 *Coadjoint Representation of $G \ltimes A$*

The space of coadjoint vectors of $G \ltimes A$ is the dual of the semi-direct sum (5.103),

$$\mathfrak{g}^* \oplus A^*. \tag{5.108}$$

Its elements are pairs (j, p) where $j \in \mathfrak{g}^*$ and $p \in A^*$, paired with adjoint vectors according to

$$\big\langle (j, p), (X, \alpha) \big\rangle = \langle j, X \rangle + \langle p, \alpha \rangle \tag{5.109}$$

where the first pairing $\langle \cdot, \cdot \rangle$ on the right-hand side is that of $\mathfrak{g}^*$ with $\mathfrak{g}$ while the second one pairs A^* with A. Note that A^* is precisely the space of momenta (see Sect. 4.1), while $\mathfrak{g}^*$ is dual to infinitesimal rotations and may be seen as a space of angular momentum vectors. This is consistent with the general interpretation of coadjoint vectors as conserved quantities (see Sect. 5.1.2) and justifies the notation (j, p).

The coadjoint representation of $G \ltimes A$ acts on the space (5.108). In order to write it down, it is convenient to introduce the following notation:

Definition The *cross product* of translations and momenta is the bilinear map $A \times A^* \to \mathfrak{g}^* : (\alpha, p) \mapsto \alpha \times p$ given for any $X \in \mathfrak{g}$ by

$$\langle \alpha \times p, X \rangle \equiv \langle p, \Sigma_X \alpha \rangle\,. \tag{5.110}$$

The notation is justified by the fact that $\times$ coincides with the vector product when $G \ltimes A$ is the Euclidean group in three dimensions.

With this notation the coadjoint action of $G \ltimes A$ is given by

$$\left\langle \mathrm{Ad}^*_{(f,\alpha)}(j,p),(X,\beta)\right\rangle \overset{(5.10)}{=} \left\langle (j,p),\mathrm{Ad}_{(f,\alpha)^{-1}}(X,\beta)\right\rangle \overset{(5.105)}{=} \left\langle (j,p),\Big(\mathrm{Ad}_{f^{-1}}X,\sigma_{f^{-1}}\beta+\Sigma_{\mathrm{Ad}_{f^{-1}}X}\sigma_{f^{-1}}\alpha\Big)\right\rangle. \tag{5.111}$$

We can use $\Sigma_X(\sigma_f\alpha)=\sigma_f(\Sigma_{\mathrm{Ad}_{f^{-1}}X}\alpha)$ to rewrite this as

$$\left\langle \mathrm{Ad}^*_{(f,\alpha)}(j,p),(X,\beta)\right\rangle \overset{(5.109)}{=} \langle j,\mathrm{Ad}_{f^{-1}}X\rangle+\left\langle p,\sigma_{f^{-1}}\beta+\sigma_{f^{-1}}\Sigma_X\alpha\right\rangle. \tag{5.112}$$

In the first term of the right-hand side we recognize the coadjoint representation of G; the part of the second term involving β is the transformation law (4.16) of momenta; the last term involves the cross product (5.110) of $\sigma^*_f p$ with α. Collecting all these terms and removing the argument (X,β), we conclude that the coadjoint representation of $G \ltimes A$ is

$$\boxed{\mathrm{Ad}^*_{(f,\alpha)}(j,p)=\left(\mathrm{Ad}^*_f j+\alpha\times\sigma^*_f p\,,\ \sigma^*_f p\right)} \tag{5.113}$$

where we keep the notation $\sigma^*_f p$ instead of the simpler $f\cdot p$ to avoid confusion. Note that the coadjoint action of the translation group A affects only angular momenta, since the transformation of p only involves $f\in G$. The translation α contributes a term $\alpha\times\sigma^*_f p$, which for trivial f boils down to the cross product $\alpha\times p$; this contribution can be identified with a combination of *orbital angular momentum* and the *centre of mass vector*, while the *spin angular momentum* is contained in $\mathrm{Ad}^*_f j$. We will return to this interpretation below.

From (5.113) we obtain the coadjoint representation (5.11) of $\mathfrak{g}\Subset_\Sigma A$:

$$\mathrm{ad}^*_{(X,\alpha)}(j,p)=\left(\mathrm{ad}^*_X j+\alpha\times p\,,\ \Sigma^*_X p\right), \tag{5.114}$$

where $\Sigma^*_X p\equiv -p\circ\Sigma_X$. We will use this formula below when dealing with the Kirillov–Kostant symplectic form.

Remark We shall see in Chap. 9 that the space of asymptotically Minkowskian solutions of Einstein's equations in three dimensions spans (a subset of) the space of the coadjoint representation of the BMS_3 group. Each metric will then be labelled by a pair (j,p), where j and p are certain functions on the celestial circle that can be interpreted as the angular momentum aspect and the Bondi mass aspect, respectively.

5.4.3 Coadjoint Orbits

Let us now classify the coadjoint orbits of a semi-direct product. This may be seen as a classification of all classical particles, analogous to the quantum classification worked out in Sect. 4.1. The coadjoint orbit of (j,p) is the set

$$\mathcal{W}_{(j,p)} = \left\{ \text{Ad}^*_{(f,\alpha)}(j, p) \middle| (f, \alpha) \in G \ltimes A \right\} \tag{5.115}$$

embedded in $\mathfrak{g}^* \oplus A^*$, with $\text{Ad}^*_{(f,\alpha)}(j, p)$ given by (5.113). In order to classify all such orbits, we will assume that the orbits (4.18) and little groups (4.19) of induced representations are known. Due to the second entry of the right-hand side of (5.113), involving only $\sigma^*_f p$, each $\mathcal{W}_{(j,p)}$ is a fibre bundle over the orbit $\mathcal{O}_p$. The fibre above $q = \sigma^*_f p$ is the set

$$\left\{ \left(\text{Ad}^*_g \text{Ad}^*_f j + \alpha \times q, q \right) \middle| g \in G_q,\ \alpha \in A \right\}.$$

It remains to understand the geometry of these fibres and the relation between fibres at different points. Note that in the degenerate case $p = 0$ the orbit $\mathcal{W}_{(j,0)}$ is simply the coadjoint orbit of j under G; in particular $\mathcal{W}_{(0,0)}$ contains only one point. Accordingly we take $p \neq 0$ until the end of this section.

Warm-Up: Scalar Orbits

We start by describing scalar orbits, that is, coadjoint orbits that contain points with vanishing angular momentum $j = 0$. The terminology is justified by the fact that each orbit is a homogeneous phase space invariant under $G \ltimes A$, whose quantization yields the Hilbert space of a particle transforming under a unitary representation of $G \ltimes A$. Saying that an orbit contains points with $j = 0$ then means that there exists a frame where the particle's spin vanishes, i.e. that the particle is scalar.

So let us describe an orbit $\mathcal{W}_{(0,p)}$. With $j = 0$ the first entry of the right-hand side of (5.113) reduces to

$$\alpha \times \sigma^*_f p. \tag{5.116}$$

Keeping $q = \sigma^*_f p$ fixed, the set spanned by angular momenta of this form is

$$A \times q \equiv \{\alpha \times q | \alpha \in A\} \subset \mathfrak{g}^* \tag{5.117}$$

and coincides with the set of orbital angular momenta that can be reached by acting with translations on a particle with momentum q. The geometric interpretation of (5.117) is as follows. Recall first that the tangent space of $\mathcal{O}_p$ at q can be identified with the space of "small displacements" of q generated by infinitesimal boosts:

$$T_q\mathcal{O}_p = \left\{ \Sigma^*_X q \middle| X \in \mathfrak{g} \right\} \subset A^*. \tag{5.118}$$

Here $\Sigma^*_X q = 0$ if and only if X belongs to the Lie algebra $\mathfrak{g}_q$ of the little group G_q, so (5.118) is isomorphic to the coset space $\mathfrak{g}/\mathfrak{g}_q$. It follows that the cotangent space $T^*_q\mathcal{O}_p$ at q is the annihilator of $\mathfrak{g}_q$ in $\mathfrak{g}^*$,

$$T^*_q\mathcal{O}_p = \mathfrak{g}^0_q \equiv \left\{ j \in \mathfrak{g}^* \middle| \langle j, X \rangle = 0\ \ \forall X \in \mathfrak{g}_q \right\} \subset \mathfrak{g}^*, \tag{5.119}$$

which provides the sought-for interpretation:

Lemma The cotangent space (5.119) coincides with the set (5.117):

$$T_q^*\mathcal{O}_p = \mathfrak{g}_q^0 = A \times q. \tag{5.120}$$

Proof Let $X \in \mathfrak{g}$ be an infinitesimal rotation leaving q invariant, i.e. $\Sigma_X^* q = 0$. One then has $\langle \alpha \times q, X\rangle = 0$ for any translation α, so $\alpha \times q$ belongs to the annihilator $\mathfrak{g}_q^0$. By (5.119) this implies that the span $A \times q$ is contained in $T_q^*\mathcal{O}_p$. To prove (5.120) we need to show the opposite inclusion, i.e. that any element of the annihilator $\mathfrak{g}_q^0$ can be written as $\alpha \times q$ for some $\alpha \in A$. To see this, consider the linear function

$$\tau_q : A \to \mathfrak{g}_q^0 : \alpha \mapsto \alpha \times q \tag{5.121}$$

mapping a translation on the associated orbital angular momentum. The rank of this map is $\dim(A) - \dim[\mathrm{Ker}(\tau_q)]$, where

$$\mathrm{Ker}(\tau_q) = \left\{\alpha \in A \middle| \langle \Sigma_X^* q, \alpha\rangle = 0 \;\forall\; [X] \in \mathfrak{g}/\mathfrak{g}_q\right\}. \tag{5.122}$$

The elements of this kernel are translations constrained by $\dim(\mathfrak{g}) - \dim(\mathfrak{g}_q)$ independent conditions (the subtraction of $\dim(\mathfrak{g}_q)$ comes from the quotient by $\mathfrak{g}_q$). This implies that $\dim[\mathrm{Ker}(\tau_q)] = \dim(A) - \dim(\mathfrak{g}) + \dim(\mathfrak{g}_q)$, from which we conclude that the rank of τ_q is

$$\dim[\mathrm{Im}(\tau_q)] = \dim(\mathfrak{g}) - \dim(\mathfrak{g}_q) = \dim(\mathfrak{g}_q^0).$$

It follows that τ_q is surjective, which was to be proven. ■

We have just shown that the span (5.117) at each $q \in \mathcal{O}_p$ is the cotangent space of $\mathcal{O}_p$ at q. Since $j = 0$, this analysis exhausts all points of $\mathcal{W}_{(0,p)}$ and we conclude that

the scalar coadjoint orbits of $G \ltimes A$
are cotangent bundles of momentum orbits. (5.123)

In mathematical terms we would write $\mathcal{W}_{(0,p)} = T^*\mathcal{O}_p$. In particular, if we have classified all momentum orbits of $G \ltimes A$, then we already know the classification of all scalar coadjoint orbits $\mathcal{W}_{(0,p)}$. Note that the map (5.121) allows us to express the stabilizer of $(0, p)$ in a compact way: it is a semi-direct product

$$\text{Stabilizer of } (j, p) \;=\; G_p \ltimes \mathrm{Ker}(\tau_p) \tag{5.124}$$

where G_p is the little group of p.

Spinning Orbits

We now turn to spinning orbits, which generally contain no point with vanishing total angular momentum. To begin, we pick a coadjoint vector (j, p) and restrict our attention to rotations f that belong to the little group G_p. The resulting span is

$$\left\{ \left(\mathrm{Ad}^*_f j + \alpha \times p, p\right) \Big| f \in G_p, \ \alpha \in A \right\} \tag{5.125}$$

and is a subset of the full orbit (5.115). In general $\mathrm{Ad}^*_f(j) \neq j$ because the little group G_p need not be included in the stabilizer of j for the coadjoint action of G. Noting that the cross product (5.110) satisfies the property $\mathrm{Ad}^*_f(\alpha \times p) = \sigma_f \alpha \times \sigma^*_f p$, and using the fact that f fixes p, we rewrite (5.125) as

$$\left\{ \left(\mathrm{Ad}^*_f \left(j + \beta \times p\right), p\right) \big| f \in G_p, \ \beta \in A \right\} \tag{5.126}$$

where β is related to the α of (5.125) by $\beta = \sigma_{f^{-1}} \alpha$. In particular we have

$$\text{Stabilizer of } (j, p) \ = \ (G_j \cap G_p) \ltimes \mathrm{Ker}(\tau_p) \tag{5.127}$$

where G_j is the stabilizer of j for the coadjoint action of G and all the remaining notation is as before. This extends (5.124) to the case $j \neq 0$.

The rewriting (5.126) allows us to see that translations along β can modify at will all components of j that point along directions in the annihilator $\mathfrak{g}^0_p$. The only piece of j that is left unchanged by the action of translations is its restriction to $\mathfrak{g}_p$,

$$j\big|_{\mathfrak{g}_p} \equiv j_p \,. \tag{5.128}$$

Accordingly the set (5.126) is diffeomorphic to a product

$$\underbrace{\left\{\mathrm{Ad}^*_f j_p | f \in G_p\right\}}_{\mathcal{W}_{j_p}} \times \underbrace{\left\{\alpha \times p | \alpha \in A\right\}}_{T^*_p \mathcal{O}_p}, \tag{5.129}$$

where we recognize the cotangent space (5.120) and where $\mathcal{W}_{j_p} \subset \mathfrak{g}^*_p$ denotes the coadjoint orbit of $j_p \in \mathfrak{g}^*_p$ under the little group G_p. This is in fact our main conclusion: when $\mathcal{W}_{(j,p)}$ is seen as a fibre bundle over $\mathcal{O}_p$, the fibre at p is a product (5.129) of the cotangent space of $\mathcal{O}_p$ at p with the coadjoint orbit of the projection j_p of j under the action of the little group of p.

Inspecting (5.129), note in particular how the little group orbit $\mathcal{W}_{j_p}$ factorizes from the cotangent space $A \times p$ due to translations. This splitting is reminiscent of the representation (4.28) of $G_p \ltimes A$, where the operators representing $f \in G_p$ and $\alpha \in A$ live on very different footings (and actually commute). Recall that we used this representation to induce an irreducible representation (4.29) of the full group $G \ltimes A$. What we see in (5.129) is the classical analogue of this little group representation; upon quantization, the sub-orbit (5.129) will precisely produce a representation of the form (4.28), and its extension to the full orbit $\mathcal{W}_{(j,p)}$ will correspond to the induction (4.29). In particular the projection (5.128) is a classical definition of spin. We shall return to this below.

The arguments that led from (5.125) to the result (5.129) can be run at any other point q on $\mathcal{O}_p$, except that the little group is G_q instead of G_p. Thus the fibre above any point $q = \sigma^*_f p \in \mathcal{O}_p$ is a product of the cotangent space of $\mathcal{O}_p$ at q with the G_q-coadjoint orbit $\mathcal{W}_{(\mathrm{Ad}^*_f j)_q}$, where $(\mathrm{Ad}^*_f j)_q$ denotes the restriction of $\mathrm{Ad}^*_f j$ to $\mathfrak{g}_q$. But little groups at different points of $\mathcal{O}_p$ are isomorphic: if one chooses standard boosts $g_q \in G$ such that $\sigma^*_{g_q}(p) = q$, then $G_q = g_q \cdot G_p \cdot g_q^{-1}$ and $\mathfrak{g}_q = \mathrm{Ad}_{g_q}\mathfrak{g}_p$. Therefore $\mathcal{W}_{(\mathrm{Ad}^*_f j)_q}$ is diffeomorphic to $\mathcal{W}_{j_p}$ for any $q = \sigma^*_f p \in \mathcal{O}_p$; the relation between the fibres above q and p is given by the coadjoint action of $G \ltimes A$.

Classification of Coadjoint Orbits

The conclusions of the previous paragraph can be used to classify the orbits of $G \ltimes A$. We start with some terminology:

Definition Let (j, p) be a coadjoint vector of the semi-direct product $G \ltimes_\sigma A$. The corresponding *bundle of little group orbits* is

$$\mathcal{B}_{(j,p)} \equiv \left\{ \left((\mathrm{Ad}^*_f j)_{\sigma^*_f p}, \sigma^*_f p \right) \middle| f \in G \right\}. \tag{5.130}$$

According to our earlier observations, the bundle of little group orbits associated with (j, p) is really the same as the coadjoint orbit $\mathcal{W}_{(j,p)}$, except that the cotangent spaces at each point of $\mathcal{O}_p$ are "neglected" since translations do not appear in (5.130). Thus $\mathcal{B}_{(j,p)}$ is a fibre bundle over $\mathcal{O}_p$, the fibre F_q at $q \in \mathcal{O}_p$ being a coadjoint orbit of the little group G_q. The relation between fibres at different points of $\mathcal{O}_p$ is given by the coadjoint action of $G \ltimes A$, or explicitly

$$(k, q) \in F_q \quad \text{iff} \quad \exists\, f \in G \text{ such that } k = \left(\mathrm{Ad}^*_f j\right)_q \text{ and } q = \sigma^*_f p.$$

Conversely, suppose that two elements $p \in A^*$ and $j_0 \in \mathfrak{g}^*_p$ are given. The group G can be seen as a principal G_p-bundle over $\mathcal{O}_p$, equipped with a natural G_p-action by multiplication from the left in each fibre. In addition G_p acts on the coadjoint orbit $\mathcal{W}_{j_0}$, so one can define an action of G_p on $G \times \mathcal{W}_{j_0}$ by

$$(f, k) \in G \times \mathcal{W}_{j_0} \stackrel{g \in G_p}{\longmapsto} \left(g \cdot f, \mathrm{Ad}^*_g(k)\right).$$

The corresponding bundle of little group orbits is defined as the associated bundle

$$\mathcal{B}_{(j_0,p)} \equiv \left(G \times \mathcal{W}_{j_0}\right)/G_p. \tag{5.131}$$

Thus one can associate a bundle of little group orbits (5.130) with each coadjoint orbit of $G \ltimes A$; conversely, starting from any bundle of little group orbits as defined in (5.131), one can build a coadjoint orbit of $G \ltimes A$ by choosing any $j \in \mathfrak{g}^*$ such that $j_p = j_0$ and taking the orbit $\mathcal{W}_{(j,p)}$. In other words the classification of coadjoint orbits of $G \ltimes A$ is equivalent to the classification of bundles of little group orbits [18, 20].

These arguments yield the complete picture of coadjoint orbits of $G \ltimes A$:

$$\begin{gathered}\textit{the coadjoint orbit } \mathcal{W}_{(j,p)} \textit{ is a fibre bundle over } \mathcal{O}_p, \textit{ where}\\ \textit{the fibre at } q \in \mathcal{O}_p \textit{ is a product of the cotangent space } T^*_q\mathcal{O}_p\\ \textit{with a coadjoint orbit of the little group } G_q.\end{gathered} \tag{5.132}$$

Equivalently, $\mathcal{W}_{(j,p)}$ is a fibre bundle over the cotangent bundle $T^*\mathcal{O}_p$, the fibre above $(q, \alpha \times q) \in T^*\mathcal{O}_p$ being a coadjoint orbit of G_q. To exhaust all coadjoint orbits of $G \ltimes A$, one can proceed as follows:

1. Pick an element $p \in A^*$ and compute its momentum orbit $\mathcal{O}_p$ under the action σ^* of G; let G_p be the corresponding little group.
2. Pick $j_p \in \mathfrak{g}^*_p$ and compute its coadjoint orbit under the action of G_p.

The set of all orbits $\mathcal{O}_p$ and of all coadjoint orbits of the corresponding little groups classifies the coadjoint orbits of $G \ltimes A$. Put differently, suppose one has classified the following objects:

1. The orbits of G for the action σ^*, with an exhaustive set of orbit representatives $p_\lambda \in A^*$ and corresponding little groups G_λ, with $\lambda \in \mathcal{I}$ some index such that $\mathcal{O}_{p_\lambda}$ and $\mathcal{O}_{p_{\lambda'}}$ are disjoint whenever $\lambda \neq \lambda'$;
2. The coadjoint orbits of each G_λ, with an exhaustive set of orbit representatives $j_{\lambda,\mu} \in \mathfrak{g}^*_\lambda$, $\mu \in \mathcal{J}_\lambda$ being some index such that $\mathcal{W}_{j_{\lambda,\mu}}$ and $\mathcal{W}_{j_{\lambda,\mu'}}$ are disjoint whenever $\mu \neq \mu'$.

Then the set

$$\left\{ \left(j_{\lambda,\mu}, p_\lambda\right) \middle| \, \lambda \in \mathcal{I}, \mu \in \mathcal{J}_\lambda \right\} \subset \mathfrak{g}^* \oplus A^* \tag{5.133}$$

is an exhaustive set of orbit representatives for the coadjoint orbits of $G \ltimes A$. The (generally continuous) indices λ, μ label the orbits uniquely. This algorithm is a classical analogue of the classification of representations described in Sect. 4.1, since it classifies the phase spaces of all "particles" associated with $G \ltimes A$.

5.4.4 Geometric Quantization and Particles

We now describe the quantization of coadjoint orbits of semi-direct products and argue that it yields Hilbert spaces of one-particle states as described in Chap. 4.

A Remark on Cotangent Bundles

Before studying quantization we briefly digress on cotangent bundles and their canonical symplectic form $\omega = -d\theta$, where θ is the Liouville one-form (5.17). Our goal is to rewrite the symplectic form on $T^*\mathcal{Q}$ in a simpler way. For a sufficiently small open neighbourhood U of $q \in \mathcal{Q}$, the preimage $\pi^{-1}(U)$ is diffeomorphic to the product $U \times T^*_q\mathcal{Q}$. Hence the tangent space $T_{(q,\alpha)}T^*\mathcal{Q}$ can be written as a direct sum

$$T_{(q,\alpha)}T^*\mathcal{Q} \;\cong\; T_q\mathcal{Q} \oplus T_\alpha T^*_q\mathcal{Q} \;\cong\; T_q\mathcal{Q} \oplus T^*_q\mathcal{Q} \tag{5.134}$$

which justifies writing its elements as $\mathcal{V} = (v, \beta)$, where $v \in T_q\mathcal{Q}$ and $\beta \in T_q^*\mathcal{Q}$. The differential of (5.16) at (q, α) then reads $\pi_{*(q,\alpha)}(v, \beta) = v$ and the Liouville one-form (5.17) reduces to

$$\theta_{(q,\alpha)}(v, \beta) = \langle \alpha, v \rangle. \tag{5.135}$$

Accordingly one finds that the canonical symplectic form $\omega = -d\theta$ is

$$\omega_{(q,\alpha)}\left((v, \beta), (w, \gamma)\right) = \langle \gamma, v \rangle - \langle \beta, w \rangle \tag{5.136}$$

which is just a more intrinsic rewriting of the standard $\omega = dq \wedge dp$. As we now show, this reformulation is useful for coadjoint orbits of semi-direct products.

Quantization

Suppose we wish to quantize a coadjoint orbit $\mathcal{W}_{(j,p)}$ of $G \ltimes A$; let ω be its Kirillov–Kostant symplectic form (5.29). Since the Lie bracket in $\mathfrak{g} \Subset A$ is (5.106), the symplectic form evaluated at the point $\left(\text{Ad}^*_f j + \alpha \times q, q\right) \equiv (\kappa, q)$ in $\mathcal{W}_{(j,p)}$ reads

$$\begin{aligned} &\omega_{(\kappa,q)}\left(\text{ad}^*_{(X,\beta)}(\kappa, q), \text{ad}^*_{(Y,\gamma)}(\kappa, q)\right) = \\ &= \left\langle \text{Ad}^*_f j, [X, Y]\right\rangle + \left\langle \alpha \times q, [X, Y]\right\rangle + \left\langle \gamma \times q, X\right\rangle - \left\langle \beta \times q, Y\right\rangle. \end{aligned} \tag{5.137}$$

In the two last terms of this expression we recognize the Liouville symplectic form (5.136) on the cotangent bundle $T^*\mathcal{O}_p$ when $\alpha \times q$ is seen as an element of $T_q^*\mathcal{O}_p$ thanks to (5.120). On the other hand the first term of (5.137) looks like the natural symplectic form (5.29) on the G-coadjoint orbit of j. In particular, when X and Y belong to the Lie algebra $\mathfrak{g}_q$ of the little group at q, the second term in (5.137) vanishes and the first one reduces to

$$\left\langle \text{Ad}^*_f j, [X, Y]\right\rangle = \left\langle (\text{Ad}^*_f j)_q, [X, Y]\right\rangle$$

where we use the notation (5.128). This is the natural symplectic form on the coadjoint orbit $\mathcal{W}_{(\text{Ad}^*_f j)_q}$, so if we see $\mathcal{W}_{(j,p)}$ as a fibre bundle over $T^*\mathcal{O}_p$ with typical fibre $\mathcal{W}_{j_p}$, restricting the symplectic form (5.137) to a fibre gives back the symplectic form on the little group's coadjoint orbit. This observation actually follows from a more general result, which states that the coadjoint orbits of a semi-direct product are obtained by *symplectic induction* from the coadjoint orbits of its little groups. Symplectic induction is the classical analogue of the method of induced representations that yields irreducible unitary representations of semi-direct products. We will not dwell on the details of this construction and refer e.g. to [20, 24] for a much more thorough treatment.

For quantization to be possible, the symplectic form (5.137) must be integral in the sense (5.59). But the Liouville two-form (5.136) is exact, so its de Rham cohomology class vanishes and demanding that (5.137) be integral reduces to demanding integrality of the symplectic form on the coadjoint orbit of the little group. We conclude (see e.g. [21] for the proof):

Theorem Let $G \ltimes A$ be a semi-direct product, (j, p) a coadjoint vector with coadjoint orbit $\mathcal{W}_{(j,p)}$. Then $\mathcal{W}_{(j,p)}$ is prequantizable if and only if the corresponding G_p-coadjoint orbit $\mathcal{W}_{j_p}$ is prequantizable.

Provided the little group orbit $\mathcal{W}_{j_p}$ is quantizable, one obtains a unitary representation $\mathcal{R}$ of the little group G_p acting on polarized sections of a line bundle over $\mathcal{W}_{j_p}$. These sections are spin states; as in Sect. 4.1, we denote their Hilbert space by $\mathcal{E}$. Upon declaring that the polarized sections on $T^*\mathcal{O}_p$ depend only on the coordinates of the momentum orbit $\mathcal{O}_p$, polarized sections on the whole orbit $\mathcal{W}_{(j,p)}$ can be seen as $\mathcal{E}$-valued wavefunctions in momentum space. Assuming that there exists a quasi-invariant measure μ on $\mathcal{O}_p$, the Hilbert space $\mathscr{H}$ obtained by quantizing $\mathcal{W}_{(j,p)}$ becomes a tensor product (3.8) of $\mathcal{E}$ with the space of square-integrable functions $\mathcal{O}_p \to \mathbb{C}$. This exactly reproduces the construction of Sect. 4.1.

Recovering Induced Representations

As the last step of quantization, we now need to understand how the group $G \ltimes A$ acts on polarized sections, or equivalently what differential operators represent the Lie algebra $\mathfrak{g} \Subset A$ on sections. Recall that these operators take the general form (5.60) where $\mathcal{J}$ is a momentum map (5.33) while ξ_X is an infinitesimal generator (5.30) for the Lie algebra element X. In the present case X is replaced by a pair $(X, \alpha) \in \mathfrak{g} \Subset A$. Furthermore, since the phase space is a coadjoint orbit, the momentum map is an inclusion (5.38) and the infinitesimal generator is $\xi_{(X,\alpha)} = \text{ad}^*_{(X,\alpha)}$.

Let us describe this in more detail in the scalar case $j = 0$, so that $\mathcal{W}_{(0,p)} = T^*\mathcal{O}_p$. Then the Kirillov–Kostant symplectic form coincides with the canonical symplectic form on $T^*\mathcal{O}_p$ and the operator (5.60) representing a Lie algebra element (X, α) is

$$\hat{\mathcal{J}}_{(X,\alpha)}\Big|_{(\beta\times q,q)} = -i\hbar\,\text{ad}^*_{(X,\alpha)}(\beta \times q, q) + \langle q, \alpha\rangle$$

when evaluated at a point $(\beta \times q, q)$ belonging to $T^*\mathcal{O}_p$. Polarized sections are functions $\Psi : \mathcal{W}_{(0,p)} \to \mathbb{C} : (\beta \times q, q) \mapsto \Psi(q)$ since they only depend on momenta $q \in \mathcal{O}_p$. Upon acting on such a function the operator $\hat{\mathcal{J}}_{(X,\alpha)}$ yields

$$\hat{\mathcal{J}}_{(X,\alpha)} \cdot \Psi(q) = -i\hbar\,(\Sigma^*_X q) \cdot \Psi + \langle q, \alpha\rangle\Psi(q) \tag{5.138}$$

where $\Sigma^*_X q \in T_q\mathcal{O}_p$ acts on Ψ according to

$$(\Sigma^*_X q) \cdot \Psi \equiv -\frac{d}{dt}\Psi(\sigma^*_{e^{-tX}} q)\Big|_{t=0}.$$

Thus all observables $\hat{\mathcal{J}}_{(X,\alpha)}$ are polarized and can be quantized so as to satisfy (5.48).

Formula (5.138) describes the action of Hermitian operators $\hat{\mathcal{J}}_{(X,\alpha)}$ on wavefunctions $\Psi : \mathcal{O}_p \to \mathbb{C}$, provided the measure μ on $\mathcal{O}_p$ is invariant under G. It can be rewritten as

$$\left(\hat{\mathcal{J}}_{(X,\alpha)}\cdot\Psi\right)(q)=i\hbar\frac{d}{dt}\left[e^{-i\langle q,t\alpha\rangle/\hbar}\,\Psi(e^{-tX}\cdot q)\right]\Big|_{t=0}$$

and thus corresponds by differentiation to the *finite* transformation law

$$\big(\mathcal{T}[(f,\alpha)]\Psi\big)(q)=e^{-i\langle q,\alpha\rangle/\hbar}\,\Psi(f^{-1}\cdot q) \tag{5.139}$$

where the map $\mathcal{T}$ is a representation of $G\ltimes A$ such that

$$\mathcal{T}\big[(e^{tX},t\alpha)\big]=\exp\left[-\frac{it}{\hbar}\hat{\mathcal{J}}_{(X,\alpha)}\right].$$

When the measure μ on $\mathcal{O}_p$ defining the scalar product of wavefunctions is invariant under G, formula (5.139) is a unitary representation of $G\ltimes A$ that coincides (up to a sign due to different conventions) with a scalar induced representation (4.30). We have thus recovered induced representations by quantization! The argument can be generalized to spinning representations and to quasi-invariant measures [21, 24], although we will not prove it here. Thus we conclude:

Theorem Let $G\ltimes A$ be a semi-direct product, $\mathcal{W}_{(j,p)}$ one of its coadjoint orbits. Then the unitary representation of $G\ltimes A$ obtained by geometric quantization of $\mathcal{W}_{(j,p)}$ is an induced representation of the form (4.30) with momentum orbit $\mathcal{O}_p$ and spin j_p.

Remark This theorem says nothing about the exhaustivity of the procedure: it does not guarantee that *all* induced representations can be obtained by quantization. In fact it is easy to work out explicit examples where certain induced representations cannot follow from geometric quantization, for instance if the little group is not connected. In this sense geometric quantization is somewhat weaker than the full theory of induced representations exposed in Sect. 4.1.

5.4.5 World Lines

Geometric actions for semi-direct products can be obtained following the general method described in Sect. 5.3. As we now show they can be interpreted as world line actions describing the motion of a point particle (generally with spin) in "space-time" A. We will rely on the Sigma model picture (5.84).

We start by evaluating the left Maurer–Cartan form (5.69) for a semi-direct product with multiplication (4.6). In order to describe a vector tangent to $G\ltimes A$ at the point (f,α), consider a path in $G\ltimes A$ given by

$$\gamma(t)=\big(g(t),\beta(t)\big) \tag{5.140}$$

with $g(0) = f$, $\beta(0) = \alpha$ and $\dot{\gamma}(0) \equiv v$. Using the group operation (4.6) in $G \ltimes A$, we then find

$$\Theta_{(f,\alpha)}(v) \overset{(5.69)}{=} \frac{d}{dt}\Big[\big(f^{-1}g(t), \sigma_{f^{-1}}\beta(t) - \sigma_{f^{-1}}\alpha\big)\Big]\Big|_{t=0} = \big(\Theta_f \oplus \sigma_{f^{-1}}\big)(v)\,, \tag{5.141}$$

where on the far right-hand side Θ denotes the Maurer–Cartan form on G. The direct sum refers to the fact that the tangent space $T_{(f,\alpha)}(G \ltimes A)$ is isomorphic to $T_f G \oplus A$. Using (5.141) we can now write the Sigma model action (5.84) associated with the orbit of a coadjoint vector $(j, p) \in \mathfrak{g}^* \oplus A^*$:

$$S[f(t), \alpha(t)] \overset{(5.109)}{=} \int_0^T dt\big\langle j, \Theta_{f(t)}(\dot{f}(t))\big\rangle + \int_0^T dt\big\langle \sigma^*_{f(t)}p, \dot{\alpha}(t)\big\rangle. \tag{5.142}$$

This can be recast in intrinsic terms as

$$S[f(t), \alpha(t)] = \int_{f(t)} \big\langle j, \Theta\big\rangle + \int_{(f(t),\alpha(t))} \big\langle \sigma^* p, d\alpha\big\rangle \tag{5.143}$$

where $\langle \sigma^* p, d\alpha\rangle$ is the one-form on $G \ltimes A$ that gives $\langle \sigma^*_f p, \beta\rangle$ when evaluated at (f, α) and acting on a vector (v, β). Note that this is just the sum of the Sigma model action (5.84) on G with a purely kinetic scalar action functional

$$S_{\text{scalar}}[f(t), \alpha(t)] = \int_{(f(t),\alpha(t))} \big\langle \sigma^* p, d\alpha\big\rangle \tag{5.144}$$

describing a point particle propagating in A along a path $\alpha(t)$ with momentum $q(t) = \sigma^*_{f(t)}p$. In particular the group A is now interpreted as "space-time". Expression (5.144) also has a gauge symmetry with gauge group (5.124), and it is invariant under redefinitions of the time parameter. As in (5.67), adding a Hamiltonian generally spoils reparameterization symmetry. In the example of the Poincaré group below the condition $p(t) \in \mathcal{O}_p$ will be a constraint generating time reparameterizations. Note that this condition only applies to momenta $q(t) \in A^*$, while the position of the particle, $\alpha(t) \in A$, is completely unconstrained.

5.5 Relativistic World Lines

In this section we study coadjoint orbits of Poincaré groups and show that the corresponding geometric actions describe world lines of relativistic particles. At the end we also turn to Galilean world lines and show that the corresponding partition functions coincide with Bargmann characters. These topics have been studied previously in a number of references. The papers [25, 26] deal with the classification problem (see also [27]); the books [28–30] describe particles in terms of quantization of Poincaré coadjoint orbits; finally the papers [31–34] describe the relation between world line actions and propagators of relativistic fields.

5.5.1 Coadjoint Orbits of Poincaré

The classification of coadjoint orbits of the Poincaré group is an application of the general algorithm described in Sect. 5.4.3: all of them are fibre bundles over momentum orbits, the fibre being a coadjoint orbit of the corresponding little group. Since momentum orbits have been classified in Sect. 4.2, the classification of coadjoint orbits is straightforward. Quantizing any coadjoint orbit yields an irreducible, unitary representation of the Poincaré group, i.e. the Hilbert space of a relativistic particle.

As an example consider the (double cover of the) Poincaré group in three dimensions, (4.93). Its momentum orbits coincide with $\mathrm{SL}(2,\mathbb{R})$ coadjoint orbits, and the little groups are stabilizers of $\mathrm{SL}(2,\mathbb{R})$ coadjoint vectors. All stabilizers are one-dimensional and Abelian, except for the trivial orbit whose little group is $\mathrm{SL}(2,\mathbb{R})$. This implies that all Poincaré coadjoint orbits are cotangent bundles of momentum orbits, except in the case $p=0$ for which $\mathcal{W}_{(j,0)}$ coincides with the coadjoint orbit of j under $\mathrm{SL}(2,\mathbb{R})$. The set of coadjoint orbit representatives for Poincaré can be obtained by following the algorithm outlined above (5.133).

5.5.2 Scalar World Lines

Let us consider a massive scalar coadjoint orbit of the Poincaré group in space-time dimension D. We wish to work out the corresponding Sigma model action (5.144). We refer to [31–34] for a similar approach and for spinning generalizations.

The action principle describing a scalar world line is (5.144). We choose a basis e_μ of $\mathbb{R}^D$ such that each translation can be written as $\alpha=\alpha^\mu e_\mu$. The dual basis consists of momenta $(e^\mu)^*$ such that $\langle p,\alpha\rangle = p_\mu\alpha^\mu$ for $p=p_\mu(e^\mu)^*$. The argument of the action functional (5.144) is a path in $G\ltimes A$, which we denote $(f(\tau),x(\tau))$ in order to distinguish the time parameter τ along the world line from the time coordinate $t=x^0$. With the coordinates p_μ just described we have $\big(\sigma^*_{f(\tau)}p\big)_\mu\equiv p_\mu(\tau)$ for some orbit representative p, and the action becomes

$$S[p(\tau),x(\tau)]=\int_0^T d\tau\; p_\mu(\tau)\dot{x}^\mu(\tau)\qquad\text{with a constraint}\qquad p_\mu(\tau)p^\mu(\tau)=-M^2\;\;\forall\,\tau,$$

where indices are raised and lowered using the Minkowski metric. The constraint accounts for the fact that momenta must belong to a massive orbit. It can be incorporated in the action thanks to a Lagrange multiplier $N(\tau)$:

$$S[p(\tau), x(\tau), N(\tau)] = \int_0^T d\tau \Big[p_\mu(\tau)\dot{x}^\mu(\tau) - N(\tau)\left(p_\mu(\tau)p^\mu(\tau) + M^2\right)\Big]. \tag{5.145}$$

The equations of motion enforce the constraint

$$\phi \equiv p_\mu p^\mu + M^2 = 0 \tag{5.146}$$

and describe a point particle propagating in space-time with constant momentum:

$$\dot{p}_\mu = 0, \qquad \dot{x}^\mu = 2Np^\mu. \tag{5.147}$$

Note how the non-trivial dynamics emerges from the fact that momenta span an orbit, even though we haven't included any Hamiltonian.

To rewrite (5.145) in Lagrangian form, we use the second equation of motion in (5.147) to express momenta in terms of velocities:

$$p^\mu = \frac{\dot{x}^\mu}{2N}\,. \tag{5.148}$$

Contracting this with p_μ and using the mass shell constraint (5.146) then gives

$$-\,M^2 = \frac{\dot{x}^\mu \dot{x}_\mu}{4N^2}\,. \tag{5.149}$$

Since our goal is to describe a massive particle, its trajectory must be time-like so we require that $\dot{x}^\mu$ remains inside the light-cone at any time τ, which gives $\dot{x}^\mu \dot{x}_\mu < 0$. This implies that (5.149) has two real solutions N; we choose the positive one,

$$N = \frac{\sqrt{-\dot{x}^\mu \dot{x}_\mu}}{2M}\,. \tag{5.150}$$

Together with (5.148) this defines an invertible Legendre transformation from the space of positions and velocities $\{(x^\mu, \dot{x}^\mu)\}$ to the space of positions and constrained momenta supplemented with a Lagrange multiplier,

$$\Big\{(x^\mu, p_\mu, N)\Big| x \in \mathbb{R}^D,\ \ p \in \mathbb{R}^D \ \text{ such that } \ p^2 = -M^2,\ \ N > 0\Big\}.$$

Upon expressing p and N in terms of $\dot{x}$ thanks to this correspondence, the Hamiltonian action (5.145) can be rewritten as

$$S[x(\tau), \dot{x}(\tau)] = -M \int_0^T d\tau \sqrt{-\dot{x}^\mu \dot{x}_\mu}. \tag{5.151}$$

This is an action functional describing the dynamics of a scalar relativistic particle, with the Lagrangian $-M\sqrt{-\eta_{\mu\nu}\dot{x}^\mu\dot{x}^\nu}$. We have thus recovered the metric structure of space-time from the coadjoint orbits of its isometry group.

One can also run the argument in reverse and recover the Hamiltonian action from the Lagrangian one. In doing so one discovers that the mass shell condition is a primary constraint generating time reparameterizations while $N(\tau)$ is a lapse function along the world line. The canonical Hamiltonian then reads $\mathcal{H} = 2N(p^2 + M^2)$ and vanishes on the constraint surface, as usual for generally covariant systems.

Remark Starting from the Hamiltonian action (5.145), one can evaluate the associated transition amplitude as a path integral. This computation was performed in [31–34] and the result turns out to coincide with the Feynman propagator of a free scalar field with mass M. This observation is one of the starting points of the *world line formalism* of quantum field theory [35–37], where scattering amplitudes are reformulated in terms of point particles propagating in space-time.

*5.5.3 Galilean World Lines**

Here we study coadjoint orbits of the Bargmann group (4.103) and write down world line actions for scalar non-relativistic particles. We also show how these actions account for Bargmann characters.

Scalar World Lines

The classification of coadjoint orbits of the Bargmann group follows from the general considerations of Sect. 5.4.3, combined with the classification of momentum orbits and little groups described in Sect. 4.4. In what follows we study the geometric action associated with one such coadjoint orbit with mass $M > 0$ and spin $j = 0$. The orbit then is a cotangent bundle $T^*\mathcal{O}_p$, where $\mathcal{O}_p$ is a massive momentum orbit (4.115). The corresponding representation describes a scalar non-relativistic particle.

In order to write down the action (5.144) we use the same trick as in (5.145) to express the integrand in components and absorb the constraint $p(\tau) \in \mathcal{O}_p$ with a Lagrange multiplier $N(\tau)$. (The time parameter along the world line is once again denoted as τ, in order to distinguish it from the time coordinate t.) Using the pairing (4.107) the world line action reads

$$S\big[x(\tau), t(\tau), p(\tau), E(\tau), N(\tau)\big] = \int_0^T d\tau \left[p_i\dot{x}^i - E\dot{t} - N\left(\frac{p^2}{2M} - E\right)\right] \tag{5.152}$$

where $i = 1, \ldots, D-1$. In principle we should also include a time-dependent central term $\lambda(\tau)$ (recall the last entry of (4.105)), but one readily verifies that its contribution to the action is a boundary term so we neglect it from now on. This being said, note that the presence of the central extension is crucial in giving rise to the constraint $E \approx p^2/2M$ obtained by varying N. The equations of motion obtained by varying E give $N = \dot{t}$, so we can once more interpret $N(\tau)$ as a lapse function along the world line. Plugging the solution of the equations of motion of N and E into (5.152), we get

$$S\big[x(\tau), t(\tau), p(\tau)\big] = \int_0^T d\tau \left[p_i \dot{x}^i - \frac{p^2}{2M}\dot{t}\right],$$

which we recognize as the action of a free non-relativistic particle moving in $\mathbb{R}^{D-1}$, written in a reparameterization-invariant way (see e.g. Chap. 4 of [38]). By expressing the action as an integral over the "real time" $t = t(\tau)$, we find

$$S[x(t), p(t)] = \int_0^T dt \left[p_i \dot{x}^i - \frac{p^2}{2M}\right] \tag{5.153}$$

where the dot now denotes differentiation with respect to t.

Path Integrals and Characters

From now on we take $D = 3$ for simplicity. Our goal is to plug the action (5.153) into a path integral so as to recover the Bargmann character (4.123) for $r = 1$. Note that the steps leading from the original Hamiltonian action (5.152) to the quadratic action (5.153) all go through in the path integral since they amount to integrating out variables on which the action depends linearly.

We wish to evaluate the rotating partition function of a massive Galilean particle,

$$Z(\beta, \theta) = \text{Tr}\left(e^{-\beta\hat{H}+i\theta\hat{J}}\right), \tag{5.154}$$

where $\hat{H} = \hat{p}^2/2M$ is the Hamiltonian and $\hat{J}$ is the angular momentum operator

$$\hat{J} = \hat{x}^1\hat{p}_2 - \hat{x}^2\hat{p}_1. \tag{5.155}$$

The trace (5.154) can be interpreted as the partition function of a free non-relativistic particle in a frame that rotates at imaginary angular velocity $i\theta/\beta$. There are at least two equivalent ways to evaluate it. The first is to compute a time-sliced path integral

$$Z(\beta, \theta) = \int\limits_{x(\beta)=x(0)} \mathcal{D}x\mathcal{D}p \, \exp\left[-\int_0^\beta d\tau \left(-ip_j\dot{x}^j + \frac{p^2}{2M} - i\theta(x^1p_2 - x^2p_1)\right)\right] \tag{5.156}$$

where $\mathcal{D}x\mathcal{D}p$ is the standard path integral measure of quantum mechanics. In the argument of the exponential we recognize the Euclidean section of (5.153) supplemented by a term proportional to θJ. Expression (5.156) may thus be seen as the canonical partition function (3.50) of a system with effective Hamiltonian $\hat{H}_{\text{eff}} = \hat{H} - \frac{i\theta}{\beta}\hat{J}$. The second way is to realize that the operator $\hat{J}$ generates rotations in the plane. Thus if we introduce a basis of states $|x^1, x^2\rangle$ localized at (x^1, x^2), the trace (5.154) is a (finite-dimensional) integral

$$Z(\beta, \theta) = \int_{\mathbb{R}^2} dx^1 dx^2 \big\langle R_\theta \cdot (x^1, x^2) \big| e^{-\beta H} \big| x^1, x^2 \big\rangle \tag{5.157}$$

where $R_\theta \cdot (x^1, x^2)$ denotes the action of a rotation by θ on the vector (x^1, x^2). From this second viewpoint, the partition function is a trace over transition amplitudes between initial and final states that are rotated with respect to each other. Since transition amplitudes can be written as path integrals, expression (5.157) is a path integral in disguise and takes the same form as (5.156) up to two key differences: (i) the term $i\theta J$ no longer appears in the exponential, and (ii) the periodicity condition on paths is $x(\beta) = R_\theta \cdot x(0)$ instead of $x(\beta) = x(0)$.

The two methods just described give identical results, but we pick the second one for simplicity. Recall that the propagator of a free massive particle on a plane is (in Dirac notation)

$$\big\langle x', t \big| e^{-iHt} \big| x, 0 \big\rangle = \frac{M}{2\pi i t} \exp\left[\frac{iM|x' - x|^2}{2t}\right] \tag{5.158}$$

where $|\cdot|$ is the Euclidean norm. From this we find the Euclidean propagator

$$\big\langle R_\theta \cdot x, t \big| e^{-\beta H} \big| x, 0 \big\rangle = \frac{M}{2\pi\beta} \exp\left[-\frac{M}{2\beta}(1 - \cos\theta) x^2\right] \tag{5.159}$$

where $x^2 \equiv |x|^2$. To obtain the partition function (5.157) we integrate (5.159):

$$Z(\beta, \theta) = \int_{\mathbb{R}^2} d^2x \, \frac{M}{2\pi\beta} \, \exp\left[-\frac{M}{2\beta}(1 - \cos\theta) x^2\right]. \tag{5.160}$$

For $\theta \neq 0$ (modulo 2π) this is just a Gaussian integral and the result is precisely a character (4.123) with $r = 1$. We conclude that the space obtained by quantizing a massive coadjoint orbit of the Bargmann group coincides with the Hilbert space of a free, massive, non-relativistic particle.

Remark Having seen the computation of the trace of $e^{-\beta H + i\theta J}$ in Bargmann representations, one may wonder if the result can be analytically continued to the grand canonical partition function

$$Z(\beta, \Omega) = \text{Tr}\left(e^{-\beta(H - \Omega J)}\right) \tag{5.161}$$

where Ω is a *real* angular velocity, describing the thermodynamics of a system in a real rotating frame. This corresponds to taking $\theta = -i\beta\Omega$ purely imaginary in (5.154). If we were to evaluate (5.161), we would be led to expression (5.159) with $1 - \cos\theta = 1 - \cosh(\beta\Omega) < 0$, which is a serious problem: the integral (5.160) would diverge. Intuitively this divergence is due to the fact that free particles move all over space without any potential that prevents them from escaping to infinity when put in a rotating frame. This divergence is typical of rotating characters in flat space and can also be seen in the Poincaré characters of Sect. 4.2. By contrast, the partition function (5.161) of a two-dimensional harmonic oscillator is well-defined as long as the angular velocity Ω is smaller than the oscillator's natural frequency.

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Part II
Virasoro Symmetry and AdS_3 Gravity

In this part, we initiate the study of infinite-dimensional symmetry groups by analysing the group of diffeomorphisms of the circle, whose central extension is the Virasoro group. Upon defining the latter, we classify its coadjoint orbits, i.e. orbits of CFT stress tensors under conformal transformations in two dimensions. As an application, we show how Virasoro symmetry is realized in asymptotically Anti-de Sitter gravity in three dimensions and interpret unitary representations of the Virasoro algebra from a gravitational perspective. Note that Virasoro coadjoint orbits will play a key role for BMS_3 particles in part III, as they will coincide with their supermomentum orbits.

Chapter 6
The Virasoro Group

In the first part of this thesis we have introduced some general tools for dealing with symmetries in quantum mechanics. Our goal is to eventually apply these tools to the BMS_3 group in three dimensions. Accordingly, in this chapter and the two next ones we address a necessary prerequisite for these considerations by studying the central extension of the group of diffeomorphisms of the circle, i.e. the Virasoro group. The latter is part of the asymptotic symmetry group of many gravitational systems, where it essentially consists of conformal transformations of celestial circles. It also accounts for the symmetries of two-dimensional conformal field theories and thus illuminates certain aspects of holography in general, and AdS_3/CFT_2 in particular.

A word of caution is in order at the outset regarding the interpretation of the Virasoro group from a gravitational viewpoint. While diffeomorphisms in general relativity are generally thought of as gauge redundancies, the group $\text{Diff}(S^1)$ that we shall study here should by no means be understood in that sense. On the contrary, it should be interpreted as a *global* space-time symmetry group on a par with $SL(2,\mathbb{R})$ or the Poincaré group. In fact, in the BMS_3 case, $\text{Diff}(S^1)$ will be an infinite-dimensional extension of the Lorentz group in three dimensions. Accordingly this chapter and the next one may be seen as a detailed investigation of a group that extends Lorentz symmetry in an infinite-dimensional way.

Our plan for this chapter is the following. In Sect. 6.1 we define the group $\text{Diff}(S^1)$ of diffeomorphisms of the circle as an infinite-dimensional Lie group, and we describe its adjoint representation, its Lie algebra $\text{Vect}(S^1)$, and its coadjoint representation. Section 6.2 is devoted to its cohomology; in particular we introduce the Gelfand-Fuks cocycle and its integral, the Bott-Thurston cocycle, which respectively define the Virasoro algebra and the Virasoro group. In Sect. 6.3 we study the Schwarzian derivative, which will lead to a unified picture of Virasoro cohomology. Finally, in Sect. 6.4 we define the Virasoro group and work out its adjoint and coadjoint representations; the latter coincides with the transformation law of two-dimensional CFT stress tensors under conformal transformations.

B. Oblak, *BMS Particles in Three Dimensions*, Springer Theses,
DOI 10.1007/978-3-319-61878-4_6

Regarding references, the holy book on the Virasoro group is [1] while [2] is a pedagogical introduction to infinite-dimensional group theory. Some familiarity with two-dimensional CFT may come in handy at this stage; see e.g. [3–5].

6.1 Diffeomorphisms of the Circle

In this section we study the elementary properties of the group $\text{Diff}(S^1)$. We first briefly mention issues related to infinite-dimensional Lie groups, then define $\text{Diff}(S^1)$ and show that its Lie algebra consists of vector fields on the circle. We also introduce densities on the circle, i.e. primary fields, display the coadjoint representation of $\text{Diff}(S^1)$, and discuss certain properties of the exponential map.

6.1.1 Infinite-Dimensional Lie Groups

The diffeomorphisms of any manifold depend on an infinity of parameters and therefore span an infinite-dimensional group. One would like this group to be smooth in a certain sense, which leads to the problem of defining infinite-dimensional Lie groups and manifolds. Here we review this question in broad terms; we refer e.g. to [6] for a much more complete presentation.

In the same way that any finite-dimensional manifold looks locally like $\mathbb{R}^n$, one would like to find the prototypical infinite-dimensional topological vector space $\mathbb{V}$ such that infinite-dimensional manifolds be locally homeomorphic to $\mathbb{V}$. As it turns out, taking $\mathbb{V}$ to be a *Fréchet space* leads to a well-defined theory of differentiation and smoothness, which can then be used to define Fréchet manifolds. Roughly speaking, Fréchet spaces are vector spaces that generalize Banach spaces. For example the space $C^\infty(\mathcal{M})$ of smooth functions on a finite-dimensional manifold $\mathcal{M}$ is a Fréchet space (but *not* a Banach space). A *Lie-Fréchet group* then is a group endowed with a structure of Fréchet manifold such that multiplication and inversion are smooth. For instance the group $\text{Diff}(\mathcal{M})$ of diffeomorphisms of a compact finite-dimensional manifold $\mathcal{M}$ is a Lie-Fréchet group. From now on we refer to infinite-dimensional Lie-Fréchet groups simply as "infinite-dimensional groups".

Infinite-dimensional manifolds are strikingly different from finite-dimensional ones in many respects. For example the notion of "tangent vectors" is ambiguous in infinite dimension, and the lack of existence/uniqueness theorems makes other seemingly obvious definitions fail, such as the notion of integral curves. We will encounter a similarly counter-intuitive phenomenon below, when explaining that the exponential that maps vector fields on diffeomorphisms is not locally surjective, in contrast with its finite-dimensional counterpart.

In the remainder of this section we deal with the group of diffeomorphisms of the circle as an infinite-dimensional Lie(-Fréchet) group. In particular we will think of its Lie algebra as its tangent space at the identity, identified with the space of left-invariant vector fields, from which the remaining definitions will follow.

6.1.2 The Group of Diffeomorphisms of the Circle

We consider the unit circle $S^1 = \left\{e^{i\varphi} \in \mathbb{C} \middle| \varphi \in [0, 2\pi[\right\}$. Its fundamental group is isomorphic to $\mathbb{Z}$ and its universal cover is the real line $\mathbb{R}$, with a projection

$$\mathsf{p} : \mathbb{R} \to S^1 : \varphi \mapsto e^{i\varphi} \tag{6.1}$$

depicted in Fig. 2.1. This allows us to think of S^1 as the quotient $\mathbb{R}/2\pi\mathbb{Z}$ of the real line by the equivalence relation $\varphi \sim \varphi + 2\pi$, since the kernel of p consists of translations of $\mathbb{R}$ by integer multiples of 2π.

Diffeomorphisms in the Complex Plane

A *diffeomorphism of the circle* is a smooth bijection $F : S^1 \to S^1$ whose inverse is also smooth. We denote the group of all such maps by $\text{Diff}(S^1)$, with the group operation given by composition:

$$F \cdot G \equiv F \circ G \qquad \forall\, F, G \in \text{Diff}(S^1). \tag{6.2}$$

$\text{Diff}(S^1)$ is an infinite-dimensional Lie group that inherits its smooth structure from that of the Fréchet manifold of smooth maps $S^1 \to S^1$. Given an orientation on S^1, diffeomorphisms may preserve it or break it. In particular the set of diffeomorphisms that preserve orientation is a subgroup of $\text{Diff}(S^1)$, denoted $\text{Diff}^+(S^1)$ and called the group of *orientation-preserving diffeomorphisms of the circle*. We will prove below that $\text{Diff}^+(S^1)$ is connected.

For practical purposes it is useful to describe diffeomorphisms of the circle in terms of the 2π-periodic coordinate φ of (6.1). A diffeomorphism then is a map $F : e^{i\varphi} \mapsto F(e^{i\varphi})$ where $F(e^{i\varphi})$ has unit norm. As an example one can verify that the set of transformations of the form

$$F(e^{i\varphi}) = \frac{Ae^{i\varphi} + B}{\bar{B}e^{i\varphi} + \bar{A}}\,, \qquad |A|^2 - |B|^2 = 1 \tag{6.3}$$

is a subgroup of $\text{Diff}^+(S^1)$ isomorphic to the connected Lorentz group in three dimensions, $\text{SO}(2,1)^{\uparrow} \overset{(4.83)}{\cong} \text{PSL}(2,\mathbb{R})$. Rigid rotations are given by $A = e^{i\theta/2}$ and $B = 0$:

$$F(e^{i\varphi}) = e^{i(\varphi+\theta)} = e^{i\theta}e^{i\varphi}. \tag{6.4}$$

Similarly, the typical orientation-changing diffeomorphism is the parity transformation

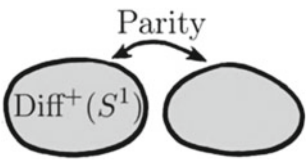

Fig. 6.1 The two connected components of Diff(S^1) are related by parity. Compare with the connected components of the Lorentz group in Fig. 4.2

$$F(e^{i\varphi}) = e^{-i\varphi}. \tag{6.5}$$

Any parity-changing diffeomorphism of the circle can be written as the composition of (6.5) with an orientation-preserving transformation. There appears to be no analogue of time-reversal in Diff(S^1). All in all, Diff$^+(S^1)$ is an infinite-dimensional cousin of the connected Lorentz group in three dimensions, $\mathrm{SO}(2,1)^\uparrow$, while Diff($S^1$) extends the orthochronous Lorentz group $\mathrm{O}(2,1)^\uparrow$. See also Fig. 6.1.

Diffeomorphisms in Real Coordinates

Given a diffeomorphism $F : S^1 \to S^1$, there exists a diffeomorphism $f : \mathbb{R} \to \mathbb{R}$ of the real line such that

$$F(e^{i\varphi}) = e^{if(\varphi)}, \qquad \text{i.e.} \qquad F \circ \mathsf{p} = \mathsf{p} \circ f \tag{6.6}$$

in terms of the projection (6.1). In order for f to be compatible with the periodicity of φ, we must require that $f(\varphi + 2\pi) = f(\varphi) \pm 2\pi$, where the plus sign corresponds to an orientation-preserving diffeomorphism while the minus sign corresponds to an orientation-changing one. In this language the rotation (6.4) corresponds to $f(\varphi) = \varphi + \theta$ while the parity transformation (6.5) is $f(\varphi) = -\varphi$.

Definition A smooth map $f : \mathbb{R} \to \mathbb{R}$ is $2\pi\mathbb{Z}$-*equivariant* if $f(\varphi + 2\pi) = f(\varphi) + 2\pi$. Any such map can be written as $f(\varphi) = \varphi + u(\varphi)$, where u is 2π-periodic.

In these terms, any orientation-preserving diffeomorphism F of the circle is a projection (6.6) of a $2\pi\mathbb{Z}$-equivariant diffeomorphism f of the real line, that is, a smooth function $f : \mathbb{R} \to \mathbb{R}$ such that

$$\boxed{f'(\varphi) > 0\,, \qquad f(\varphi + 2\pi) = f(\varphi) + 2\pi} \tag{6.7}$$

for any $\varphi \in \mathbb{R}$, where prime denotes differentiation with respect to φ. The group operation (6.2) then becomes

$$f \cdot g = f \circ g \tag{6.8}$$

where f, g correspond to F, G according to (6.6). From now on we always describe Diff(S^1) in terms of diffeomorphisms of $\mathbb{R}$ satisfying the properties (6.7). By the way, this is why we have kept writing group elements as "f" throughout this thesis.

Note that the diffeomorphism F does not determine f uniquely: one can add to $f(\varphi)$ an arbitrary constant multiple of 2π without affecting $F = e^{if}$. This ambiguity can be removed by requiring e.g. that $f(0)$ belongs to the interval $[0, 2\pi[$. One says that f is a *lift* of F, and there are infinitely many lifts for a given F. For our purposes it is only important that giving f determines $F = e^{if}$ uniquely, so that we can consistently write all orientation-preserving diffeomorphisms of the circle in the form (6.7).

6.1.3 Topology of Diff(S^1)

Lemma The group $\text{Diff}^+(S^1)$ of orientation-preserving diffeomorphisms is connected, and $\text{Diff}(S^1)$ has two connected components related by parity.

Proof Let $f(\varphi)$ be a diffeomorphism of $\mathbb{R}$ that satisfies (6.7), and consider the corresponding diffeomorphism of the circle given by (6.6). We wish to show that there exists a continuous path that connects f to the identity. Consider therefore the one-parameter family of functions

$$f_t(\varphi) = (1-t)f(\varphi) + t\varphi\,, \qquad t \in [0,1]\,. \tag{6.9}$$

For each t, f_t satisfies (6.7) and therefore defines a diffeomorphism of the circle. At $t = 0$ it coincides with f while at $t = 1$ it is the identity. See Fig. 6.2. ∎

For the purposes of representation theory it is important to know the fundamental group of $\text{Diff}^+(S^1)$, as it determines whether $\text{Diff}^+(S^1)$ has topological central extensions. In that context the key result is the following:

Lemma $\text{Diff}^+(S^1)$ is homotopic to a circle, so its fundamental group is

$$\pi_1\left(\text{Diff}^+(S^1)\right) \cong \mathbb{Z}. \tag{6.10}$$

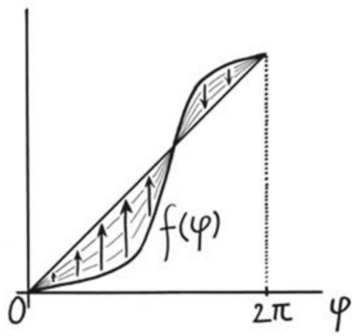

Fig. 6.2 The homotopy (6.9) turns a diffeomorphism $f(\varphi)$ (here leaving the point $\varphi = 0$ fixed) into the identity. It implies both that the group $\text{Diff}^+(S^1)$ is connected and that it is homotopic to a circle

Its universal cover $\widetilde{\text{Diff}}^+(S^1)$ is the group of $2\pi\mathbb{Z}$-equivariant diffeomorphisms of $\mathbb{R}$, with the projection given by (6.6).

Proof We follow [7]. The key to the proof is to realize that $\text{Diff}^+(S^1)$ is homotopic to its subgroup $\text{Isom}^+(S^1)$ of orientation-preserving isometries of the circle (for the standard flat metric). Since $\text{Isom}^+(S^1)$ is a group U(1) of rigid rotations, it will follow that $\text{Diff}^+(S^1)$ has the homotopy type of a circle and therefore has a fundamental group $\mathbb{Z}$. So let us prove the homotopy equivalence $\text{Diff}^+(S^1) \sim \text{Isom}^+(S^1)$. Call $\text{Diff}_0^+(S^1)$ the group of orientation-preserving diffeomorphisms of the circle leaving the point $\varphi = 0$ fixed. Since isometries of S^1 are rotations, there exists a decomposition

$$\text{Diff}^+(S^1) = \text{Diff}_0^+(S^1) \cdot \text{Isom}^+(S^1). \tag{6.11}$$

Indeed, any diffeomorphism of the circle is the composition of a rigid rotation with a diffeomorphism leaving $\varphi = 0$ fixed; both $\text{Diff}_0^+(S^1)$ and $\text{Isom}^+(S^1)$ are groups and their intersection only contains the identity. Now note that any diffeomorphism preserving $\varphi = 0$ admits a unique lift f such that $f(0) = 0$ and $f(2\pi) = 2\pi$, so we can think of $\text{Diff}_0^+(S^1)$ as the set of $2\pi\mathbb{Z}$-equivariant diffeomorphisms of $\mathbb{R}$ that fix the point $\varphi = 0$; this identification is one-to-one. It only remains to observe that the maps (6.9) define a homotopy whose effect at $t = 1$ is to retract the whole $\text{Diff}_0^+(S^1)$ on the identity. As a result $\text{Diff}_0^+(S^1)$ is homotopic to a point, and so by (6.11) $\text{Diff}^+(S^1)$ is homotopic to a circle. Unwinding this circle gives rise to the group of $2\pi\mathbb{Z}$-equivariant diffeomorphisms of $\mathbb{R}$, which therefore span the universal cover of $\text{Diff}^+(S^1)$. ■

This lemma confirms the interpretation of $\text{Diff}^+(S^1)$ as an infinite-dimensional analogue of $\text{PSL}(2, \mathbb{R})$, since the latter is also homotopic to a circle (see Sect. 4.3). In particular formula (6.11) is the $\text{Diff}(S^1)$ analogue of the Iwasawa decomposition (4.81) of $\text{SL}(2, \mathbb{R})$. Since $\text{Diff}_0^+(S^1)$ has the homotopy type of a point, the group $\text{Diff}^+(S^1)$ may be seen as an infinite-dimensional cylinder $S^1 \times \mathbb{R}^\infty$ where S^1 consists of rigid rotations while $\mathbb{R}^\infty$ is spanned by infinite-dimensional generalizations of boosts. Note that property (6.10) implies the existence of topological projective representations of $\text{Diff}(S^1)$. Applied to BMS_3, it will imply that the spin of massive particles is not quantized (as in the Poincaré group in three dimensions).

In what follows we focus on the universal cover $\widetilde{\text{Diff}}^+(S^1)$ rather than $\text{Diff}(S^1)$ or $\text{Diff}^+(S^1)$, except if explicitly stated otherwise. To reduce clutter we will abuse notation by writing $\text{Diff}(S^1)$ for the universal cover, instead of the more accurate notation $\widetilde{\text{Diff}}^+(S^1)$. Accordingly, from now on elements of $\text{Diff}(S^1)$ are diffeomorphisms f, g, etc. of the real line satisfying the properties (6.7). The inverse of f will be denoted f^{-1} and is such that $f(f^{-1}(\varphi)) = f^{-1}(f(\varphi)) = \varphi$.

6.1.4 Adjoint Representation and Vector Fields

We can now look for the Lie algebra of $\text{Diff}(S^1)$, which is identified with the tangent space at the identity and corresponds to infinitesimal diffeomorphisms. It is intuitively clear that this algebra is a space of functions, since a diffeomorphism close to the identity can be written as

$$f(\varphi) = \varphi + \epsilon X(\varphi) \tag{6.12}$$

where ϵ is "small" and $X(\varphi)$ is a function on the circle. A more subtle problem is to determine the adjoint action of diffeomorphisms on this Lie algebra, and to deduce the expression of the Lie bracket. In order to work this out we pick a path $\gamma : \mathbb{R} \to \text{Diff}(S^1) : t \mapsto \gamma_t$ such that γ_0 is the identity and

$$\gamma_t(\varphi) = \varphi + tX(\varphi) + \mathcal{O}(t^2) \tag{6.13}$$

for small t. The adjoint representation is defined by (5.6), so we find

$$\big(\text{Ad}_f(X)\big)(\varphi) = \frac{d}{dt}\Big[f\big(\gamma_t\left(f^{-1}(\varphi)\right)\big)\Big]\Big|_{t=0} \overset{(6.13)}{=} \frac{d}{dt}\Big[f\big(f^{-1}(\varphi) + tX\left(f^{-1}(\varphi)\right)\big)\Big]\Big|_{t=0}. \tag{6.14}$$

Since t is "small" in this expression, we can Taylor expand

$$f\left(f^{-1}(\varphi) + tX\left(f^{-1}(\varphi)\right)\right) = \varphi + tX\left(f^{-1}(\varphi)\right) f'\left(f^{-1}(\varphi)\right) + \mathcal{O}(t^2) \tag{6.15}$$

where we have used $f(f^{-1}(\varphi)) = \varphi$. The derivative of the latter equation implies

$$f'(f^{-1}(\varphi)) = \frac{1}{(f^{-1})'(\varphi)} \tag{6.16}$$

which can be plugged into (6.15) and thus provides the adjoint representation

$$\big(\text{Ad}_f(X)\big)(\varphi) = \frac{X\left(f^{-1}(\varphi)\right)}{(f^{-1})'(\varphi)}. \tag{6.17}$$

This formula is the transformation law of $X(\varphi)$ under a diffeomorphism $\varphi \mapsto f(\varphi)$. It shows in particular that $X(\varphi)$ in (6.12) is *not* just a function on the circle, due to the derivative of f^{-1} in (6.17). Using (6.16), we can also rewrite it by evaluating the left-hand side at $f(\varphi)$ rather than φ:

$$\big(\text{Ad}_f(X)\big)(f(\varphi)) = f'(\varphi)X(\varphi) \tag{6.18}$$

We recognize here the transformation law of the component $X(\varphi)$ of a vector field

$$X(\varphi)\frac{\partial}{\partial\varphi} \tag{6.19}$$

under a diffeomorphism f. We shall denote by $\text{Vect}(S^1)$ the space of smooth vector fields on S^1, whose elements will be written as X, Y, etc. We have just shown that $\text{Vect}(S^1)$ is the Lie algebra of $\text{Diff}(S^1)$; it now remains to find the Lie bracket.

Take once more a path γ_t in $\text{Diff}(S^1)$ that satisfies (6.13). Picking a vector field $Y \in \text{Vect}(S^1)$ and a point $\varphi \in [0, 2\pi]$, let us evaluate

$$(\text{ad}_X(Y))(\varphi) \stackrel{(5.8)}{=} \frac{d}{dt}\big(\text{Ad}_{\gamma_t}(Y)\big)\big|_{t=0}(\varphi) \stackrel{(6.17)}{=} \frac{d}{dt}\left(\frac{Y(\gamma_t^{-1}(\varphi))}{(\gamma_t^{-1})'(\varphi)}\right)\bigg|_{t=0}. \tag{6.20}$$

Here (6.13) implies $(\gamma_t^{-1})'(\varphi) = 1 - tX'(\varphi)$ as well as $Y(\gamma_t^{-1}(\varphi)) = Y(\varphi) - tX(\varphi)Y'(\varphi)$ to first order in t. Plugging these expressions in (6.20) we obtain $\text{ad}_X Y = -XY' + YX'$, where it is understood that both sides are evaluated at the same point φ. We conclude that the Lie bracket of $\text{Vect}(S^1)$, seen as the Lie algebra of the group $\text{Diff}(S^1)$, is the *opposite* of the standard Lie bracket of vector fields:

$$[X, Y]_{\text{Lie algebra}} = -[X, Y]_{\text{Vector fields}}. \tag{6.21}$$

This is in fact a common phenomenon: as a consequence of (5.31), the group $\text{Diff}(\mathcal{M})$ of diffeomorphisms of a (compact, finite-dimensional) manifold $\mathcal{M}$ is a Lie-Fréchet group whose Lie algebra is the space $\text{Vect}(\mathcal{M})$ endowed with the *opposite* of the standard Lie bracket of vector fields [1, 2, 8]. Thus:

Proposition The Lie algebra of the group $\text{Diff}(S^1)$ is the space $\text{Vect}(S^1)$ of vector fields on the circle, with the Lie bracket (6.21) given by the opposite of the standard Lie bracket of vector fields.

In what follows we will bluntly neglect the sign subtlety and endow $\text{Vect}(S^1)$ with the usual bracket

$$[X, Y] = \big(X(\varphi)Y'(\varphi) - Y(\varphi)X'(\varphi)\big)\frac{\partial}{\partial\varphi}. \tag{6.22}$$

This is a harmless abuse of conventions and may be seen as an alternative definition of the Lie bracket for groups of diffeomorphisms. With that abuse the Lie algebra of $\text{Diff}(S^1)$ becomes $\text{Vect}(S^1)$ with the *usual* Lie bracket of vector fields.

Witt Algebra

Since all functions on the circle can be expanded in Fourier series, any vector field is a (generally infinite) complex linear combination of generators

$$\ell_m \equiv e^{im\varphi}\partial_\varphi, \qquad m \in \mathbb{Z}. \tag{6.23}$$

The brackets (6.22) of these generators can be written as

$$i[\ell_m, \ell_n] = (m - n)\ell_{m+n}, \tag{6.24}$$

where one may recognize the *Witt algebra* of conformal field theory. It is an infinite-dimensional extension of the $\mathfrak{sl}(2, \mathbb{R})$ algebra (5.90) spanned by $\ell_{-1}, \ell_0, \ell_1$. The latter consists of vector fields $X(\varphi)\partial_\varphi$ with $X(\varphi) = X_0 + X_1 \cos\varphi + X_2 \sin\varphi$ for $X_\mu \in \mathbb{R}$, and generates diffeomorphisms of the form (6.3). In particular the constant vector field ℓ_0 generates rigid rotations of the circle.

6.1.5 Primary Fields on the Circle

Formula (6.18) gives the transformation law of vector fields on the circle under diffeomorphisms. Similarly, a one-form $\alpha(\varphi)d\varphi$ would transform as $\alpha \mapsto f \cdot \alpha$, where

$$(f \cdot \alpha)(f(\varphi)) = \frac{\alpha(\varphi)}{f'(\varphi)}\,, \tag{6.25}$$

while a function $\alpha(\varphi)$ would simply transform as $(f \cdot \alpha)(f(\varphi)) = \alpha(\varphi)$. Vector fields, one-forms and functions can all be seen as sections of suitable vector bundles on the circle, which suggests that they can be generalized to sections of tensor product bundles such as $TS^1 \otimes \cdots \otimes TS^1$ or $T^*S^1 \otimes \cdots \otimes T^*S^1$.

Definition A *density* of weight $h \in \mathbb{R}$ on the circle is an expression of the form

$$\alpha = \alpha(\varphi)(d\varphi)^h \tag{6.26}$$

where $\alpha(\varphi)$ is a smooth function on the circle; it acts on the tangent space $T_\varphi S^1$ according to $\langle \alpha(\varphi)d\varphi^h, V\partial_\varphi \rangle \equiv \alpha(\varphi)V^h$. We denote by $\mathcal{F}_h(S^1)$ the vector space of densities of weight h.

When h is an integer, a density of weight h is a section of

$$\underbrace{T^*S^1 \otimes \cdots \otimes T^*S^1}_{|h| \text{ times}} \quad \text{if } h \geq 0, \qquad \text{or} \qquad \underbrace{TS^1 \otimes \cdots \otimes TS^1}_{|h| \text{ times}} \quad \text{if } h < 0.$$

In particular, a density is a vector field when $h = -1$, a one-form when $h = 1$, and a function when $h = 0$. The definition (6.26) generalizes these notions to arbitrary real values of h. The notation α is justified by the fact that in the BMS_3 group, supertranslations will be densities with weight -1.

Expression (6.26) suggests that the density $\alpha(\varphi)(d\varphi)^h$ is a coordinate-independent quantity. Indeed, under a diffeomorphism $f : \varphi \mapsto f(\varphi)$, it transforms as

$$(f \cdot \alpha)(\varphi) \equiv \big((f^{-1})'(\varphi)\big)^h \,\alpha(f^{-1}(\varphi)) \tag{6.27}$$

or equivalently as

$$(f \cdot \alpha)(f(\varphi)) \equiv \frac{\alpha(\varphi)}{(f'(\varphi))^h}\,. \tag{6.28}$$

This reduces to (6.18) for $h = -1$ and to (6.25) for $h = 1$. If we think of $f(\varphi)$ as a "conformal transformation" of the circle, then Eq. (6.28) coincides with the transformation law of a (chiral) primary field of weight h. It provides a representation of $\text{Diff}(S^1)$ in the space $\mathcal{F}_h(S^1)$. This representation is infinite-dimensional and generally non-unitary because the would-be "scalar product"

$$\int_0^{2\pi} d\varphi\, \alpha(\varphi)\beta(\varphi)\,, \qquad \alpha, \beta \in \mathcal{F}_h(S^1) \tag{6.29}$$

is not left invariant by (6.28) for generic values of h. The only exception is the case of spinor fields, $h = 1/2$. One can think of (6.28) as a $\text{Diff}(S^1)$ generalization of the various finite-dimensional (but non-unitary) irreducible representations of the Lorentz group. The number h can then be thought of as a spin label, in the same way that finite-dimensional Lorentz representations correspond to transformation laws of relativistic fields with definite spin. (Beware: the word "spin" here does not refer to the notion of "spin" encountered in representations of semi-direct products. These two notions are related in that the Lorentz spin of a quantum field determines the Poincaré spin of the corresponding particles, but they are nevertheless different concepts.)

From (6.28) one can read off the transformation law of densities under infinitesimal diffeomorphisms, that is, under vector fields on the circle. Taking $f(\varphi) = \varphi + \epsilon X(\varphi)$ in (6.28) one finds, to first order in ϵ,

$$(f \cdot \alpha)(\varphi) = \alpha(\varphi) - \epsilon\big[X(\varphi)\alpha'(\varphi) + h\,\alpha(\varphi)X'(\varphi)\big]. \tag{6.30}$$

We then define

$$X \cdot \alpha(\varphi) \equiv -\frac{(f \cdot \alpha)(\varphi) - \alpha(\varphi)}{\epsilon} \tag{6.31}$$

and obtain

$$X \cdot \alpha = X\alpha' + h\,\alpha X'. \tag{6.32}$$

As before, one may recognize here the infinitesimal transformation law of a primary field α of weight h under an conformal transformation generated by X.

6.1.6 Coadjoint Representation of Diff(S^1)

Dual Spaces

We mentioned around (6.29) that the integral of the product of two densities with the same weight is generally not invariant under diffeomorphisms. However, there does exist a $\text{Diff}(S^1)$-invariant pairing of densities. Indeed, consider the space $\mathcal{F}_h(S^1)$ of densities of weight h. Its dual space consists of all linear forms

$$p : \mathcal{F}_h(S^1) \to \mathbb{R} : \alpha \mapsto \langle p, \alpha \rangle \,. \tag{6.33}$$

Since $\mathcal{F}_h(S^1)$ is infinite-dimensional, its dual space is pathological: the map (6.33) need not be continuous and therefore does not preserve the differentiable structure of $\mathcal{F}_h(S^1)$. Accordingly one generally restricts attention to the space of continuous linear forms (6.33); the latter coincides with the space of distributions on the circle. In addition, for concrete computations it is much more convenient to consider only *regular* distributions, that is, distributions that can be written in the form

$$\langle p, \alpha \rangle = \frac{1}{2\pi} \int_0^{2\pi} d\varphi \, p(\varphi)\alpha(\varphi) \tag{6.34}$$

where $p(\varphi)$ is a smooth function on the circle. We will call the space of such distributions the *smooth* or *regular dual* of $\mathcal{F}_h(S^1)$. As a vector space, it is isomorphic to the space $C^\infty(S^1)$ of smooth functions on the circle. Note that any distribution can be obtained as the limit of a sequence of regular distributions, so in this sense we are not missing anything even when restricting attention to regular distributions. The regular dual of $\mathcal{F}_h(S^1)$ will be denoted as $\mathcal{F}_h(S^1)^*$. The notation in (6.33) is justified by the fact that, in BMS_3, p will be an infinite-dimensional supermomentum vector dual to supertranslations.

Since $\mathcal{F}_h(S^1)$ carries a representation (6.28), it is natural to ask how the dual representation (4.16) acts on the regular dual. By definition, one has $\langle f \cdot p, \alpha \rangle = \left\langle p, f^{-1} \cdot \alpha \right\rangle$ for any diffeomorphism f, any density $\alpha \in \mathcal{F}_h(S^1)$ and any smooth distribution $p \in \mathcal{F}_h(S^1)^*$. Using (6.27) and the pairing (6.34), we get

$$\langle f \cdot p, \alpha \rangle = \frac{1}{2\pi} \int_0^{2\pi} d\varphi \, p(\varphi)(f'(\varphi))^h \alpha(f(\varphi)) \,. \tag{6.35}$$

If now we rewrite $\langle f \cdot p, \alpha \rangle$ as an integral (6.34) with the integration variable φ replaced by $f(\varphi)$, the condition that $\langle f \cdot p, \alpha \rangle$ matches the right-hand side of (6.35) for any α readily provides

$$(f \cdot p)(f(\varphi)) = \frac{p(\varphi)}{(f'(\varphi))^{1-h}} \,.$$

This is the transformation law (6.28) with h replaced by $1 - h$:

Proposition There is an isomorphism $\mathcal{F}_h(S^1)^* \cong \mathcal{F}_{1-h}(S^1)$ which is compatible with the natural action of $\mathrm{Diff}(S^1)$ on these spaces. In addition, the pairing between $\mathcal{F}_h(S^1)$ and $\mathcal{F}_{1-h}(S^1)$ is $\mathrm{Diff}(S^1)$-invariant in the sense that $\langle f \cdot p, f \cdot \alpha \rangle = \langle p, \alpha \rangle$ for all $f \in \mathrm{Diff}(S^1)$ and all densities $p \in \mathcal{F}_{1-h}(S^1)$, $\alpha \in \mathcal{F}_h(S^1)$.

Thus the duals of densities with weight h are densities with weight $1 - h$, and vice-versa. One can apply this to the examples encountered above:

- the duals of vector fields ($h = -1$) are quadratic densities ($h = 2$);
- the duals of functions ($h = 0$) are one-forms ($h = 1$);

- the duals of spinor fields ($h = 1/2$) are spinor fields (i.e. $\mathcal{F}_{1/2}(S^1)$ is self-dual).

Note that, for all values of h except $h = 1/2$, the conformally invariant pairing (6.34) is *not* a scalar product since its arguments are densities whose transformation laws under $\text{Diff}(S^1)$ differ. This should be contrasted with the finite-dimensional examples encountered in Chap. 4, where the existence of an invariant bilinear form led to the equivalence of σ^* and σ. We shall see below that this difference is crucial for coadjoint orbits of the Virasoro group (Sect. 7.1) and hence for the supermomentum orbits of the BMS_3 group (see part III).

Coadjoint Representation

Since the adjoint representation of $\text{Diff}(S^1)$ is the transformation law (6.18) of vector fields, we now know that the *coadjoint* representation of $\text{Diff}(S^1)$ is the transformation law of quadratic densities:

$$(\text{Ad}^*_f p)(f(\varphi)) = \frac{p(\varphi)}{(f'(\varphi))^2} \tag{6.36}$$

This can also be written infinitesimally as

$$\text{ad}^*_X p = Xp' + 2X'p. \tag{6.37}$$

In CFT terminology, vector fields are infinitesimal conformal transformations and their duals are (quasi-)primary fields with weight $h = 2$, that is, CFT stress tensors. Indeed formula (6.36) is the transformation law of a stress tensor $p(\varphi)$ if we think of the map $\varphi \mapsto f(\varphi)$ as a conformal transformation. Similarly, the duals of functions are primary fields with weight $h = 1$, i.e. currents. From now on we sometimes refer to $\text{Diff}(S^1)$-invariance as "conformal invariance". Note that at this point we haven't included any central charge yet; this will change once we turn to the Virasoro group.

6.1.7 *Exponential Map and Vector Flows*

We mentioned above that each vector field $X(\varphi)\partial_\varphi$ may be seen as an infinitesimal diffeomorphism; let us make this more precise. Given a vector field $X(\varphi)\partial_\varphi$, its integral curves are paths $\varphi(t)$ on the circle that satisfy the evolution equation

$$\dot{\varphi}(t) = X(\varphi(t)). \tag{6.38}$$

In particular, when $t = \epsilon$ is "small" one finds that $\varphi(\epsilon)$ takes the form (6.12) with initial condition $\varphi(0) = \varphi$. Equation (6.38) is an ordinary differential equation in one dimension and $X(\varphi)$ is smooth, so given an initial condition $\varphi(0)$, the solution exists and is unique. We define the *flow* of X as the one-parameter family of diffeomorphisms that maps a "time" t and an initial condition φ on the point $\varphi(t)$ obtained by solving (6.38) with this initial condition. If we call this solution $\tilde{\varphi}(t,\varphi)$, then the flow of X is

$$\phi_X : \mathbb{R} \times S^1 \to S^1 : (t,\varphi) \mapsto \tilde{\varphi}(t,\varphi). \tag{6.39}$$

For example the flow of the constant vector field $X(\varphi) = 1$ is given by $\tilde{\varphi}(t) = \varphi + t$ and consists of rigid rotations by t, as already anticipated above. Using the notion of flow, one can define an exponential map for Diff(S^1):

Definition The *exponential map* of the group Diff(S^1) is

$$\exp : \text{Vect}(S^1) \to \text{Diff}(S^1) : X(\varphi)\partial_\varphi \mapsto \exp[X] \equiv \phi_X(1,\cdot) \tag{6.40}$$

where ϕ_X is the flow (6.39) of X. In other words the diffeomorphism $\exp[X](\varphi)$ is obtained by requiring that the equality

$$\int_\varphi^{\exp[X](\varphi)} \frac{d\phi}{X(\phi)} = 1 \tag{6.41}$$

holds for any initial condition φ.

In any finite-dimensional Lie group, the exponential map (5.3) is a local diffeomorphism, so any group element belonging to a suitable neighbourhood of the identity can be written as the exponential of an element of the Lie algebra. However, this is not so for groups of diffeomorphisms: one can show that the exponential map (6.40) does not define a local chart on Diff(S^1) in that it is neither locally injective, nor locally surjective. The idea of the proof is to build and explicit family of diffeomorphisms that are arbitrarily close to the identity but cannot be written as exponentials of vector fields. This being said, the exponential map is always well-defined on a Lie-Fréchet group, even when it is not locally surjective. See [1, 2] for details.

6.2 Virasoro Cohomology

As emphasized in Chap. 2, cohomology is crucial for quantum-mechanical applications: it measures the possible "deformations" of a group structure (e.g. central extensions), which typically do occur in quantum mechanics. When an algebra is finite-dimensional and semi-simple, Whitehead's lemma (2.23) ensures that there are essentially no deformations; the same is true of the Poincaré group (4.40). By contrast, we have seen how crucial cohomology is for the Galilei group (4.103), since its central extension gives rise to the notion of mass.

With this motivation, the present section is devoted to the cohomology of Diff(S^1) and its Lie algebra. These considerations will eventually lead to the definition of the Virasoro algebra, so we refer to them as "Virasoro cohomology". We will start by describing the real cohomology groups of Vect(S^1) and of Diff(S^1), then turn to cohomologies whose cochains are primary fields on the circle. The results summarized here are discussed at greater length in [1].

Remark We use the notation and conventions of Chap. 2, and all cochains are required to be smooth. Lie algebra cochains are denoted by lowercase sans serif letters such as b, c, s, etc. while group cochains are denoted by uppercase letters B, C, S, etc.

6.2.1 The Gelfand-Fuks Cocycle

Here we derive the first and second real cohomology groups of Vect(S^1); in particular we introduce the Gelfand-Fuks cocycle, which will eventually give rise to the Virasoro algebra. We also describe higher-degree real cohomology groups.

Cohomology in Degrees One and Two

The computation of the first cohomology of Vect(S^1) is immediate: since any vector field can be written as the bracket (6.22) of two other vector fields, the first cohomology group (2.19) of Vect(S^1) vanishes:

$$\mathcal{H}^1\big(\text{Vect}(S^1)\big) = 0. \tag{6.42}$$

In other words there is no non-trivial real one-cocycle on Vect(S^1). The second cohomology of Vect(S^1) is far more interesting:

Theorem The second real cohomology space of Vect(S^1) is one-dimensional. It is generated by the class of the *Gelfand-Fuks cocycle*

$$\mathsf{c}(X, Y) \equiv -\frac{1}{24\pi}\int_0^{2\pi} d\varphi X(\varphi)Y'''(\varphi) \tag{6.43}$$

whose expression in the basis (6.23) is

$$\mathsf{c}(\ell_m, \ell_n) = -i\frac{m^3}{12}\delta_{m+n,0}\,. \tag{6.44}$$

Proof Let c be a real two-cocycle on Vect(S^1). Then $\mathsf{dc} = 0$ where d is the Chevalley–Eilenberg differential (2.15) for the trivial representation $\mathscr{T}$. In terms of the basis (6.23), the statement $\mathsf{dc} = 0$ is tantamount to

$$\mathsf{c}([\ell_m, \ell_n], \ell_p) + \mathsf{c}([\ell_n, \ell_p], \ell_m) + \mathsf{c}([\ell_p, \ell_m], \ell_n) = 0 \tag{6.45}$$

for all integers m, n, p. Taking $p = 0$ and using the antisymmetry of c we get

$$\mathsf{c}(\ell_0, [\ell_m, \ell_n]) = \mathsf{c}([\ell_0, \ell_m], \ell_n) + \mathsf{c}(\ell_m, [\ell_0, \ell_n]). \tag{6.46}$$

Here we can interpret the left-hand side as the differential of the one-cochain $\mathsf{k} = \mathsf{c}(\ell_0, \cdot)$, so the left-hand side is exact while the right-hand side is a Lie derivative[1]

$$\mathsf{c}([\ell_0, \ell_m], \ell_n) + \mathsf{c}(\ell_m, [\ell_0, \ell_n]) = \big((\mathsf{i}_{\ell_0} \circ \mathsf{d} + \mathsf{d} \circ \mathsf{i}_{\ell_0}) \cdot \mathsf{c}\big)(\ell_m, \ell_n) = (\mathcal{L}_{\ell_0}\mathsf{c})(\ell_m, \ell_n) \tag{6.47}$$

where we used $\mathsf{dc} = 0$. Since the left-hand side of (6.46) is exact, we conclude that Lie derivation with respect to ℓ_0 leaves the cohomology class of c invariant. (In geometric terms ℓ_0 generates rotations, so this says that the cohomology class of c is invariant under rotations.) This allows us to turn c into a rotation-invariant cocycle. Indeed, let b be a one-cochain and define $\tilde{\mathsf{c}} \equiv \mathsf{c} + \mathsf{db}$, which has the same cohomology class as c. The Lie derivative of $\tilde{\mathsf{c}}$ with respect to ℓ_0 now is

$$\mathcal{L}_{\ell_0}\tilde{\mathsf{c}} = \mathcal{L}_{\ell_0}\mathsf{c} + \mathcal{L}_{\ell_0}\mathsf{db} \overset{(6.46)}{=} \mathsf{dk} + \mathsf{d}(\mathsf{i}_{\ell_0}(\mathsf{db})) = \mathsf{d}\big(\mathsf{k} + \mathsf{db}(\ell_0, \cdot)\big). \tag{6.48}$$

In order to make $\tilde{\mathsf{c}}$ invariant under rotations, we need to choose b such that (6.48) vanishes. One verifies that the definition

$$\mathsf{b}(\ell_m) = \frac{i}{m}\mathsf{c}(\ell_0, \ell_m) \qquad \text{for } m \neq 0 \tag{6.49}$$

satisfies this requirement for any $\mathsf{b}(\ell_0)$. Thus, from now on we work only with the rotation-invariant cocycle $\tilde{\mathsf{c}}$ and we rename it into c for simplicity. Then we have $\mathsf{c}(\ell_0, \ell_m) = 0$, and Eq. (6.46) becomes

$$\mathsf{c}([\ell_0, \ell_m], \ell_n) + \mathsf{c}(\ell_m, [\ell_0, \ell_n]) = 0 \tag{6.50}$$

for all integers m, n. The Lie brackets (6.24) then yield

$$(m + n)\,\mathsf{c}(\ell_m, \ell_n) = 0 \tag{6.51}$$

and thus imply that $\mathsf{c}(\ell_m, \ell_n) = 0$ whenever $m + n$ is non-zero. Writing $\mathsf{c}(\ell_m, \ell_n) = c_m \delta_{m+n,0}$ for some coefficients $c_m = -c_{-m}$, we are left with the task of determining the c_m's with $m > 0$. Returning to the cocycle identity (6.45) with $p = -m - 1$ and using once more the brackets (6.24), we find

$$c_{m+1} = \frac{(2 + m)c_m - (2m + 1)c_1}{m - 1} \tag{6.52}$$

for $m \geq 2$. This shows that all c_m's are determined recursively by c_1 and c_2. In particular, we now know that the cohomology space $\mathcal{H}^2\big(\text{Vect}(S^1)\big)$ is at most two-dimensional; the choices $c_m = m^3$ and $c_m = m$ are indeed two linearly independent solutions of the recursion relations (6.52). Now note that, if c is a coboundary $\mathsf{c} = \mathsf{dk}$ for some one-cochain k, then

[1] We denote by i the interior product of cochains.

$$\mathsf{c}(\ell_m, \ell_n) = \mathsf{dk}(\ell_m, \ell_n) \overset{(2.15)}{=} \mathsf{k}([\ell_m, \ell_n]) \overset{(6.24)}{=} -i(m-n)\mathsf{k}(\ell_{m+n})$$

so that $\mathsf{c}(\ell_m, \ell_{-m}) = -2im\mathsf{k}(\ell_0)$ always depends linearly on m. Accordingly, the solution $c_m = m$ of the recursion relations (6.52) yields a trivial cocycle, while $c_m = m^3$ is non-trivial. We conclude that, up to a coboundary, any non-trivial two-cocycle on $\text{Vect}(S^1)$ reads

$$\mathsf{c}(\ell_m, \ell_n) = \mathcal{N} m^3 \delta_{m+n,0} \tag{6.53}$$

for some normalization $\mathcal{N} \neq 0$. In particular, $\mathcal{H}^2\big(\text{Vect}(S^1)\big)$ is one-dimensional. ■

Higher Degree Cohomologies

The real cohomology groups of $\text{Vect}(S^1)$ increase in complexity as their degree becomes higher. Since we will not need any degree higher than two, we restrict ourselves here to a qualitative description of the result (details can be found in [1]).

The first step is to fix the kind of cochains one wants to study. For $\text{Vect}(S^1)$ it is natural to consider *local* real cochains

$$\mathsf{c} : \text{Vect}(S^1)^k \to \mathbb{R} : (X_1, \ldots, X_k) \mapsto \mathsf{c}(X_1, \ldots, X_k) \tag{6.54}$$

that take the form of an integral over S^1 of some "cochain density" $\mathscr{C}$:

$$\mathsf{c}(X_1, \ldots, X_k) = \int_0^{2\pi} d\varphi\, \mathscr{C}\left(X_1(\varphi), X_1'(\varphi), \ldots, X_1^{(n_1)}(\varphi), \ldots, X_k(\varphi), \ldots, X_k^{(n_k)}(\varphi)\right).$$

Here the word *local* is used in the same sense as in field theory. The Gelfand-Fuks cocycle (6.43) is of that form, with a density $\mathscr{C}(X, Y) \propto XY'''$. With this restriction on the allowed cochains, one can study the resulting cohomology groups $\mathcal{H}^k_{\text{loc}}\big(\text{Vect}(S^1)\big)$. The result is as follows:

Proposition The real local cohomology groups $\mathcal{H}^k_{\text{loc}}\big(\text{Vect}(S^1)\big)$ are all trivial except if k is equal to 0, 2 or 3, in which case the cohomology group is one-dimensional:

$$\mathcal{H}^k_{\text{loc}}\big(\text{Vect}(S^1)\big) = \begin{cases} \mathbb{R} & \text{if } k \in \{0, 2, 3\} \\ 0 & \text{otherwise.} \end{cases} \tag{6.55}$$

The generator of $\mathcal{H}^0_{\text{loc}}$ is the class of any non-zero constant function on $\text{Vect}(S^1)$; that of $\mathcal{H}^2_{\text{loc}}$ is the Gelfand-Fuks cocycle (6.43). Finally $\mathcal{H}^3_{\text{loc}}$ is generated by the class of the *Godbillon-Vey cocycle*

$$\int_0^{2\pi} d\varphi \det \begin{pmatrix} X & Y & Z \\ X' & Y' & Z' \\ X'' & Y'' & Z'' \end{pmatrix} \tag{6.56}$$

where it is understood that the integrand is evaluated at φ.

Note that the *full*, generally non-local, cohomology groups of $\text{Vect}(S^1)$ do not coincide with (6.55) because they contain classes of wedge products of the Gelfand-Fuks and Godbillon-Vey cocycles. For example $\mathsf{c} \wedge \mathsf{c}$ is a non-trivial, non-local four-cocycle on $\text{Vect}(S^1)$ when c is the Gelfand-Fuks cocycle.

Remark In this work the Godbillon-Vey cocycle (6.56) will be unimportant. However, it does play a key role in a specific context, as it was shown in [9] that it is responsible for the unique non-trivial gauge-invariant deformation of a higher-spin Chern-Simons action in three dimensions with gauge algebra $\text{Vect}(S^1) \Subset_{\text{ad}^*} \text{Vect}(S^1)^*$.

6.2.2 *The Bott-Thurston Cocycle*

We now turn to the low-degree real cohomology groups of the universal cover $\widetilde{\text{Diff}}^+(S^1)$ of the group of orientation-preserving diffeomorphisms of the circle. We show in particular how one can build a non-trivial two-cocycle corresponding to Gelfand-Fuks by integration, and known as the *Bott-Thurston cocycle*. The latter will eventually lead to the definition of the Virasoro group. As before, we abuse notation by denoting the universal cover $\widetilde{\text{Diff}}^+(S^1)$ simply as $\text{Diff}(S^1)$.

Cocyclic Recipes

We start by describing a general algorithm for building two-cocycles on a group [1, 10]. Let $\mathcal{M}$ be an orientable manifold endowed with a volume form μ. For any orientation-preserving diffeomorphism $f : \mathcal{M} \to \mathcal{M}$, we define a function $\mathsf{T}[f^{-1}]$ on $\mathcal{M}$ by

$$f^*\mu \equiv e^{\mathsf{T}[f^{-1}]}\mu. \tag{6.57}$$

This function can be thought of as a modified derivative of f. It appears to have no standard name in the literature but we will use it repeatedly below in the case $\mathcal{M} = S^1$, so from now on we refer to $\mathsf{T}[f^{-1}]$ as the *twisted derivative* of f.

Lemma The map $\mathsf{T} : \text{Diff}^+(\mathcal{M}) \to C^\infty(\mathcal{M}) = f \mapsto \mathsf{T}[f]$ defined by (6.57) is a $C^\infty(\mathcal{M})$-valued one-cocycle on $\text{Diff}^+(\mathcal{M})$, where the action of diffeomorphisms on functions is given by

$$(f \cdot \mathcal{F})(p) = \mathcal{F}(f^{-1}(p)) \tag{6.58}$$

for $f \in \text{Diff}^+(\mathcal{M})$, $\mathcal{F} \in C^\infty(\mathcal{M})$ and $p \in \mathcal{M}$.

Proof We need to show that $\mathsf{dT} = 0$ with the Chevalley–Eilenberg differential (2.31) and the representation $\mathcal{T}$ given by the action (6.58) of diffeomorphisms on functions.[2] If f, g are orientation-preserving diffeomorphisms, one readily verifies from the definition (6.57) that T satisfies the cocycle property (2.32). ∎

[2]The fact that the same letter denotes the cocycle T and the representation $\mathcal{T}$ is merely a notational coincidence.

Now let us consider another recipe, seemingly unrelated to and just as random as the previous one. Take two vector spaces $\mathbb{V}$ and $\mathbb{W}$ acted upon by a group G according to representations $\mathcal{S}$ and $\mathcal{T}$, respectively, and let $\Omega : \mathbb{V} \times \mathbb{V} \to \mathbb{W} : (v, v') \mapsto \Omega(v, v')$ be an antisymmetric bilinear map such that

$$\Omega\big(\mathcal{S}[f]v, \mathcal{S}[f]v'\big) = \mathcal{T}[f]\,\Omega(v, v') \tag{6.59}$$

for any group element $f \in G$ and all $v, v' \in \mathbb{V}$. Finally, let $\mathsf{T} : G \to \mathbb{V}$ be a $\mathbb{V}$-valued one-cocycle on G with respect to the representation $\mathcal{S}$.

Lemma The map

$$\mathsf{C} : G \times G \to \mathbb{W} : (f, g) \mapsto \mathsf{C}(f, g) \equiv \Omega\big(\mathsf{T}[f], \mathsf{T}[fg]\big) \tag{6.60}$$

is a $\mathbb{W}$-valued two-cocycle on G.

Proof We need to show that $\mathsf{dC} = 0$ for the Chevalley–Eilenberg differential (2.31), given that $\mathbb{W}$ is acted upon by G according to the representation $\mathcal{T}$. Using the fact that Ω is bilinear and antisymmetric together with property (6.59), one readily verifies by brute force that this is indeed the case. ■

The Bott-Thurston Cocycle

The two constructions just described can be used to define a non-trivial two-cocycle on the group $\text{Diff}(S^1)$. We will first use (6.57) to define a one-cocycle on $\text{Diff}(S^1)$, then plug it into (6.60) for a well chosen map Ω to obtain the desired two-cocycle.

We consider the circle S^1 endowed with the flat volume form $\mu = d\varphi$. Under a diffeomorphism $\varphi \mapsto f(\varphi)$ we have $(f^*\mu)_\varphi = d(f(\varphi)) = f'(\varphi)d\varphi = e^{\log(f'(\varphi))}\mu$, so (6.57) provides a $C^\infty(S^1)$-valued twisted derivative

$$\mathsf{T}[f](\varphi) \equiv \log\big[(f^{-1})'(\varphi)\big], \tag{6.61}$$

which is a one-cocycle. We use square brackets to denote the argument of T because the latter is a functional on $\text{Diff}(S^1)$; then $\mathsf{T}[f](\varphi)$ is the function $\mathsf{T}[f]$ evaluated at φ.

To apply the construction (6.60), we also need to find a bilinear antisymmetric map $\Omega : C^\infty(S^1) \times C^\infty(S^1) \to \mathbb{R}$ which is invariant under $\text{Diff}(S^1)$ in the sense that (6.59) holds when $\mathcal{T}$ is the trivial representation while $\mathcal{S}$ is the action (6.58) of $\text{Diff}(S^1)$ on functions. A natural guess is

$$\Omega : C^\infty(S^1) \times C^\infty(S^1) \to \mathbb{R} : (\mathcal{F}, \mathcal{G}) \mapsto \int_0^{2\pi} d\varphi\, \mathcal{F}(\varphi)\mathcal{G}'(\varphi) = \int_{S^1} \mathcal{F} d\mathcal{G}\,, \tag{6.62}$$

which is manifestly antisymmetric (integrate by parts) and reparameterization-invariant (the integrand is analogous to the $p\dot{q}$ of Hamiltonian actions). Applying the prescription (6.60) with the one-cocycle (6.61), we obtain the following real two-cocycle:

Definition The *Bott-Thurston cocycle* on $\text{Diff}(S^1)$ is [11]

$$\mathsf{C}(f,g) \equiv -\frac{1}{48\pi}\int_{S^1}\mathsf{T}[f]\,d\mathsf{T}[f\circ g] \tag{6.63}$$

$$\overset{(6.61)}{=}-\frac{1}{48\pi}\int_0^{2\pi}d\varphi\,\log\left[(f^{-1})'(\varphi)\right]\left(\log\left[((f\circ g)^{-1})'\right]\right)'(\varphi) \tag{6.64}$$

where d denotes the exterior derivative on the circle.

By construction, the Bott-Thurston cocycle satisfies the cocycle identity (2.9),

$$\mathsf{C}(f,gh)+\mathsf{C}(g,h)=\mathsf{C}(fg,h)+\mathsf{C}(f,g)\,. \tag{6.65}$$

This will be instrumental in ensuring that C yields a well-defined centrally extended group. For future applications it is useful to rewrite (6.64) in a slightly simpler way, which relies on the following result:

Lemma If T is the twisted derivative (6.61), then for all $f, g \in \text{Diff}(S^1)$ one has

$$\int_{S^1}\mathsf{T}[f]d\mathsf{T}[f\circ g]=\int_{S^1}\mathsf{T}[(f\circ g)^{-1}]d\mathsf{T}[g^{-1}]\,. \tag{6.66}$$

Proof We use two key properties: the first is the fact that T is a one-cocycle with respect to the action of $\text{Diff}(S^1)$ on $C^\infty(S^1)$, so

$$\mathsf{T}[f\circ g]=\mathsf{T}[f]+\mathsf{T}[g]\circ f^{-1}, \tag{6.67}$$

and the second is a property that follows from the definition (6.61) and Eq. (6.16):

$$\mathsf{T}[f]\circ f=-\mathsf{T}[f^{-1}]. \tag{6.68}$$

We then find that (6.63) can be rewritten as

$$\int_{S^1}\mathsf{T}[f]\,d\mathsf{T}[f\circ g]=-\int_{S^1}\mathsf{T}[f^{-1}]\,d\mathsf{T}[g]=\int_{S^1}\mathsf{T}[(f\circ g)^{-1}]\,d\mathsf{T}[g^{-1}]\,,$$

which was to be proven. ■

Thanks to this lemma we can write the Bott-Thurston cocycle (6.64) in a more convenient way, without f^{-1}'s all around the place:

$$\mathsf{C}(f,g)=-\frac{1}{48\pi}\int_{S^1}\log(f'\circ g)\,d\log(g')=-\frac{1}{48\pi}\int_{S^1}\mathsf{T}[(f\circ g)^{-1}]d\mathsf{T}[g^{-1}]. \tag{6.69}$$

This is the definition that we will be using from now on.

At this stage the Bott-Thurston cocycle seems to be coming out of the blue. However it turns out that (6.69) is, in fact, a very natural quantity. We will explain this in greater detail in Sect. 6.3, but for now we simply note the following relation:

Proposition The Bott-Thurston cocycle (6.69) is the integral of the Gelfand-Fuks cocycle (6.43) in the sense of formula (2.35):

$$\mathsf{c}(X,Y) = -\frac{d^2}{dt\,ds}\Big[\mathsf{C}\left(e^{tX},e^{sY}\right) - \mathsf{C}\left(e^{sY},e^{tX}\right)\Big]\Big|_{t=0,\,s=0}. \tag{6.70}$$

In particular, the Bott-Thurston cocycle is non-trivial.

Proof We consider infinitesimal diffeomorphisms $f(\varphi) = \varphi + tX(\varphi) + \mathcal{O}(t^2)$ and $g(\varphi) = \varphi + sY(\varphi) + \mathcal{O}(s^2)$. Then $\log\left(f'\circ g\right) = tX'$ and $\log(g') = sY'$ so that

$$\mathsf{C}(e^{tX},e^{sY}) \overset{(6.69)}{=} -\frac{1}{48\pi}\int_0^{2\pi} d\varphi\, tX'(\varphi)sY''(\varphi)$$

to first order in t, s. Relation (6.70) follows. It also follows that the Bott-Thurston cocycle is non-trivial, since the Gelfand-Fuks cocycle is non-trivial. ■

Remark The bilinear map (6.62) is a non-trivial two-cocycle on the Abelian Lie algebra $C^\infty(S^1)$ of smooth functions on the circle. It defines a central extension of $C^\infty(S^1)$ that can be interpreted in several ways: either as an infinite-dimensional Heisenberg algebra, or as a $\mathfrak{u}(1)$ Kac–Moody algebra. This kind of central extension occurs for instance in the realm of warped conformal field theories [12, 13].

6.2.3 Primary Cohomology of Vect(S^1)

Here we study some of the cohomology groups of Vect(S^1) in spaces of densities (i.e. primary fields). As in the real-valued case described earlier we consider the cohomology defined by *local* cochains, which in the present case take the form

$$\mathsf{c}[X_1,\ldots,X_k] = \mathscr{C}\big[X_1(\varphi),X_1'(\varphi),\ldots,X_1^{(n_1)}(\varphi),\ldots,X_k(\varphi),\ldots,X_k^{(n_k)}(\varphi)\big](d\varphi)^h$$

for some weight h. The functional $\mathscr{C}$ depends on the X_i's and finitely many of their derivatives, all evaluated at the same point φ. We denote the corresponding cohomology spaces by $\mathcal{H}^k\big(\text{Vect}(S^1),\mathcal{F}_\lambda(S^1)\big)$. In order to avoid technical considerations we state the results without proof and refer to [1] for details.

Theorem If the weight h is not a non-negative integer, then

$$\mathcal{H}^k\big(\text{Vect}(S^1),\mathcal{F}_h(S^1)\big) = 0 \quad \text{for all}\, k\in\mathbb{N}. \tag{6.71}$$

In particular $\mathcal{H}^2\big(\text{Vect}(S^1), \text{Vect}(S^1)\big) = 0$, so there exists no non-trivial deformation of $\text{Vect}(S^1)$.

The result (6.71) implies that the non-trivial primary cohomology of $\text{Vect}(S^1)$ is localized only on non-negative integers with weights $h \in \mathbb{N}$. Here we briefly describe the non-trivial first cohomology groups ($k = 1$) for the cases $h = 0, 1, 2$ that will be useful below.

The case $h = 0$ corresponds to one-cochains taking values in the space of functions on the circle. It turns out that there are exactly two linearly independent, non-trivial one-cocycles in that case, namely $\tilde{\mathsf{c}}[X](\varphi) = X(\varphi)$ and

$$\mathsf{t}[X](\varphi) = X'(\varphi)\,. \tag{6.72}$$

The latter may be recognized as the infinitesimal cocycle corresponding to the twisted derivative (6.61) by differentiation.

At weight $h = 1$ we are in the realm of cochains taking values in the space $\Omega^1(S^1)$ of one-forms; in particular one can show that the corresponding first cohomology is one-dimensional, generated by the (class of the) one-cocycle

$$\mathsf{w}[X](\varphi) = X''(\varphi)d\varphi\,. \tag{6.73}$$

The notation w is because this cocycle is relevant to certain aspects [13] of warped conformal symmetry [12].

Finally, when $h = 2$, cochains take their values in the space $\mathcal{F}_2(S^1)$ of quadratic densities on the circle. In particular one can show that the first cohomology space is one-dimensional, generated by the (class of the) *infinitesimal Schwarzian derivative*

$$\mathsf{s}[X](\varphi) = X'''(\varphi)d\varphi^2. \tag{6.74}$$

One can go on and similarly classify all cohomology groups with higher weight h. Since we will not need these results here, we refrain from displaying them (see e.g. [14, 15]). Instead, we now relate the cocycles (6.72)–(6.74) to one-cocycles on $\text{Diff}(S^1)$.

6.2.4 Primary Cohomology of Diff(S^1)

The complete classification of density-valued cohomology groups of $\text{Diff}(S^1)$ is beyond the scope of this presentation, so we refer to [1] for a more detailed discussion. Here we simply note that the Lie algebra one-cocycles mentioned above can be integrated to non-trivial group one-cocycles:

- The cocycle (6.72) can be integrated to the twisted derivative (6.61), which we used to build the Bott-Thurston cocycle. Indeed, for $f(\varphi) = \varphi + \epsilon X(\varphi)$, formula (6.61) reduces to $\mathsf{T}[f] = -\epsilon\,\mathsf{t}[X]$.

- The cocycle (6.73) can be integrated to

$$\mathsf{W}[f](\varphi) = d\log[(f^{-1})'(\varphi)] = d\,\mathsf{T}[f](\varphi) \tag{6.75}$$

where d denotes the exterior derivative on the circle. As mentioned above this "warped derivative" has been used recently [13] to describe certain aspects of warped conformal field theories. Note that in these terms the Bott-Thurston cocycle (6.63) is $\mathsf{C}(f, g) \propto \int \mathsf{T}[f] \otimes \mathsf{W}[f \circ g]$.

The one-cocycle (6.74) can similarly be related to the $\mathcal{F}_2$-valued Schwarzian derivative on Diff(S^1), although the integration is somewhat less trivial than in the two cases just described. The Schwarzian derivative is crucial for our upcoming considerations, so the whole next section is devoted to it.

6.3 On the Schwarzian Derivative

Definition Let $f \in$ Diff(S^1). Then the *Schwarzian derivative*[3] of f at φ is

$$\mathsf{S}[f](\varphi) \equiv \frac{f'''(\varphi)}{f'(\varphi)} - \frac{3}{2}\left(\frac{f''(\varphi)}{f'(\varphi)}\right)^2. \tag{6.76}$$

Many references use the notation $\{f; \varphi\}$, but we will stick to $\mathsf{S}[f](\varphi)$ instead.

In this section we investigate the many properties of the Schwarzian derivative. We will start by showing that it is (related to) a one-cocycle on Diff(S^1) taking its values in the space $\mathcal{F}_2(S^1)$ of quadratic densities, and that it corresponds to the Lie algebra cocycle (6.74) by differentiation. We will then show that it is related to the Bott-Thurston cocycle by the so-called Souriau construction. We will also describe the remarkable symmetry properties of the Schwarzian derivative under the PSL(2, $\mathbb{R}$) subgroup of Diff(S^1), and obtain as a by-product the expression of Lorentz transformations in terms of diffeomorphisms of the circle. (In Chap. 9 these transformations will turn out to be actual Lorentz transformations on the celestial circle.)

6.3.1 The Schwarzian Derivative is a Cocycle

Here we show that the Schwarzian derivative is the one-cocycle corresponding to (6.74) by integration. Note that the relation between (6.76) and (6.74) is obvious: upon taking $f(\varphi) = \varphi + \epsilon X(\varphi)$ in (6.76), one finds $\mathsf{S}[f] = \epsilon X'''$ to first order in

[3]The name refers to H. Schwarz, who first introduced the object (6.76); it is the same Schwarz as in the Cauchy-Schwarz inequality.

ϵ. The non-trivial problem is showing that the Schwarzian derivative is actually a cocycle:

Proposition The Schwarzian derivative (6.76) defines a one-cocycle

$$\text{Diff}(S^1) \to \mathcal{F}_2(S^1) : f \mapsto \mathsf{S}[f^{-1}](\varphi)d\varphi^2$$

valued in the space of quadratic densities on the circle.

Proof We start by noting that the definition (6.76) implies

$$\mathsf{S}[f \circ g] = \text{Ad}^*_{g^{-1}}\mathsf{S}[f] + \mathsf{S}[g] = \mathsf{S}[g] + (g')^2\mathsf{S}[f] \circ g\,, \tag{6.77}$$

where Ad^* denotes the coadjoint representation (6.36) of $\text{Diff}(S^1)$. Upon defining $\tilde{\mathsf{S}}[f] \equiv \mathsf{S}[f^{-1}]d\varphi^2$, one obtains a map that associates a quadratic density with any diffeomorphism f, and which satisfies

$$\tilde{\mathsf{S}}[f \circ g] = \tilde{\mathsf{S}}[f] + ((f^{-1})')^2\tilde{\mathsf{S}}[g] \circ f^{-1} = \tilde{\mathsf{S}}[f] + \text{Ad}^*_f\tilde{\mathsf{S}}[g] \tag{6.78}$$

by virtue of (6.77). This is precisely the cocycle identity (2.32). ■

The Souriau Construction

We now study the relation between the Schwarzian derivative and the Bott-Thurston cocycle, which follows from the so-called Souriau construction.

Definition Let G be a Lie group with Lie algebra $\mathfrak{g}$, $\mathsf{C} : G \times G \to \mathbb{R}$ a real two-cocycle on G. Then the *Souriau cocycle* associated with C is the map $G \mapsto \mathfrak{g}^* : f \mapsto \mathsf{S}[f^{-1}]$ defined by

$$\frac{d}{dt}\Big[\mathsf{C}(f, e^{tX}) + \mathsf{C}(fe^{tX}, f^{-1})\Big]\Big|_{t=0} \equiv -\frac{1}{12}\langle\mathsf{S}[f], X\rangle \tag{6.79}$$

for any $f \in G$ and any adjoint vector $X \in \mathfrak{g}$. (The normalization is chosen so that S eventually coincides with the Schwarzian derivative.)

Proposition The Souriau cocycle is a one-cocycle on G valued in the space of coadjoint vectors.

Proof Since C is a real two-cocycle, it is clear that the left-hand side of (6.79) defines a real linear function of $X \in \mathfrak{g}$, that is, a coadjoint vector. The latter only depends on f so we can certainly write it as $\mathsf{S}[f]$, which defines the map S. The problem is to show that the map $f \mapsto \mathsf{S}[f^{-1}]$ is actually a one-cocycle. For this we let $X \in \mathfrak{g}$, pick two group elements $f, g \in G$, and write

$$\big\langle\mathsf{S}[(fg)^{-1}], X\big\rangle \overset{(6.79)}{=} \frac{d}{dt}\Big[\mathsf{C}(g^{-1}f^{-1}, e^{tX}) + \mathsf{C}(g^{-1}f^{-1}e^{tX}, fg)\Big]\Big|_{t=0}. \tag{6.80}$$

On the other hand, if Ad^* denotes the coadjoint representation of G, we have

$$\left\langle \text{Ad}^*_f \mathsf{S}[g^{-1}] + \mathsf{S}[f^{-1}], X \right\rangle = \tag{6.81}$$
$$\stackrel{(6.79)}{=} \frac{d}{dt} \left[\mathsf{C}(g^{-1}, e^{t\text{Ad}_{f^{-1}}X}) + \mathsf{C}(g^{-1}e^{t\text{Ad}_{f^{-1}}X}, g) + \mathsf{C}(f^{-1}, e^{tX}) + \mathsf{C}(f^{-1}e^{tX}, f) \right] \Big|_{t=0}.$$

Using the cocycle identity (6.65) together with property (5.7), one can then show by brute force that (6.81) coincides with the right-hand side of (6.80). ■

In the case of the group $\text{Diff}(S^1)$, the Souriau construction yields the Schwarzian derivative from the Bott-Thurston cocycle. Let us check this explicitly: taking $g(\varphi) = \varphi + tX(\varphi)$ in (6.69), one finds

$$\mathsf{C}(f, e^{tX}) = -\frac{t}{48\pi} \int_0^{2\pi} d\varphi \left[\frac{f'''}{f'} - \left(\frac{f''}{f'} \right)^2 \right] X(\varphi),$$
$$\mathsf{C}(f \circ e^{tX}, f^{-1}) = (t\text{-independent}) - \frac{t}{48\pi} \int_0^{2\pi} d\varphi \left[\frac{f'''}{f'} - 2 \left(\frac{f''}{f'} \right)^2 \right] X(\varphi)$$

to first order in t. It then follows that relation (6.79) holds when S is the Schwarzian derivative (6.76) and $\langle \cdot, \cdot \rangle$ is the pairing (6.34) of $\text{Vect}(S^1)$ with its dual. Note that by taking $f(\varphi) = \varphi + sY(\varphi)$ with small s, the Schwarzian derivative reduces to $\mathsf{S}[f] = sY'''$. Upon differentiating with respect to s in the right-hand side of (6.79), we recover precisely the Gelfand-Fuks cocycle (6.43).

Virasoro Universality

At this point the cohomological constructions of the previous pages are starting to fit in a global picture of Virasoro cohomology: Eq. (6.70) relates the Bott-Thurston cocycle to the Gelfand-Fuks cocycle (6.43), while (6.79) relates it to the Schwarzian derivative, which in turn is the integral of the infinitesimal cocycle (6.74). In addition the integral of the latter with a vector field on the circle reproduces the Gelfand-Fuks cocycle. The common feature of all these expressions is the occurrence of third derivatives such as f''' or X''', which will indeed play a key role in the sequel (and give rise to the term m^3 in (6.44)).

In this sense, all these cocycles are really one and the same quantity, albeit expressed in very different ways. Depending on one's viewpoint, one may decide that the most fundamental quantity is the Gelfand-Fuks cocycle, or the Schwarzian derivative, or Bott-Thurston. Our point of view will be that the Bott-Thurston cocycle is the most fundamental of them all, since it yields the other ones by differentiation:

$$
\begin{array}{c}
\text{Bott-Thurston } \mathsf{C} \\
{\scriptstyle \text{Souriau}} \big\downarrow \\
\text{Schwarzian derivative } \mathsf{S} \\
{\scriptstyle \text{differentiate}} \big\downarrow \\
\text{infinitesimal Schwarzian } \mathsf{s} \\
{\scriptstyle \text{pair with} \mathrm{Vect}(S^1)} \big\downarrow \\
\text{Gelfand-Fuks } \mathsf{c}
\end{array}
$$

6.3.2 Projective Invariance of the Schwarzian

There exists a deep relation between the circle, the projective line and the Schwarzian derivative [10], which in turn leads to powerful symmetry properties under the group $\mathrm{SL}(2,\mathbb{R})/\mathbb{Z}_2 = \mathrm{PSL}(2,\mathbb{R})$. Our goal here is to explore this relation. Accordingly we start with a short detour through one-dimensional projective geometry, before recovering the Schwarzian derivative as a quantity that measures the extent to which diffeomorphisms deform the projective structure. Along the way we will encounter the expression of Lorentz transformations in terms of diffeomorphisms of the circle.

The Projective Line

Consider the plane $\mathbb{R}^2$ and define the *projective line* $\mathbb{R}P^1$ to be the space of its one-dimensional subspaces. Equivalently $\mathbb{R}P^1$ is the space of straight lines in $\mathbb{R}^2$ going through the origin, i.e. a quotient of $\mathbb{R}^2\backslash\{(0,0)\}$ by the equivalence relation

$$(x, y) \sim (x', y') \qquad \text{if} \qquad \exists\, \lambda \in \mathbb{R}^* \text{ such that } (x, y) = \lambda(x', y').$$

Denoting by $[(x, y)]$ the equivalence class of $(x, y) \in \mathbb{R}^2$, the projective line is thus

$$\mathbb{R}P^1 = \Big\{[(x, y)]\Big|(x, y) \in \mathbb{R}^2\backslash\{(0,0)\}\Big\}. \tag{6.82}$$

In topological terms the projective line is a circle centred at the origin in $\mathbb{R}^2$ with antipodal points identified. This is to say that $\mathbb{R}P^1 \cong S^1/\mathbb{Z}_2$, where $\mathbb{Z}_2$ acts on S^1 by rotations. Since any group $\mathbb{Z}_n$ acting on the circle by rotations of $2\pi/n$ is such that $S^1/\mathbb{Z}_n \cong S^1$, the projective line is actually diffeomorphic to a circle:

$$\mathbb{R}P^1 \cong S^1. \tag{6.83}$$

As a result, all considerations concerning the group of diffeomorphisms of S^1 can be recast in terms of projective geometry, and vice-versa.

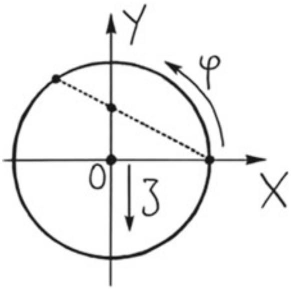

Fig. 6.3 The stereographic coordinate (6.85) is obtained by projecting the points of a unit circle in the (X, Y) plane on the Y axis, along a straight line that goes through the "east pole" $(1, 0)$. Upon writing $X = \cos\varphi$ and $Y = \sin\varphi$ and declaring that the projective coordinate ζ is *minus* the Y coordinate of the projection, one obtains (6.85). The minus sign is included so as to preserve the orientation of the circle in the sense that $d\zeta/d\varphi > 0$

The diffeomorphism (6.83) can be made explicit in terms of well chosen coordinates. Indeed, in terms of (6.82), the projective line is a union $\mathbb{R}P^1 = \{[(\zeta, 1)] | \zeta \in \mathbb{R}\} \cup \{[(1, 0)]\}$, so the projective coordinate

$$\zeta \equiv x/y \tag{6.84}$$

is a local coordinate on $\mathbb{R}P^1$ that misses only one point, namely the class of $(1, 0)$. In this sense the projective line is a real line $\mathbb{R}$ with an extra "point at infinity".

This is exactly the same situation as with the stereographic coordinate on a circle. For later convenience we define this coordinate in terms of an angular coordinate φ on the circle by

$$\zeta = -\cot(\varphi/2) = \frac{e^{i\varphi} + 1}{ie^{i\varphi} - i} \tag{6.85}$$

(see Fig. 6.3). The diffeomorphism (6.83) is then obtained by identifying this stereographic coordinate with the projective coordinate (6.84).

Projective Transformations

The projective line inherits a symmetry from the linear action of $\mathrm{GL}(2, \mathbb{R})$ on $\mathbb{R}^2$. Explicitly, an invertible matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{6.86}$$

acts on the coordinate (6.84) as a *projective transformation*

$$\zeta \mapsto \frac{a\zeta + b}{c\zeta + d}. \tag{6.87}$$

Any such transformation is independent of the determinant of (6.86), which we can therefore set to one without loss of generality. Furthermore the overall sign of the matrix is irrelevant, so the transformations (6.87) span a *projective group*

$$\mathrm{PGL}(2,\mathbb{R}) \equiv \mathrm{GL}(2,\mathbb{R})/\mathbb{R}^* \cong \mathrm{SL}(2,\mathbb{R})/\mathbb{Z}_2 \equiv \mathrm{PSL}(2,\mathbb{R}).$$

According to (4.83) this is the connected Lorentz group in three dimensions.

Upon identifying the projective coordinate (6.84) with the stereographic coordinate (6.85), the transformation (6.87) can be reformulated in terms of the angular coordinate φ. Using (6.87) and the inverse of (6.85), we find that projective transformations act on $e^{i\varphi}$ according to

$$e^{i\varphi} \mapsto e^{if(\varphi)} = \frac{Ae^{i\varphi}+B}{\bar{B}e^{i\varphi}+\bar{A}} \tag{6.88}$$

where the complex coefficients

$$A = \frac{1}{2}(a+ib-ic+d), \qquad B = \frac{1}{2}(a-ib-ic-d) \tag{6.89}$$

are such that $|A|^2-|B|^2=1$. The family of transformations (6.88) spans a subgroup $\mathrm{PSL}(2,\mathbb{R})$ of $\mathrm{Diff}(S^1)$ that we already anticipated in (6.3). In Sect. 9.1 we shall interpret that subgroup as the Lorentz group acting on null infinity. For infinitesimal parameters $A = 1+i\epsilon$ and $B=\varepsilon$ (with $\epsilon \in \mathbb{R}$ and $\varepsilon \in \mathbb{C}$), formula (6.88) becomes an infinitesimal diffeomorphism

$$f(\varphi) = \varphi + 2\epsilon + 2\,\mathrm{Re}(\varepsilon)\,\cos\varphi - 2\,\mathrm{Re}(\varepsilon)\,\sin(\varphi).$$

This is an $\mathfrak{sl}(2,\mathbb{R})$ transformation generated by the vector fields ℓ_0, ℓ_1 and ℓ_{-1} mentioned below (6.24). Conversely, any transformation (6.88) belongs to the flow of an $\mathfrak{sl}(2,\mathbb{R})$ vector field.

Remark The relation between S^1 and $\mathbb{R}P^1$ discussed here has a complex analogue $\mathbb{C}P^1 \cong S^2$, where $\mathbb{C}P^1$ is the complex projective line. In this generalization the projective coordinate (6.84) becomes a complex coordinate z and coincides with the stereographic coordinate (1.4) of S^2. The projective transformations of $\mathbb{C}P^1$ then are Möbius transformations (1.6), i.e. Lorentz transformations in four dimensions.

Cross Ratios and the Schwarzian Derivative

Given the projective line $\mathbb{R}P^1$, one may look for projective invariants, i.e. quantities that are left invariant by the transformations (6.87). For example, consider four points on $\mathbb{R}P^1$ whose projective coordinates are $\zeta_1, \zeta_2, \zeta_3, \zeta_4$ and define their *cross ratio*

$$[\zeta_1,\zeta_2,\zeta_3,\zeta_4] \equiv \frac{(\zeta_1-\zeta_3)(\zeta_2-\zeta_4)}{(\zeta_1-\zeta_2)(\zeta_3-\zeta_4)}\,.$$

This number is a projective invariant, as one can verify by direct computation. Now take a diffeomorphism $f : S^1 \to S^1$, where we think of S^1 as a projective line (6.83). This diffeomorphism can be written in terms of the projective coordinate (6.84), giving rise to a map $\zeta \mapsto \mathsf{f}(\zeta)$. The explicit correspondence between f and f follows from (6.85) and reads

$$\mathsf{f}\big(-\cot(\varphi/2)\big) = -\cot\big(f(\varphi)/2\big)\,. \tag{6.90}$$

In general, f is not a projective transformation (6.87) and therefore spoils the projective structure of S^1. This can be measured by taking a point with coordinate ζ on $\mathbb{R}P^1$ together with three other nearby points, then evaluating the change in their cross-ratio under the action of f. Let therefore $\zeta_1 = \zeta + \epsilon$, $\zeta_2 = \zeta + 2\epsilon$ and $\zeta_3 = \zeta + 3\epsilon$ be three points close to ζ. These points move under the action of f, and one can show that

$$\big[\mathsf{f}(\zeta), \mathsf{f}(\zeta_1), \mathsf{f}(\zeta_2), \mathsf{f}(\zeta_3)\big] = [\zeta, \zeta_1, \zeta_2, \zeta_3] - 2\epsilon^2\,\mathsf{S}[\mathsf{f}](\zeta) + \mathcal{O}(|\epsilon|^3)$$

where S is the Schwarzian derivative (6.76). Thus we have recovered the Schwarzian derivative as a measuring device that tells us how much the diffeomorphism f spoils the projective structure of $\mathbb{R}P^1$. From this one concludes:

Proposition Let f, g be diffeomorphisms of the projective line. Then

$$\mathsf{S}[\mathsf{f} \circ \mathsf{g}](\zeta) = \mathsf{S}[\mathsf{g}](\zeta) \tag{6.91}$$

if and only if f is a projective transformation (6.87). In particular, $\mathsf{S}[\mathsf{f}](\zeta) = 0$ if and only if $\mathsf{f}(\zeta)$ is a projective transformation of the form (6.87).

In technical terms, a one-cocycle S on a group G with a subgroup H is said to be *H-relative* if $\mathsf{S}[h] = 0$ for all $h \in H$. Thus (6.91) says that the Schwarzian derivative is a $\mathrm{PSL}(2, \mathbb{R})$-relative cocycle. In conformal field theory this corresponds to the statement that the Schwarzian derivative is blind to Möbius transformations (1.6).

Schwarzians on the Circle

All the above considerations can be reformulated in terms of the angular coordinate φ using the correspondence (6.85). Here we work out this rewriting for projective transformations (6.88). To begin, note that (6.88) precisely takes the form of a projective transformation (6.87) in terms of the coordinate $e^{i\varphi}$. Accordingly,

$$\mathsf{S}\big[e^{if(\varphi)}\big](e^{i\varphi}) = 0. \tag{6.92}$$

In order to go from (6.92) to $\mathsf{S}[f](\varphi)$ we use the cocycle identity (6.77) repeatedly. First we write

$$\mathsf{S}[f](\varphi) = \mathsf{S}[\log(e^{if(\varphi)})](\varphi) \overset{(6.77)}{=} \mathsf{S}[e^{if(\varphi)}](\varphi) + \big(if'(\varphi)e^{if(\varphi)}\big)^2\,\mathsf{S}[\log](e^{if(\varphi)})\,, \tag{6.93}$$

where $\mathsf{S}[\log](x) \overset{(6.76)}{=} \frac{1}{2x^2}$. The first term on the far right-hand side of (6.93) involves

$$\mathsf{S}[e^{if(\varphi)}](\varphi) \overset{(6.77)}{=} \mathsf{S}[e^{i\varphi}](\varphi) + \big((e^{i\varphi})'\big)^2\, \mathsf{S}[e^{if(\varphi)}](e^{i\varphi}) \overset{(6.92)}{=} \mathsf{S}[e^{i\varphi}](\varphi) \overset{(6.76)}{=} \frac{1}{2}\,,$$

which finally gives

$$\mathsf{S}[f](\varphi) = \frac{1}{2}\big[1 - \big(f'(\varphi)\big)^2\big] \tag{6.94}$$

when $f(\varphi)$ is given by (6.88). We will put this formula to use in the next chapter.

Note that these observations can be generalized to infinitely many other families of diffeomorphisms of the circle. Indeed, pick a positive integer $n \in \mathbb{N}^*$ and take formula (6.88) with φ replaced by $n\varphi$ and $f(\varphi)$ replaced by $nf(\varphi)$:

$$e^{inf(\varphi)} = \frac{Ae^{in\varphi} + B}{\bar{B}e^{in\varphi} + \bar{A}}, \quad |A|^2 - |B|^2 = 1. \tag{6.95}$$

This defines a diffeomorphism of the circle, and the family of such diffeomorphisms also spans a subgroup of $\text{Diff}(S^1)$ which is locally isomorphic to $\text{SL}(2, \mathbb{R})$. The difference with respect to the case $n = 1$ discussed above is that the corresponding Lie algebra $\mathfrak{sl}(2, \mathbb{R})$ is generated by ℓ_0, ℓ_n and ℓ_{-n}, and that the actual group spanned by such transformations is an n-fold cover of $\text{PSL}(2, \mathbb{R})$; we shall denote this cover by $\text{SL}^{(n)}(2, \mathbb{R})/\mathbb{Z}_2 \equiv \text{PSL}^{(n)}(2, \mathbb{R})$. One can also verify that the Schwarzian derivative of the diffeomorphism f defined by (6.95) satisfies

$$\mathsf{S}[f](\varphi) = \frac{n^2}{2}\left[1 - (f'(\varphi))^2\right], \tag{6.96}$$

generalizing the case $n = 1$ of (6.94).

Remark One can show that the restriction of the Bott-Thurston cocycle (6.69) to the $\text{PSL}(2, \mathbb{R})$ subgroup (6.88) coincides with the unique non-trivial two-cocycle on $\text{PSL}(2, \mathbb{R})$. The latter acts on the hyperbolic plane $\mathbb{H}^2$ by isometries of the form (6.87), where $\zeta \in \mathbb{C}$ has positive imaginary part, and the two-cocycle associates with two such transformations f, g the area of the triangle with corners i, $\mathsf{f}^{-1}(i)$ and $\mathsf{g}^{-1} \circ \mathsf{f}^{-1}(i)$. See [1] for details.

6.4 The Virasoro Group

We are now in position to describe the central extension of $\text{Diff}(S^1)$. This discussion is crucial for our purposes, as all symmetry groups of the later chapters will be variations on the basic Virasoro pattern. As announced above, our viewpoint is that the fundamental Virasoro structure is that of the group, from which the rest follows. Accordingly we start this section by reviewing general properties of centrally extended groups, which we then apply to the Virasoro group whose adjoint and coadjoint representations follow by differentiation. We also define the Virasoro algebra and end by displaying the Kirillov-Kostant Poisson bracket on its dual.

6.4.1 Centrally Extended Groups Revisited

Let $\widehat{G}$ be a central extension of some Lie group G, with group operation (2.11) in terms of some two-cocycle C. Here we work out its adjoint and coadjoint representations.

Adjoint Representation

Since the group $\widehat{G}$ consists of pairs (f, λ) where $f \in G$ and $\lambda \in \mathbb{R}$, its Lie algebra $\widehat{\mathfrak{g}}$ consists of pairs (X, λ) where $X \in \mathfrak{g}$. The adjoint representation of $\widehat{G}$ then follows from (5.6): for $X \in \mathfrak{g}$, $f \in G$ and $\lambda, \mu \in \mathbb{R}$ we find

$$\widehat{\mathrm{Ad}}_{(f,\mu)}(X, \lambda) = \frac{d}{dt}\left[\left(f \circ e^{tX} \circ f^{-1}, t\lambda + \mathsf{C}(f, e^{tX}) + \mathsf{C}(f \circ e^{tX}, f^{-1})\right)\right]\Big|_{t=0} \tag{6.97}$$

where the hat in $\widehat{\mathrm{Ad}}$ stresses that this is the adjoint representation of a centrally extended group, as opposed to that of G. Note that μ acts trivially, so we can lighten the notation by writing $\widehat{\mathrm{Ad}}_f$ instead of $\widehat{\mathrm{Ad}}_{(f,\mu)}$. This follows from the fact that $\widehat{G}$ is a central extension of G so that "central elements" (i.e. the real numbers that enter in the second slot of (f, μ)) act trivially on everything, which is a general property of centrally extended groups.

It then remains to compute the two entries on the right-hand side (6.97). The first entry yields the adjoint representation of G, while the second is precisely the expression (6.79) defining the Souriau cocycle S associated with C. We conclude that the adjoint representation of $\widehat{G}$ reads

$$\widehat{\mathrm{Ad}}_f(X, \lambda) = \left(\mathrm{Ad}_f X, \lambda - \frac{1}{12}\langle \mathsf{S}[f], X\rangle\right) \tag{6.98}$$

where the "Ad" on the right-hand side is the adjoint representation of G.

Centrally Extended Algebra

From the adjoint representation of a group one can read off the Lie brackets (5.8) of its algebra. Let therefore (X, λ) and (Y, μ) belong to the centrally extended Lie algebra $\widehat{\mathfrak{g}}$. Using (6.98) we find

$$\left[(X, \lambda), (Y, \mu)\right] = \frac{d}{dt}\left[\left(\mathrm{Ad}_{e^{tX}} Y, \mu - \frac{1}{12}\left\langle \mathsf{S}[e^{tX}], Y\right\rangle\right)\right]_{t=0}. \tag{6.99}$$

The first entry on the right-hand side is the same as expression (5.8) in G; accordingly it boils down to the standard Lie bracket of $\mathfrak{g}$, which we denote as $[X, Y]$. The second entry involves the differential of the Souriau cocycle,

$$\mathsf{s}[X] \equiv \frac{d}{dt}\,\mathsf{S}[e^{tX}]\big|_{t=0}\,, \tag{6.100}$$

paired with $Y \in \mathfrak{g}$. We therefore define a two-cocycle c on $\mathfrak{g}$ by

$$\mathsf{c}(X, Y) \equiv -\frac{1}{12}\,\langle \mathsf{s}[X], Y\rangle \tag{6.101}$$

and the bracket of $\widehat{\mathfrak{g}}$ takes the centrally extended form (2.6):

$$\Big[(X, \lambda), (Y, \rho)\Big] = \big([X, Y], \mathsf{c}(X, Y)\big) = \Big([X, Y], -\frac{1}{12}\,\langle \mathsf{s}[X], Y\rangle\Big). \tag{6.102}$$

The fact that (6.101) is indeed a two-cocycle is inherited from the Souriau cocycle. Note that the central terms λ, μ commute with everything, as they should. In terms of Lie algebra generators the bracket (6.102) takes the general form (2.27).

Coadjoint Representation

The Lie algebra $\widehat{\mathfrak{g}}$ is spanned by pairs (X, λ), so its dual consists of pairs (p, c) where p belongs to $\mathfrak{g}^*$ while $c \in \mathbb{R}$ is a real number, paired with adjoint vectors according to

$$\big\langle (p, c), (X, \lambda)\big\rangle = \langle p, X\rangle + c\lambda \tag{6.103}$$

where the pairing $\langle\cdot,\cdot\rangle$ on the right-hand side is that of $\mathfrak{g}^*$ with $\mathfrak{g}$. The number c is known as a *central charge*. The coadjoint transformation law of (p, c) follows from the definition (5.10). In particular, since central elements act trivially in the adjoint representation (6.97), we can safely write $\widehat{\text{Ad}}^*_{(f,\lambda)} \equiv \widehat{\text{Ad}}^*_f$ for any $(f, \lambda) \in \widehat{G}$, where the hat on top of Ad* indicates that we refer to a representation of the centrally extended group. If then $(X, \lambda) \in \widehat{\mathfrak{g}}$ and $(p, c) \in \widehat{\mathfrak{g}}^*$, one obtains

$$\begin{aligned}\big\langle \widehat{\text{Ad}}^*_f(p, c), (X, \lambda)\big\rangle &\overset{(6.98)}{=} \Big\langle (p, c), \Big(\text{Ad}_{f^{-1}}, \lambda - \frac{1}{12}\big\langle \mathsf{S}[f^{-1}], X\big\rangle\Big)\Big\rangle \\ &\overset{(6.103)}{=} \big\langle p, \text{Ad}_{f^{-1}}X\big\rangle + c\lambda - \frac{c}{12}\big\langle \mathsf{S}[f^{-1}], X\big\rangle\end{aligned}$$

where the pairing $\langle\cdot,\cdot\rangle$ on the right-hand side of the last equation is the centreless pairing of $\mathfrak{g}^*$ with $\mathfrak{g}$. In particular the first term is simply the coadjoint representation of G. Removing the dependence on X, we conclude that

$$\widehat{\text{Ad}}^*_f(p, c) = \Big(\text{Ad}^*_f p - \frac{c}{12}\mathsf{S}[f^{-1}], c\Big)\,. \tag{6.104}$$

Note that the central charge c is left invariant by the coadjoint representation, as it should. Crucially, it also appears in the first entry and thus affects the transformation law of p. In abstract terms, formula (6.104) is the affine G-module (2.33) associated with the Souriau cocycle.

The coadjoint action can be differentiated, which yields a representation (5.11) of the Lie algebra $\widehat{\mathfrak{g}}$. Using (6.102) with the two-cocycle (6.101) one finds

$$\widehat{\mathrm{ad}}^*_X(p,c) = \Big(\mathrm{ad}^*_X p + \frac{c}{12}\mathsf{s}[X], 0\Big) \tag{6.105}$$

where the ad^* on the right-hand side is the coadjoint representation of $\mathfrak{g}$ while s is the infinitesimal Souriau cocycle (6.100). In the remainder of this section we apply these considerations to the Virasoro group.

6.4.2 Virasoro Group

Definition The *Virasoro group* is the universal central extension of $\mathrm{Diff}(S^1)$. It is diffeomorphic to the product $\mathrm{Diff}(S^1) \times \mathbb{R}$ and its elements are pairs (f, λ) where $f \in \mathrm{Diff}(S^1)$ and $\lambda \in \mathbb{R}$, with a group operation (2.11) where C is the Bott-Thurston cocycle (6.69). Explicitly:

$$(f,\lambda)\cdot(g,\mu) = \Big(f\circ g, \lambda+\mu+\mathsf{C}(f,g)\Big). \tag{6.106}$$

We shall denote the Virasoro group by $\widehat{\mathrm{Diff}}(S^1)$.

As in the previous sections we abuse notation and terminology by simply calling "Virasoro group" what is really the universal cover of the maximal connected subgroup of the Virasoro group. It should in fact be written as $\widehat{\widetilde{\mathrm{Diff}}}{}^+(S^1)$, while we denote it by $\widehat{\mathrm{Diff}}(S^1)$ to reduce clutter.

6.4.3 Adjoint Representation and Virasoro Algebra

As a vector space, the Lie algebra of the Virasoro group is equivalent to the direct sum $\mathrm{Vect}(S^1) \oplus \mathbb{R}$. In particular, Virasoro adjoint vectors are pairs (X, λ) where $X = X(\varphi)\partial_\varphi$ is a vector field on the circle and λ is a real number. The adjoint representation of the Virasoro group follows from the group operation (6.106) and the definition (5.6). Thus the adjoint representation takes the form (6.98), where the adjoint representation of $\mathrm{Diff}(S^1)$ is the transformation law (6.18) of vector fields, while S is the Schwarzian derivative (6.76). This result will be instrumental in our definition of the centrally extended BMS_3 group in Sect. 9.2.

The adjoint representation of a group yields the Lie brackets (5.8) of its algebra. In the present case this definition leads to an awkward sign (6.21), which we absorb by declaring that the Lie bracket of the Virasoro algebra is defined by

$$\big[(X,\lambda),(Y,\mu)\big] \equiv -\frac{d}{dt}\left[\widehat{\mathrm{Ad}}_{e^{tX}}(Y,\mu)\right]_{t=0}. \tag{6.107}$$

With this definition formula (6.99) holds up to an overall minus sign on the right-hand side. Using then the infinitesimal Schwarzian derivative (6.74), the pairing (6.34) allows us to recognize the Gelfand-Fuks cocycle (6.43) in $\langle \mathsf{s}[X], Y\rangle$. Thus the Lie bracket of the algebra of the Virasoro group takes the form (6.102), or explicitly

$$\big[(X,\lambda),(Y,\mu)\big] = \big([X,Y], \mathsf{c}(X,Y)\big) \tag{6.108}$$

where $[X, Y]$ is the usual Lie bracket of vector fields.

Definition The *Virasoro algebra* is the Lie algebra $\widehat{\mathrm{Vect}}(S^1) = \mathrm{Vect}(S^1) \oplus \mathbb{R}$ endowed with the Lie bracket (6.108). It is the universal central extension of $\mathrm{Vect}(S^1)$.[4]

In the physics literature it is customary to rewrite the Virasoro algebra in a form analogous to (6.24). Let us therefore define the basis elements

$$\mathcal{L}_m \equiv (\ell_m, 0), \qquad \mathcal{Z} \equiv (0, 1) \tag{6.109}$$

where the ℓ_m's are given by (6.23). The bracket (6.108) then yields $[\mathcal{Z},\mathcal{Z}] = [\mathcal{Z},\mathcal{L}_m] = 0$, as well as

$$i[\mathcal{L}_m,\mathcal{L}_n] = i[(\ell_m,0),(\ell_n,0)] \overset{(6.108)}{=} (i[\ell_m,\ell_n], i\mathsf{c}(\ell_m,\ell_n)) \;.$$

Using the Witt algebra (6.24) and Eq. (6.44), we can rewrite this as

$$i[\mathcal{L}_m,\mathcal{L}_n] = (m-n)\mathcal{L}_{m+n} + \frac{\mathcal{Z}}{12} m^3 \delta_{m+n,0}\,, \tag{6.110}$$

which is indeed the standard expression of the Virasoro algebra [3–5]. In this form it can be seen as a central extension of the Witt algebra (6.24), with a central term involving the celebrated $m^3\delta_{m+n,0}$. As mentioned below (6.44), the latter is a remnant of the third derivative of Y in the Gelfand-Fuks cocycle (6.43), while the $\delta_{m+n,0}$ follows from the integration over the circle and reflects the fact that the cocycle is invariant under rotations.

Remark The generator $\mathcal{Z}$ of Eq. (6.109) should rightfully be called the "central charge" of the Virasoro algebra, since it is a Lie algebra element that commutes with everything. However, in keeping with the standard physics terminology, we will also use the word "central charge" to refer to the *dual* of $\mathcal{Z}$, which is just a real number c (see the coadjoint representation below). This ambiguous terminology should not lead to any confusion.

[4] Universality follows from the fact that the first real cohomology of $\mathrm{Vect}(S^1)$ vanishes.

6.4.4 Coadjoint Representation

Coadjoint Vectors

Virasoro adjoint vectors are pairs (X, λ) where X is a vector field and λ a real number. Accordingly the smooth dual of the Virasoro algebra consists of pairs (p, c) where $p = p(\varphi)d\varphi^2$ is a quadratic density and $c \in \mathbb{R}$ is a real number, paired with adjoint vectors according to the centrally extended generalization (6.103) of (6.34):

$$\big\langle (p, c), (X, \lambda) \big\rangle \equiv \frac{1}{2\pi} \int_0^{2\pi} d\varphi \, p(\varphi) X(\varphi) + c\lambda. \tag{6.111}$$

We refer to pairs (p, c) as Virasoro coadjoint vectors; they span the space $\widehat{\text{Vect}}(S^1)^*$.

It is worth mentioning that coadjoint vectors are crucial physical quantities in all theories enjoying $\text{Diff}(S^1)$ symmetry, and in particular all conformal field theories in two dimensions. Indeed the function $p(\varphi)$ is nothing but (the chiral component of) a CFT stress tensor, while c is a CFT central charge. Expression (6.111) then coincides (up to central terms) with the Noether charge associated with a symmetry generator $X(\varphi)\partial_\varphi$, seen as an infinitesimal conformal transformation. More precisely, in a CFT on a Lorentzian cylinder, the coordinate φ would be replaced by one of the two light-cone coordinates $x^\pm$ and $p(\varphi)$ would become $p(x^+)$ or $\bar{p}(x^-)$. This is consistent with the interpretation of coadjoint vectors as Noether currents, thanks to the momentum maps of Sect. 5.1.

Remark Our notation is somewhat non-standard in that we denote by $p(\varphi)$ what would normally be written as $T(\varphi)$, where T stands for the stress tensor. This choice has to do with our motivations: we shall see in Chap. 9 that Virasoro coadjoint vectors play the role of *supermomentum vectors* for the BMS_3 group. They will be infinite-dimensional generalizations of the Poincaré momenta p_μ, so the notation $p(\varphi)$ is introduced here to suggest thinking of coadjoint vectors as quantities related to energy and momentum. In fact this interpretation also holds in CFT, since a stress tensor is nothing but an energy-momentum density.

Coadjoint Representation

The transformation law of Virasoro coadjoint vectors follows from the definition (5.10). In particular, since central elements act trivially in the adjoint representation (6.97), we may write $\widehat{\text{Ad}}^*_{(f,\lambda)} \equiv \widehat{\text{Ad}}^*_f$ for any (f, λ) belonging to the Virasoro group. If then we let $(X, \lambda) \in \widehat{\text{Vect}}(S^1)$ be an adjoint vector and $(p, c) \in \widehat{\text{Vect}}(S^1)^*$ be a coadjoint one, formula (6.104) still holds upon letting S be the Schwarzian derivative. The central charge c is left invariant by the coadjoint representation, as it should, but it also affects the transformation law of $p(\varphi)$. Accordingly, from now on we often write the coadjoint representation of the Virasoro group *without* including a second slot for the central charge, since the latter is invariant. With this simplified notation formula (6.104) boils down to

$$\widehat{\mathrm{Ad}}^*_f p = \mathrm{Ad}^*_f p - \frac{c}{12}\mathsf{S}[f^{-1}] \tag{6.112}$$

For future reference it will be useful to rewrite this in detail, in terms of functions on the circle. Evaluating both sides of the equation at a point φ on the circle, we obtain

$$\big(\widehat{\mathrm{Ad}}^*_f p\big)(\varphi) = \big[(f^{-1})'(\varphi)\big]^2 p(f^{-1}(\varphi)) - \frac{c}{12}\mathsf{S}[f^{-1}](\varphi) \tag{6.113}$$

by virtue of the centreless coadjoint action (6.36). The formulas are much simpler if we evaluate Eq. (6.112) at $f(\varphi)$; using the cocycle identity (6.77) we find

$$\boxed{\big(\widehat{\mathrm{Ad}}^*_f p\big)(f(\varphi)) = \frac{1}{(f'(\varphi))^2}\left[p(\varphi) + \frac{c}{12}\mathsf{S}[f](\varphi)\right].} \tag{6.114}$$

This is a transparent expression of the coadjoint representation of the Virasoro group, with $\mathsf{S}[f](\varphi)$ given by (6.76). It is the most important equation of this chapter. We will sometimes refer to the two terms on the right-hand side as the "homogeneous term" and the "central" or "inhomogeneous term", respectively. The formula can also be recognized as the transformation law of a CFT stress tensor $p(\varphi)$ with a central charge c; in that context $p(\varphi)$ is said to be a *quasi-primary field* with weight two. In the next chapter we will classify the orbits of this action, which in Chap. 9 will turn out to be supermomentum orbits labelling BMS particles in three dimensions.

The differential (6.105) of formula (6.114) is a representation of the Virasoro algebra. Taking an infinitesimal diffeomorphism $f(\varphi) = \varphi + \epsilon X(\varphi)$, we treat the homogeneous term $\mathrm{Ad}^*_f p$ as in (6.30) and find $\mathrm{Ad}^*_f p = p - \epsilon(Xp' + 2X'p)$ to first order in ϵ (both sides of the equation are evaluated at the same point). For the Schwarzian derivative we use $\mathsf{S}[f^{-1}] = -\epsilon X'''$. Defining

$$\widehat{\mathrm{ad}}^*_X p(\varphi) \equiv -\frac{(\widehat{\mathrm{Ad}}^*_f p)(\varphi) - p(\varphi)}{\epsilon}$$

as in (6.31), we end up with the coadjoint representation of the Virasoro algebra:

$$\widehat{\mathrm{ad}}^*_X p = Xp' + 2X'p - \frac{c}{12}X''', \tag{6.115}$$

where both sides are evaluated at the same point.[5] This is the Virasoro version of Eq. (6.105). In the homogeneous term we recognize the primary transformation law (6.32), while the central term involves the infinitesimal Schwarzian (6.74).

[5]Equation (6.115) can also be written as $\widehat{\mathrm{ad}}^*_X(p, c) \equiv -(p, c) \circ \widehat{\mathrm{ad}}_X$, where the infinitesimal adjoint representation of the Virasoro algebra is defined with a sign such that that $\widehat{\mathrm{ad}}_X(Y, \mu) = \big([X, Y], \mathsf{c}(X, Y)\big)$ coincides with the bracket (6.108).

6.4.5 Kirillov-Kostant Bracket

In order to make contact with physics, let us describe the Kirillov-Kostant Poisson bracket (5.23) for the Virasoro group. In that case the bracket eats functions on $\widehat{\text{Vect}}(S^1)^*$, i.e. functionals $\mathcal{F}[\,p(\varphi), c]$. In practice, since any quadratic density $p(\varphi)d\varphi^2$ can be Fourier-expanded as

$$p(\varphi) = \sum_{m\in\mathbb{Z}} p_m e^{-im\varphi}, \tag{6.116}$$

the Fourier modes $p_m = p^*_{-m}$ define global coordinates on $\text{Vect}(S^1)^*$. Any functional $\mathcal{F}[\,p(\varphi), c]$ can then be seen as a function of the variables p_m and c, so it suffices to know the Poisson brackets of these variables in order to find the Poisson brackets of functions on $\widehat{\text{Vect}}(S^1)^*$.

Now recall the basis (6.109) of the Virasoro algebra and let $\{(\mathcal{L}_m)^*, \mathcal{Z}^*\}$ denote the corresponding dual basis, such that $\langle\mathcal{L}^*_m, \mathcal{L}_n\rangle = \delta_{mn}$ and $\langle\mathcal{Z}^*, \mathcal{Z}\rangle = 1$. Using the pairing (6.111) we find that, as coadjoint vectors,

$$(\mathcal{L}_m)^* = \big((\ell_m)^*, 0\big) = \big(e^{-im\varphi}d\varphi^2, 0\big), \qquad \mathcal{Z}^* = (0, 1). \tag{6.117}$$

Thus, when writing a quadratic density as a Fourier series (6.116), the parameters p_m, c are actually coordinates on $\widehat{\text{Vect}}(S^1)^*$ defined with respect to the basis (6.117):

$$\big(p(\varphi)d\varphi^2, c\big) = \sum_{m\in\mathbb{Z}} p_m\mathcal{L}^*_m + c\,\mathcal{Z}^*.$$

Accordingly, Eq. (5.28) implies that the Kirillov-Kostant Poisson bracket of these coordinates reproduces the Lie brackets (6.110):

$$i\{p_m, p_n\} = (m-n)\,p_{m+n} + \frac{c}{12}m^3\delta_{m+n,0}\,, \tag{6.118}$$

while all Poisson brackets involving the central charge c vanish. The key difference between (6.118) and (6.110) is that the latter is an abstract Lie bracket, while the former is its phase space realization.

The bracket (6.118) is well-known to physicists. Indeed, the standard way to introduce the Virasoro algebra in CFT textbooks is to expand the stress tensor in modes as in (6.116), and then compute their Poisson brackets. Upon quantization, the operator $\widehat{i\{\cdot,\cdot\}}$ coincides with the commutator $[\cdot,\cdot]$ and the resulting quantum commutators span a Virasoro algebra (6.110)–(6.118), generally with a non-zero central charge c.

Note that each coordinate p_m can be seen as the function on $\widehat{\text{Vect}}(S^1)^*$ that maps (p, c) on $\langle(p, c), \mathcal{L}_m\rangle$. As mentioned below (6.111), the object $\langle(p, c), \mathcal{L}_m\rangle$ may be thought of as the Noether charge associated with the symmetry generator $\mathcal{L}_m$, so the Poisson bracket (6.118) can be interpreted as a Poisson bracket of Noether charges. We shall see in Chaps. 8 and 9 that the Poisson brackets of surface charges in three-dimensional gravity coincide with the Kirillov-Kostant brackets on the dual of suitable asymptotic symmetry algebras (albeit with definite values of the central charges).

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Chapter 7
Virasoro Coadjoint Orbits

In this chapter we classify the coadjoint orbits of the Virasoro group. Aside from their usefulness in the study of conformal symmetry, they are crucial for our purposes because they will turn out to coincide with the supermomentum orbits that classify BMS_3 particles. As we shall see, despite being infinite-dimensional, these orbits behave very much like the finite-dimensional coadjoint orbits of $SL(2, \mathbb{R})$.

The plan is as follows. In Sect. 7.1 we describe the problem and explain how it can be addressed in terms of two invariant quantities, namely the conjugacy class of a certain monodromy matrix and the winding number of a related curve taking its values in a circle. We then use this approach in Sect. 7.2 to display explicit orbit representatives. Finally, Sect. 7.3 is devoted to a discussion of energy positivity in the Virasoro context.

Coadjoint orbits of the Virasoro group were first classified in [1, 2] and are described in many later papers [3–6] and textbooks [7, 8]. The presentation of this chapter follows [9].

7.1 Coadjoint Orbits of the Virasoro Group

In this section we explain the methods used to classify coadjoint orbits of the Virasoro group. We start by describing the simple (but pathological) classification of orbits at zero central charge, before discussing certain basic aspects of centrally extended orbits. We then turn to the correspondence between Virasoro coadjoint vectors and Hill's operators, which yields two invariant quantities that can be used to classify the orbits. These invariants are (i) the conjugacy class of a monodromy matrix and (ii) the winding number of a curve on the real line whose target space is a circle.

B. Oblak, *BMS Particles in Three Dimensions*, Springer Theses,
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7.1.1 Centerless Coadjoint Orbits

We start our investigation with a problem that is much simpler than the full classification of coadjoint orbits of the Virasoro group $\widehat{\text{Diff}}(S^1)$, namely the classification of orbits at vanishing central charge, $c = 0$. Those are orbits of the centreless group $\text{Diff}(S^1) = \widetilde{\text{Diff}}^+(S^1)$, whose coadjoint representation is given by Eq. (6.36).

Let us pick a coadjoint vector $\big(p(\varphi)d\varphi^2, c = 0\big)$ and denote its coadjoint orbit by $\mathcal{W}_{(p,0)}$. For now, suppose for simplicity that $p(\varphi)$ is strictly positive for all $\varphi \in [0, 2\pi]$. One can then verify that the integral

$$\sqrt{M} \equiv \frac{1}{2\pi}\int_0^{2\pi} d\varphi\sqrt{p(\varphi)} \tag{7.1}$$

is invariant under the action (6.36) of $\text{Diff}(S^1)$ on p. This actually follows from the fact that p is a quadratic density, so its square root is a one-form and can be integrated on the circle in a $\text{Diff}(S^1)$-invariant way. With this notation the diffeomorphism

$$f(\varphi) \equiv \int_0^{\varphi} d\phi\sqrt{\frac{p(\phi)}{M}} \tag{7.2}$$

maps $p(\varphi)$ on the constant coadjoint vector $f \cdot p = M$. Thus any strictly positive coadjoint vector p belongs to the orbit of a constant M given by (7.1), which is the "mass" associated with $p(\varphi)$. The stabilizer of $p(\varphi) = M$ is the set of diffeomorphisms f such that $M = M/(f'(\varphi))^2$. Since we set $f' > 0$ to preserve orientation, the only solution is $f' = 1$ and the stabilizer of $p = M$ is the group U(1) of rigid rotations $f(\varphi) = \varphi + \theta$. Thus the orbit of any strictly positive coadjoint vector is diffeomorphic to the quotient space $\text{Diff}(S^1)/S^1$. The same analysis applies, up to signs, to strictly negative coadjoint vectors. Note that $\text{Diff}(S^1)/S^1$ has codimension one in $\text{Diff}(S^1)$.

Of course, coadjoint vectors may well vanish at certain points of the circle, and in particular they can change sign; the previous analysis must then be modified. For example, suppose $p(\varphi)$ is everywhere non-negative, but vanishes at the point $\varphi = 0$. We will say that $p(\varphi)$ has a "double zero" at $\varphi = 0$, since both $p(\varphi)$ and $p'(\varphi)$ vanish there. Then the integral (7.1) is still invariant on the orbit of p, but it is no longer true that $p(\varphi)$ can be mapped on a constant because the corresponding would-be diffeomorphism (7.2) is degenerate: its derivative vanishes at $\varphi = 0$. We conclude that the orbit of p is now specified by two invariant statements: first, the fact that the integral of $\sqrt{p}$ takes the value (7.1), and second, the fact that $p(\varphi)$ has one double zero. More generally, if $p(\varphi)$ is everywhere non-negative but has N double zeros at the points $\varphi = \varphi_1, \ldots, \varphi_N$, then the N integrals

$$\int_{\varphi_i}^{\varphi_{i+1}} d\varphi\sqrt{p(\varphi)}$$

(where $\varphi_{N+1} \equiv \varphi_1$) are invariants specifying the orbit of p. The orbit is labelled by the values of these integrals together with their ordering (which is $\text{Diff}^+(S^1)$-invariant) and the statement that all its elements have exactly N double zeros. In particular, the orbit has codimension N in $\text{Diff}(S^1)$ since it is specified by N continuous parameters.

A similar treatment can be applied to coadjoint vectors that change sign on the circle, i.e. functions $p(\varphi)$ having *simple* zeros (where p' does not vanish). The number of such points is always even since $p(\varphi)$ is 2π-periodic, so let us suppose $p(\varphi)$ has $2N'$ simple zeros. Then the integral of $\sqrt{|p(\varphi)|}$ between any two consecutive zeros is $\text{Diff}(S^1)$-invariant as before, so the orbit of p is specified by the $2N'$ values of these integrals, by their ordering and by the sign of $p(\varphi)$ on one of the intervals where it does not vanish. From this we also deduce the general classification of orbits for quadratic densities with a finite number of zeros: if $p(\varphi)$ has N double zeros and $2N'$ simple zeros, its orbit is specified by the values of $N + 2N'$ integral invariants of the form

$$\int_{\varphi_i}^{\varphi_{i+1}} d\varphi \sqrt{|p(\varphi)|} \tag{7.3}$$

(where φ_i and φ_{i+1} are any two consecutive zeros), together with the ordering of these invariants, the specification of whether the points φ_i and φ_{i+1} are simple or double zeros, and the sign of p on a given interval, say $[\varphi_1, \varphi_2]$. The orbit of p then has codimension $N + 2N'$ in $\text{Diff}(S^1)$; in particular there exist orbits with arbitrarily high codimension.

As we can see here, centreless coadjoint orbits are somewhat messy: they can be specified by an arbitrarily large number of parameters. Besides, we haven't even discussed the case of coadjoint vectors $p(\varphi)$ that vanish on a whole open set in S^1 — these have an infinite-dimensional little group and their orbits have infinite codimension in $\text{Diff}(S^1)$. In particular, coadjoint orbits can have arbitrary (even or odd) codimension in $\text{Diff}(S^1)$. This is in sharp contrast with finite-dimensional Lie groups, where all coadjoint orbits are even-dimensional since they are symplectic manifolds. In the case of $\text{Diff}(S^1)$, coadjoint orbits are still symplectic, but they need not satisfy "codimension parity": the fact that a given orbit has codimension N does not imply that there are no orbits with codimension $N \pm 1$. We shall see that this pathology does not occur when the Virasoro central charge is non-zero, where all orbits have codimension one or three.

7.1.2 Basic Properties of Centrally Extended Orbits

Let us turn to coadjoint orbits of the Virasoro group at *non-zero* central charge $c \neq 0$. From now on we pick some non-zero value for c and we stick to it; for definiteness we take $c > 0$, although all our considerations also apply to $c < 0$ after a few straightforward sign modifications. In principle our goal is to address the following problems:

1. Classify all Virasoro coadjoint orbits with central charge c.
2. Find a non-redundant, exhaustive set of orbit representatives.
3. Given a coadjoint vector $p(\varphi)d\varphi^2$ (at central charge c), write down the diffeomorphism $f \in \text{Diff}(S^1)$ that maps it on one of the orbit representatives.

If we manage to satisfy these criteria, we will have fully classified the orbits of the Virasoro group (at non-zero central charge).

While this task was relatively easy in the centreless case thanks to the integral invariants (7.3), it turns out to be much more complicated in the centrally extended case; the remainder of this chapter is devoted to orbits at non-zero central charge, where the classification will rely on elaborate techniques involving monodromy matrices. For now we simply describe the most elementary aspects of some of these orbits.

Stabilizers

Suppose we are given a coadjoint vector $\big(p(\varphi)d\varphi^2, c\big)$. Since the central charge is invariant, the orbit of (p, c) under the Virasoro group can be represented as

$$\mathcal{W}_{(p,c)} = \left\{\widehat{\text{Ad}}^*_f p \,\middle|\, f \in \text{Diff}(S^1)\right\}, \tag{7.4}$$

where $\widehat{\text{Ad}}^*_f p$ is given by (6.114). It is an infinite-dimensional manifold, so obtaining information on its geometry sounds at first like an impossible task. Accordingly, instead of actually trying to picture the orbit as such, let us look for the stabilizer G_p of p, which is a subgroup of $\text{Diff}(S^1)$ such that

$$\mathcal{W}_{(p,c)} \cong \widehat{\text{Diff}}(S^1)/(G_p \times \mathbb{R}) \cong \text{Diff}(S^1)/G_p\,. \tag{7.5}$$

The stabilizer consists of diffeomorphisms $f(\varphi)$ such that

$$p(f(\varphi)) = \frac{1}{(f'(\varphi))^2}\left[p(\varphi) + \frac{c}{12}\mathsf{S}[f](\varphi)\right]. \tag{7.6}$$

Given $p(\varphi)$, this is a highly non-linear differential equation for $f(\varphi)$; if we could actually solve it, we would know the stabilizer.

To make things simpler let us look only for the Lie algebra of the stabilizer, rather than the stabilizer itself. This algebra is spanned by vector fields X that leave $p(\varphi)$ invariant, which according to (6.115) amounts to the requirement

$$Xp' + 2X'p - \frac{c}{12}X''' = 0\,. \tag{7.7}$$

This is already a lot easier than Eq. (7.6): it is a *linear* third order equation for the function $X(\varphi)$, assuming that the function $p(\varphi)$ is known. A number of important consequences follow from this equation. The first is that, for non-zero c, it admits at most three linearly independent solutions:

Lemma The stabilizer of $p(\varphi)d\varphi^2$ at non-zero central charge is at most three-dimensional.

This is already a sharp difference with respect to the centreless case, where stabilizers had arbitrarily high dimension. If we were on a line rather than a circle, we would actually conclude from (7.7) that the stabilizer is *always* three-dimensional; but the requirement of periodicity restricts the space of allowed solutions X for a given p, as we shall see momentarily.

Constant Coadjoint Vectors

It is worth exploring the solutions of (7.7) in the simple case where $p(\varphi) = p_0$ is a constant. The equation then reduces to

$$X''' - \frac{24p_0}{c}X' = 0\,,$$

whose general solution is a sum of exponentials

$$X(\varphi) = A + B\,e^{\sqrt{\frac{24p_0}{c}}\,\varphi} + C\,e^{-\sqrt{\frac{24p_0}{c}}\,\varphi} \tag{7.8}$$

where A is real while B and C are generally complex coefficients, being understood that $\sqrt{24p_0/c}$ is purely imaginary when $p_0 < 0$. For generic values of p_0, the only 2π-periodic solution of this type is a constant $X(\varphi) = \text{const}$. In that case the stabilizer is one-dimensional, and consists of rigid rotations of the circle. But there also exist exceptional values of p_0 whose stabilizer is larger, namely

$$p_0 = -\frac{n^2c}{24} \tag{7.9}$$

where $n \in \mathbb{N}^*$ is a positive integer. At such values the exponentials in (7.8) are $e^{\pm in\varphi}$ and the corresponding vector field X is automatically 2π-periodic (and real upon setting $C = B^*$). Thus, for exceptional constants (7.9), the stabilizer is three-dimensional. Its Lie algebra is isomorphic to $\mathfrak{sl}(2,\mathbb{R})$; we will see below that the stabilizer itself is an n-fold cover of $\mathrm{PSL}(2,\mathbb{R})$. In particular, orbits of generic constants p_0 are radically different from orbits of exceptional constants (7.9). The situation is depicted in Fig. 7.1.

If one thinks of $\mathrm{Diff}(S^1)$ as a group of conformal transformations and identifies $p(\varphi)$ with the stress tensor of a CFT, one can recognize in (7.9) with $n = 1$ the vacuum value of a stress tensor on the cylinder:

$$p_{\text{vac}} = -\frac{c}{24}\,. \tag{7.10}$$

We shall see below that this interpretation is indeed correct, as the coadjoint orbit of p_{vac} turns out to be the lowest-lying orbit with energy bounded from below and has its energy minimum at p_{vac}. By contrast, the points (7.9) with $n \geq 2$ belong to orbits with energy unbounded from below.

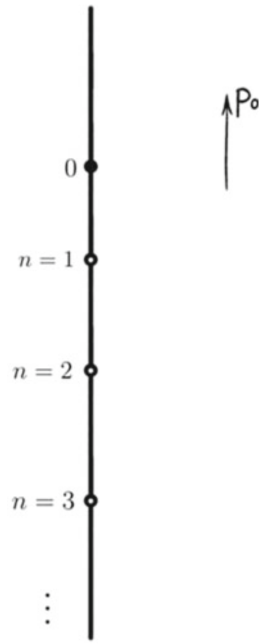

Fig. 7.1 The map of Virasoro coadjoint orbits with constant representatives. The *open dots* labelled by $n = 1, 2, 3, \ldots$ indicate the location of the exceptional points $-c/24$, $-4c/24$, $-9c/24$, etc

7.1.3 Hill's Equation and Monodromy

At this point we have seen the most basic features of Virasoro orbits at $c \neq 0$; we now describe the first step of the complete classification by explaining the relation between orbits and monodromy matrices for solutions of Hill's equations. To lighten the notation, from now on we write $\widehat{\mathrm{Ad}}^*_f p \equiv f \cdot p$.

Until the next section it will be convenient to think of the coordinate φ as spanning the real line $\mathbb{R}$, without identification $\varphi \sim \varphi + 2\pi$ (this will be justified below). Accordingly we now reinstate the notation $\widetilde{\mathrm{Diff}}^+(S^1)$ for the universal cover of the group of orientation-preserving diffeomorphisms of the circle and we think of it as a subgroup of $\mathrm{Diff}^+(\mathbb{R})$. Functions on the circle then are 2π-periodic functions on $\mathbb{R}$. We also sometimes use the words "conformally invariant" or "conformally equivalent" to refer to objects that are $\widetilde{\mathrm{Diff}}^+(S^1)$-invariant or $\widetilde{\mathrm{Diff}}^+(S^1)$-equivalent, respectively.

Virasoro Symmetry of Hill's Equation

The key idea of the classification is the following: given a coadjoint vector (p, c), we can associate with it a differential operator

$$\Delta_{(p,c)} \equiv -\frac{c}{6}\frac{\partial^2}{\partial\varphi^2} + p(\varphi) \tag{7.11}$$

where φ is a coordinate on the real line, $p(\varphi)$ is 2π-periodic, and the operator $\Delta_{(p,c)}$ acts on suitable densities on the real line. The normalization in front of ∂_φ^2 is chosen so as to ensure that the operator has good transformation properties under $\widetilde{\text{Diff}}^+(S^1)$, as we shall see below. Note that the crucial term ∂_φ^2 disappears if $c = 0$, which is why the considerations that follow apply exclusively to orbits with non-zero central charge.

Definition Let (p, c) be a Virasoro coadjoint vector with $c \neq 0$. The associated *Hill's equation* is the second-order, linear differential equation

$$-\frac{c}{6}\psi''(\varphi) + p(\varphi)\psi(\varphi) = 0 \tag{7.12}$$

for the real-valued function $\psi(\varphi)$ on the real line. With the notation (7.11) this is just the statement $\Delta_{(p,c)} \cdot \psi = 0$.

Hill's equation may be seen as a non-relativistic Schrödinger equation on the real line for a "wavefunction" $\psi(\varphi)$ with a periodic "potential energy" $p(\varphi)$, up to the fact that ψ is real and need not be square-integrable.[1] Thus we can associate an equation (7.12) with each coadjoint vector (p, c), and vice-versa.

The transformation law of (p, c) under $\widetilde{\text{Diff}}^+(S^1)$ determines that of Hill's operator (7.11). Using (6.112) and the centreless transformation law (6.36), we find

$$\Delta_{(f\cdot p,c)}(\varphi) = -\frac{c}{6}\partial_\varphi^2 + p(f^{-1}(\varphi))\big((f^{-1})'(\varphi)\big)^2 - \frac{c}{12}\mathsf{S}[f^{-1}](\varphi)\,, \tag{7.13}$$

which is indeed very different from the original operator (7.11). The key point, however, is that the associated Hill's equation (7.12) can be made conformally invariant by choosing a suitable transformation law for $\psi(\varphi)$:

Lemma If $\psi(\varphi)$ is a density with weight $-1/2$ on the real line, then Hill's equation (7.12) is invariant under $\widetilde{\text{Diff}}^+(S^1)$.

Proof To simplify formulas, let us act on Hill's operator with a diffeomorphism f^{-1} rather than f so that $\big(f^{-1}\cdot\psi\big)(\varphi) = \psi(f(\varphi))(f'(\varphi))^{-1/2}$. Then f^{-1} maps the left-hand side of Hill's equation (7.12) on

$$\begin{aligned}-\frac{c}{6}\partial_\varphi^2\big[\psi(f(\varphi))(f'(\varphi))^{-1/2}\big] + p(f(\varphi))\psi(f(\varphi))(f'(\varphi))^{3/2}\\ -\frac{c}{12}\mathsf{S}[f](\varphi)\,(f'(\varphi))^{-1/2}\psi(f(\varphi))\end{aligned} \tag{7.14}$$

where the term with a second derivative can be written as

$$\partial_\varphi^2\big[\psi(f(\varphi))(f'(\varphi))^{-1/2}\big] = \psi''(f(\varphi))(f'(\varphi))^{3/2} - \frac{1}{2}\psi(f(\varphi))(f'(\varphi))^{-1/2}\mathsf{S}[f](\varphi)\,.$$

[1] Note that Hill's operator (7.11) coincides with a Sturm-Liouville operator with periodic potential.

Here the term involving the Schwarzian derivative cancels that of (7.14); the latter expression can therefore be rewritten as

$$\big(f'(\varphi)\big)^{3/2}\left[-\frac{c}{6}\psi''(\varphi)+p(\varphi)\psi(\varphi)\right].$$

Provided ψ solves (7.12), this vanishes. ■

As a consequence of this lemma, the map that associates Hill's equations with Virasoro coadjoint vectors (p, c) is $\widetilde{\text{Diff}}^+(S^1)$-invariant. Thus, Hill's equation is an invariant associated with each coadjoint orbit of the Virasoro group, and classifying Virasoro orbits is equivalent to classifying all $\widetilde{\text{Diff}}^+(S^1)$-inequivalent Hill's equations.

Monodromy

Hill's equation (7.12) is a second-order linear differential equation, so its solutions span a two-dimensional vector space. Let ψ_1 and ψ_2 be two linearly independent solutions. We define their *Wronskian* as

$$W \equiv \det\begin{pmatrix}\psi_1 & \psi_2\\ \psi_1' & \psi_2'\end{pmatrix} = \psi_1\psi_2' - \psi_2\psi_1'\,. \tag{7.15}$$

The Wronskian is constant on the real line ($W' = 0$) by virtue of Hill's equation. Furthermore W does not vanish since ψ_1 and ψ_2 are linearly independent. (Conversely, if the Wronskian does not vanish, then the solutions ψ_1, ψ_2 are linearly independent.) Thus we can always choose

$$W[\psi_1, \psi_2] = -1\,. \tag{7.16}$$

We will refer to this equality as the "Wronskian condition" and to the solutions that satisfy it as being "normalized". Note that the Wronskian (7.15) is invariant under $\widetilde{\text{Diff}}^+(S^1)$ when the ψ_i's transform as densities of weight $-1/2$, regardless of them solving Hill's equation:

$$W[f\cdot\psi_1, f\cdot\psi_2](\varphi) = W[\psi_1,\psi_2](f^{-1}(\varphi))\,.$$

In particular, for solutions of Hill's equation, the Wronskian is constant:

$$W[f\cdot\psi_1, f\cdot\psi_2] = W[\psi_1,\psi_2] \qquad \text{when } \psi_1, \psi_2 \text{ solve (7.12).} \tag{7.17}$$

Hill's equation is a differential equation on the real line $\varphi \in \mathbb{R}$ with a 2π-periodic potential $p(\varphi)$. Its solutions need not be periodic, but they do satisfy certain constraints due to the periodicity of $p(\varphi)$:

Lemma Let $p(\varphi)$ be 2π-periodic and let ψ_1, ψ_2 be linearly independent solutions of Hill's equation (7.12). Then there exists a *monodromy matrix* $\mathsf{M} \in \text{SL}(2, \mathbb{R})$ such that, for any $\varphi \in \mathbb{R}$,

$$\begin{pmatrix}\psi_1(\varphi+2\pi)\\ \psi_2(\varphi+2\pi)\end{pmatrix} = \mathsf{M}\cdot\begin{pmatrix}\psi_1(\varphi)\\ \psi_2(\varphi)\end{pmatrix}. \tag{7.18}$$

Proof Let ψ_1, ψ_2 be two linearly independent solutions of (7.12), and define $\tilde{\psi}_i(\varphi) \equiv \psi_i(\varphi + 2\pi)$ for $i = 1, 2$. Then the Wronskian associated with $\tilde{\psi}_{1,2}$ takes the same value as that of $\psi_{1,2}$; furthermore, the functions $\tilde{\psi}_i$ solve the same Hill's equation as the functions ψ_i since $p(\varphi)$ is 2π-periodic. This implies that there exists a real matrix M such that (7.18) holds for any $\varphi \in \mathbb{R}$. Since the ψ_i's and the $\tilde{\psi}_i$'s have the same Wronskian, M must have unit determinant. ∎

Thus we can associate a monodromy matrix with any Virasoro coadjoint vector and any pair of (normalized) solutions of the corresponding Hill's equation. From now on we use the notation

$$\Psi \equiv \begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} \tag{7.19}$$

for the "solution vector" associated with the basis of solutions ψ_1, ψ_2. Relation (7.18) then becomes $\Psi(\varphi+2\pi) = \mathsf{M}\cdot\Psi(\varphi)$. If we were to choose another normalized basis of solutions, say $(\phi_1, \phi_2) = \Phi^t$, there would be a linear relation $\Phi = S \cdot \Psi$ between solution vectors, for some matrix $S \in \mathrm{SL}(2, \mathbb{R})$. Accordingly the monodromy matrix M_Φ associated with Φ would be related to the monodromy M_Ψ of Ψ by $\mathsf{M}_\Phi = S\mathsf{M}_\Psi S^{-1}$. Thus the monodromy matrix changes by conjugation in $\mathrm{SL}(2, \mathbb{R})$ under changes of bases of normalized solutions. In particular, the *conjugacy class* of M,

$$[\mathsf{M}] \equiv \left\{ S\mathsf{M}S^{-1} \middle| S \in \mathrm{SL}(2, \mathbb{R}) \right\},$$

is invariant under changes of bases. It depends only on the function $p(\varphi)$, and not on the choice of solutions Ψ.

We have shown above that Hill's equation is invariant under $\widetilde{\mathrm{Diff}}^+(S^1)$ in the sense that, if ψ solves the equation with a potential $p(\varphi)$, then $f \cdot \psi$ solves the same equation with a potential $f \cdot p$. In addition we have seen in (7.17) that this transformation preserves the Wronskian condition, so that normalized solutions remain normalized under $\widetilde{\mathrm{Diff}}^+(S^1)$. Accordingly, if Ψ is a normalized solution vector for the potential p, then $f \cdot \Psi$ is a normalized solution vector for $f \cdot p$. And now comes the key argument: since both Hill's equation and the action of $\widetilde{\mathrm{Diff}}^+(S^1)$ on Ψ are linear, the monodromy matrix of $f \cdot \Psi$ coincides with that of Ψ. We therefore conclude:

Theorem Let $c \neq 0$ and denote by $[\mathsf{M}]_{(p,c)}$ the conjugacy class of any monodromy matrix M associated with the Hill's equation (7.12) specified by $p(\varphi)$ and c. Then there is a well-defined map

$$\{\text{Virasoro orbits at central charge } c\} \to \{\text{Conjugacy classes of } \mathrm{SL}(2, \mathbb{R})\} \tag{7.20}$$

that associates with a coadjoint orbit $\mathcal{W}_{(p,c)}$ the equivalence class $[\mathsf{M}]_{(p,c)}$ of the corresponding monodromy matrix. In particular, Virasoro coadjoint vectors with the same central charge but non-conjugate monodromy matrices do not belong to the same orbit.

This result illustrates the power of Hill's operators. It provides a rough classification of Virasoro orbits by allowing us to distinguish orbits with non-conjugate monodromies and may be seen as an infinite-dimensional analogue of the classification of coadjoint orbits of $\mathrm{SL}(2,\mathbb{R})$ according to the value of the "mass squared". In particular the trace $\mathrm{Tr}(\mathsf{M})$ is a conformally invariant quantity. However, the classification is not precise in that two orbits whose monodromy matrices are conjugate may well be different: the map (7.20) need not be injective (and we shall see below that it is not). To make further progress we need to investigate Hill's equation in more detail.

For future reference, note the following: thanks to the fact that the conjugacy class of the monodromy matrix is independent of the choice of a solution vector Ψ for Hill's equation associated with (p, c), one can write its trace as a Wilson loop

$$\mathrm{Tr}(\mathsf{M}) = \mathrm{Tr}\left(P\ \exp\left[\int_0^{2\pi} d\varphi \begin{pmatrix} 0 & 1 \\ 6p(\varphi)/c & 0 \end{pmatrix}\right]\right) \tag{7.21}$$

where P denotes path ordering. This quantity is conformally invariant, so one can replace (p, c) by any coadjoint vector (q, c) belonging to its orbit without affecting the value of (7.21). In particular, if (p, c) belongs to the orbit of a constant coadjoint vector (p_0, c) with positive p_0, the trace reads

$$\mathrm{Tr}(\mathsf{M}) = 2\cosh\left(2\pi\sqrt{\frac{6p_0}{c}}\right). \tag{7.22}$$

The same formula holds for negative p_0, with $\cosh(ix) = \cos(x)$. We will put it to use in Sect. 10.1 when defining the mass of BMS_3 particles.

Hill's Equation and Stabilizers

The stabilizer of a coadjoint vector (p, c) consists of diffeomorphisms that satisfy (7.6). Let us see how this information is related to Hill's equation (7.12). First note that, if ψ_1 and ψ_2 are linearly independent solutions of Hill's equation, then the combinations

$$\psi_1^2, \quad \psi_1\psi_2, \quad \psi_2^2 \tag{7.23}$$

all solve the stabilizer equation (7.7). These products are generally not 2π-periodic and therefore do not represent vector fields on the circle, but one can show that there always exist either one or three 2π-periodic linear combinations of these products. This confirms our earlier observation that the stabilizer of all orbits is either one- or three-dimensional. Note that, being $-1/2$-densities on the circle, the products (7.23) were bound to be densities of weight -1, i.e. vector fields.

Let us now see how the stabilizer G_p of (p, c) is described in the Hill language. If $f \in G_p$ and if Ψ is a normalized solution vector of Hill's equation associated with (p, c), the action of G_p on Ψ is such that $f \cdot \Psi$ provides another normalized solution vector for the same equation. Accordingly there exists some (constant) $\mathrm{SL}(2,\mathbb{R})$ matrix A_f such that

$$f \cdot \Psi = A_f^{-1}\Psi. \tag{7.24}$$

In addition we have seen that the action of $\mathrm{Diff}(S^1)$ leaves the monodromy matrix invariant, so the monodromy of $f \cdot \Psi$ coincides with the monodromy M of Ψ. Combining this statement with (7.24) we conclude that $A_f^{-1}\mathsf{M}A_f = \mathsf{M}$, which is to say that A_f belongs to the stabilizer G_{M} of M with respect to conjugation. In addition the inversion A_f^{-1} in (7.24) ensures that $A_{fg} = A_f A_g$, so we conclude:

Lemma Let (p, c) be a Virasoro coadjoint vector with $c \neq 0$, Ψ a normalized solution vector of the associated Hill's equation. Let G_p be the stabilizer of p for the coadjoint action (6.114) and let G_{M} be the stabilizer of M for conjugation. Then the map

$$\mathcal{A} : G_p \to G_{\mathsf{M}} : f \mapsto \mathcal{A}(f) \equiv A_f \tag{7.25}$$

defined by (7.24) is a homomorphism.

This map relates the stabilizer of p to that of the corresponding monodromy matrix. In particular it allows us to classify the conformally inequivalent solutions of Hill's equation at fixed (p, c). Indeed, the set of normalized solution vectors of Hill's equation at p with fixed monodromy M is in one-to-one correspondence with the elements of G_{M}, so the set of orbits of the stabilizer G_p in that set of solutions is a quotient

$$G_{\mathsf{M}}/\mathrm{Im}(\mathcal{A}) \tag{7.26}$$

where $\mathrm{Im}(\mathcal{A})$ is the image of (7.25). Two solution vectors are conformally equivalent if and only if they belong to the same orbit under G_p, i.e. if they define the same point in (7.26).

Remark The fact that the products of half-densities (7.23) solving Hill's equation produce integer densities solving the stabilizer equation (7.7) is reminiscent of the fact that the "square" of two Killing spinors is a Killing vector. This correspondence is exactly realized in three-dimensional gravity: Eq. (7.7) turns out to coincide with the Killing equation expressed in terms of a suitable component X of a vector field on space-time, while Hill's equation (7.12) corresponds to the Killing spinor equation for a suitable spinor component (see e.g. Eq. (16) in [10]).

7.1.4 Winding Number

The conjugacy class of monodromy matrices provides a continuous parameter that roughly classifies Virasoro orbits. We now describe a second invariant quantity which, combined with monodromies, will provide a precise classification of orbits. This second invariant turns out to be the discrete winding number of a path in the circle.

Let ψ_1 and ψ_2 be normalized solutions of Hill's equation (7.12). They have non-zero weight under $\widetilde{\text{Diff}}^+(S^1)$, but their ratio

$$\eta(\varphi) \equiv \frac{\psi_1(\varphi)}{\psi_2(\varphi)} \tag{7.27}$$

transforms under $\widetilde{\text{Diff}}^+(S^1)$ as a function (i.e. a zero-weight density). It blows up at the zeros of ψ_2, so it is more convenient to think of it as a curve

$$\eta : \mathbb{R} \to \mathbb{R}P^1 : \varphi \mapsto \eta(\varphi) \tag{7.28}$$

whose expression is (7.27) in terms of the projective coordinate (6.84). The points where η diverges are then mapped by η on the "point at infinity" in $\mathbb{R}P^1$. Since $\mathbb{R}P^1$ is diffeomorphic to the circle (6.83), we can also think of η as a path in S^1 whose expression in stereographic coordinates is (7.27).

Coadjoint Vectors from Projective Curves

As in (7.19) we denote the basis of solutions $\psi_{1,2}$ by Ψ. Then the quasi-periodicity (7.18) of Ψ implies a similar "projective" monodromy for $\eta(\varphi)$,

$$\eta(\varphi + 2\pi) = \frac{a\,\eta(\varphi) + b}{c\,\eta(\varphi) + d} \tag{7.29}$$

where a, b, c, d are the entries of the monodromy matrix M. If we let $\Phi = A\Psi$ be another normalized basis of solutions with $A \in \text{SL}(2, \mathbb{R})$, the curve $\tilde{\eta} = \phi_1/\phi_2$ corresponding to Φ by (7.27) is related to η by a projective transformation of the form (6.87). In particular Eq. (6.91) implies that the Schwarzian derivative of η with respect to φ is left unchanged by such a transformation. Thus the Schwarzian derivative of η is invariant under changes of (normalized) bases of solutions of Hill's equation, which is consistent with the following observation:

Lemma Let ψ_1 and ψ_2 be normalized solutions of Hill's equation (7.12) and $\eta \equiv \psi_1/\psi_2$. Then the function $p(\varphi)$ is specified by the solutions of its Hill's equation:

$$\mathsf{S}[\eta](\varphi) = -\frac{12}{c}\,p(\varphi)\,. \tag{7.30}$$

Proof By virtue of the Wronskian condition (7.16),

$$\eta' = \frac{1}{(\psi_2)^2}\,. \tag{7.31}$$

It then follows from the definition (6.76) that $\mathsf{S}[\eta](\varphi) = -2\frac{\psi_2''(\varphi)}{\psi_2(\varphi)}$, which coincides with the right-hand side of (7.30) upon using Hill's equation (7.12). ∎

This lemma says that the correspondence between Virasoro coadjoint vectors and solutions of Hill's equation goes bothways: Hill's equation specifies certain solutions, which in turn uniquely determine the periodic potential $p(\varphi)$ via (7.30). In particular the coadjoint transformation law (6.114) of p can be rewritten in terms of the scalar transformation law of (7.27) plugged into (7.30).

Winding Numbers

One can think of $\eta(\varphi)$ as a path in the circle with a "time parameter" φ. Equation (7.31) then says that $\eta'(\varphi) > 0$, so $\eta(\varphi)$ always spins around the circle in the same direction. We therefore introduce the following terminology:

Definition The *winding number* $n \in \mathbb{N}$ of $\eta(\varphi)$ is the number of laps around the circle performed by η in a "time interval" of length 2π.

We will illustrate the computation of the winding number in the next section, when describing explicit Virasoro orbit representatives. For now note that $\eta(\varphi)$ transforms under $\widetilde{\text{Diff}}^+(S^1)$ as a function, so its winding number is conformally invariant:

Proposition Let $c \neq 0$ and let $n_{(p,c)} \in \mathbb{N}$ be the winding number of the curve η associated with the Hill's equation (7.12) specified by $p(\varphi)$ and c. Then there is a well-defined map

$$\{\text{Virasoro orbits at central charge } c\} \to \mathbb{N} : \mathcal{W}_{(p,c)} \mapsto n_{(p,c)} \,. \tag{7.32}$$

In particular, Virasoro coadjoint vectors with the same central charge but different winding numbers do not belong to the same orbit.

This supplements our previous observation (7.20) that the conjugacy classes of monodromy matrices yield a rough classification of Virasoro coadjoint orbits. In fact, these two invariants together provide the complete classification of Virasoro orbits. Indeed one can show that the map that associates a pair $([\mathsf{M}], n)$ with each Virasoro coadjoint orbit is injective, provided $[\mathsf{M}]$ is the conjugacy class of the monodromy matrix and n is the winding number. Note however that the map is not surjective, as some pairs $([\mathsf{M}], n)$ do not belong to its image. We now verify this by brute force by describing orbit representatives.

7.2 Virasoro Orbit Representatives

Virasoro coadjoint orbits are classified by two parameters, one of them continuous (the conjugacy class of the monodromy M), the other discrete (the winding number n). In this section we display explicit orbit representatives for all admissible pairs $([\mathsf{M}], n)$, after a brief review of conjugacy classes in $\text{SL}(2, \mathbb{R})$. We end with a picture of orbits that extends Fig. 7.1. As before, we assume that the central charge c is positive.

7.2.1 Prelude: Conjugacy Classes of SL(2, $\mathbb{R}$)

In order to classify the conjugacy classes of SL(2, $\mathbb{R}$), we note that the trace of an SL(2, $\mathbb{R}$) matrix is invariant under conjugation; matrices with different traces cannot be conjugate. This motivates the following terminology:

$$\text{An SL}(2,\mathbb{R}) \text{ matrix } \mathsf{M} \text{ is } \begin{cases} elliptic & \text{if } |\text{Tr}(\mathsf{M})| < 2; \\ parabolic & \text{if } |\text{Tr}(\mathsf{M})| = 2; \\ hyperbolic & \text{if } |\text{Tr}(\mathsf{M})| > 2. \end{cases}$$

Each conjugacy class of SL(2, $\mathbb{R}$) is contained in one of these three families, but each family contains several conjugacy classes. The elliptic and hyperbolic families contain infinitely many conjugacy classes since they depend on a continuous parameter (the trace of M). Note that the trace of M determine the properties of its eigenvalues:

$$\begin{aligned} \mathsf{M} \text{ is elliptic} &\leftrightarrow \text{distinct complex eigenvalues;} \\ \mathsf{M} \text{ is parabolic} &\leftrightarrow \text{degenerate real eigenvalue} \pm 1; \\ \mathsf{M} \text{ is hyperbolic} &\leftrightarrow \text{distinct real eigenvalues.} \end{aligned}$$

We now determine the conjugacy classes contained in each family. The computations are very similar to those of Sect. 4.3 where we determined the orbits of momenta for the Poincaré group in three dimensions.

Lemma (elliptic family) Let M be elliptic. Then it is conjugate to a unique rotation matrix

$$\begin{pmatrix} \cos(2\pi\omega) & \sin(2\pi\omega) \\ -\sin(2\pi\omega) & \cos(2\pi\omega) \end{pmatrix} \tag{7.33}$$

where ω belongs to the set $]0, 1/2[\,\cup\,]1/2, 1[$. The stabilizer of (7.33) is the U(1) rotation subgroup (4.82) of SL(2, $\mathbb{R}$).

Proof In the elliptic family, the eigenvalues of M are complex conjugates of one another with non-zero imaginary part. Since $\det(\mathsf{M}) = 1$, they can be written as $e^{\pm 2\pi i\omega}$ where ω belongs to the open interval $]0, 1[$ without loss of generality, but differs from $1/2$. Let $v \in \mathbb{C}^2$ be an eigenvector of M such that $\mathsf{M} \cdot v = e^{2\pi i\omega} v$. This vector is complex and linearly independent of its complex conjugate $\bar{v}$; the latter is an eigenvector of M with eigenvalue $e^{-2\pi i\omega}$. Then $v + \bar{v}$ and $i(v - \bar{v})$ are linearly independent real vectors; we can choose the norm of v in such a way that the (real) matrix S expressing M in the basis $\{v + \bar{v}, i(v - \bar{v})\}$ has unit determinant. Then $S\mathsf{M}S^{-1}$ takes the form (7.33). The stabilizer consists of all matrices that commute with (7.33) and is readily seen to consist of rotations. ■

Lemma (parabolic family) Let M be parabolic. Then it is conjugate to exactly one of the following six matrices:

$$\pm\begin{pmatrix}1&0\\0&1\end{pmatrix}, \quad \pm\begin{pmatrix}1&1\\0&1\end{pmatrix}, \quad \pm\begin{pmatrix}1&-1\\0&1\end{pmatrix}. \tag{7.34}$$

The stabilizer of the first two matrices is the whole group $SL(2,\mathbb{R})$, while the stabilizer $\mathbb{R}\times\mathbb{Z}_2$ of the four remaining ones consists of triangular matrices (4.96).

Proof When M is parabolic, its eigenvalues are either both 1 or both -1. Let λ be the eigenvalue of M and let $v\in\mathbb{R}^2$ be a (real) eigenvector of M. Let v' be another vector such that $\{v, v'\}$ is a basis of $\mathbb{R}^2$, and choose the normalization of v and v' in such a way that the matrix S expressing M in this basis has unit determinant. Then

$$S\mathsf{M}S^{-1}=\begin{pmatrix}\lambda&x\\0&\lambda\end{pmatrix} \tag{7.35}$$

where $\lambda=\pm1$ and x is an arbitrary real number. For $x=0$ we find the first two matrices in the list (7.34), each of which is alone in its conjugacy class. For non-zero x, note that

$$\begin{pmatrix}y&0\\0&1/y\end{pmatrix}\begin{pmatrix}1&\pm1\\0&1\end{pmatrix}\begin{pmatrix}1/y&0\\0&y\end{pmatrix}=\begin{pmatrix}1&\pm y^2\\0&1\end{pmatrix}, \tag{7.36}$$

so for $\lambda=+1$, M is conjugate to the second matrix in (7.34) if $x>0$ and to the third one if $x<0$, in both cases with an overall plus sign. The situation is similar when $\lambda=-1$, but with the minus sign. The proof ends with the observation that all matrices in (7.34) belong to disjoint conjugacy classes. The stabilizer is obtained by direct computation. ■

Lemma (hyperbolic family) Let M be hyperbolic. Then it is conjugate to a unique matrix of the form

$$\pm\begin{pmatrix}e^{2\pi\omega}&0\\0&e^{-2\pi\omega}\end{pmatrix} \tag{7.37}$$

where ω is a strictly positive real number. Its stabilizer is the group $\mathbb{R}\times\mathbb{Z}_2$ consisting of matrices (4.95) of the same form as (7.37) but without restriction on $\omega\in\mathbb{R}$.

Proof Since M is hyperbolic, it has two distinct real eigenvalues λ and $1/\lambda$, where $\lambda\in\mathbb{R}^*$. Let v and v' be two eigenvectors of M for these eigenvalues; we can normalize them so that the matrix S expressing M in the basis $\{v, v'\}$ has unit determinant. Then $S\mathsf{M}S^{-1}$ takes the form (7.37) with $e^{2\pi\omega}=\lambda$ or $e^{2\pi\omega}=1/\lambda$. The ordering of eigenvalues can be changed thanks to the $SL(2,\mathbb{R})$ matrix $\begin{pmatrix}0&1\\-1&0\end{pmatrix}$, so we are free to pick $\omega>0$, and this specifies uniquely the conjugacy class of the matrix M. Finding the stabilizer is straightforward. ■

From now on we say that a Virasoro orbit is elliptic, parabolic or hyperbolic if the associated monodromy matrix is of one of those three types, respectively. In addition we will distinguish parabolic orbits associated with $\pm\mathbb{I}$ from parabolic orbits associated with the four other matrices in (7.34) by referring to the former as "degenerate" and to the latter as "non-degenerate".

7.2.2 Elliptic Orbits

Here we initiate the classification of Virasoro coadjoint orbits, by studying those whose monodromy is elliptic. Parabolic and hyperbolic orbits will be investigated in Sects. 7.2.3 to 7.2.6.

Finding Orbit Representatives

Let $c > 0$ and suppose that $q(\varphi)d\varphi^2$ is a quadratic density such that the monodromy of Hill's equation (7.12) is elliptic. (We denote the quadratic density by q rather than p, because the latter will eventually be the "representative" of the orbit of q.) Then we can choose a solution vector Ψ whose monodromy matrix takes the form (7.33) for some angle $2\pi\omega$ which is not an integer multiple of π. The function

$$X_q(\varphi) \equiv \psi_1^2(\varphi) + \psi_2^2(\varphi) \tag{7.38}$$

is strictly positive and 2π-periodic; it is a vector field on the circle, since it is a quadratic combination of $-1/2$-densities such as (7.23). In fact, it belongs to the Lie algebra of the stabilizer of q since it solves equation (7.7). In addition it is invariant under the action of the stabilizer of M and is therefore a well-defined functional of $q(\varphi)$, which justifies the notation X_q. Conversely, X_q determines $q(\varphi)$ since Hill's equation implies

$$q = \frac{c}{6}\,\frac{\psi_1''\psi_1 + \psi_2''\psi_2}{X_q} \overset{(7.38)}{=} \frac{c}{6}\left[\frac{1}{2}\frac{X_q''}{X_q} - \frac{1}{4}\left(\frac{X_q'}{X_q}\right)^2 - \frac{1}{X_q^2}\right]. \tag{7.39}$$

We would have obtained the same formula upon using Eq. (7.30) with $\eta = \psi_1/\psi_2$. Our goal now is to build a diffeomorphism $g_q \in \widetilde{\text{Diff}}^+(S^1)$ such that q is obtained by acting with g_q on a suitable orbit representative p. In the language of induced representations, the maps g_q will be "standard boosts" on the orbit of p.

Let us define the negative number

$$p_0 \equiv -\frac{c}{6}\left[\int_0^{2\pi} \frac{d\varphi}{X_q(\varphi)}\right]^2, \tag{7.40}$$

where the notation "p_0" will be justified below. This number is well-defined since $X_q(\varphi)$ never vanishes, and it is invariant under $\widetilde{\text{Diff}}^+(S^1)$ since X_q is a vector field. We can then define a diffeomorphism $f \in \widetilde{\text{Diff}}^+(S^1)$ by

$$f(\varphi) \equiv \frac{2\pi}{\sqrt{6|p_0|/c}} \int_0^{\varphi} \frac{d\phi}{X_q(\phi)}. \tag{7.41}$$

This quantity is the inverse of the sought-for standard boost since Eq. (7.39) can be written as

$$q(\varphi) = p_0(f'(\varphi))^2 - \frac{c}{12}\mathsf{S}[f](\varphi)\,, \tag{7.42}$$

which we recognize as the coadjoint action (6.113) of

$$g_q \equiv f^{-1} \tag{7.43}$$

on the constant coadjoint vector $p(\varphi) = p_0 < 0$. In conclusion:

Proposition Let (q, c) with $c > 0$ be a Virasoro coadjoint vector with elliptic monodromy. Then it belongs to the orbit of a constant coadjoint vector (p, c) with $p(\varphi) = p_0$, where the value of p_0 is determined by $q(\varphi)$ according to (7.40) with X_q given by (7.38) in terms of normalized solutions of the Hill's equation of (q, c). In addition, the diffeomorphism g_q defined as the inverse of (7.41) is a *standard boost* for the orbit of p in the sense that

$$g_q \cdot p = q \tag{7.44}$$

where the dot denotes the coadjoint action (6.114).

Monodromy and Winding Number

Let us now see how the parameter (7.40) is related to the monodromy matrix. At $p = p_0$, Hill's equation (7.12) reads

$$-\frac{c}{6}\psi'' - |p_0|\psi = 0 \tag{7.45}$$

where we write $p_0 = -|p_0|$ to emphasize that this is a harmonic oscillator equation with frequency

$$\omega = \sqrt{6|p_0|/c}\,. \tag{7.46}$$

A basis of solutions satisfying the Wronskian condition (7.16) is provided by

$$\psi_1(\varphi) = \frac{1}{\sqrt{\omega}}\sin(\omega\varphi), \qquad \psi_2(\varphi) = \frac{1}{\sqrt{\omega}}\cos(\omega\varphi). \tag{7.47}$$

The corresponding monodromy matrix M is readily seen to take the form (7.33) with ω given by (7.46) in terms of p_0/c. The fact that the monodromy matrix is elliptic implies that ω is *not* an integer multiple of $1/2$, which is equivalent to saying that

$$p_0 \neq -\frac{n^2c}{24}\,. \tag{7.48}$$

In the language of Sect. 7.1.2, the constant orbit representative p_0 must be *generic* in order for its orbit to be elliptic. By contrast, the exceptional orbit representatives (7.9) will turn out to have degenerate parabolic monodromy (see below).

Thus different values of p_0 generally define disjoint orbits since their monodromy matrices (7.33) are not conjugate. However, at this stage we cannot tell whether

$$p_0 \qquad \text{and} \qquad -\left(\sqrt{|p_0|}+\sqrt{\tfrac{c}{6}}\,N\right)^2 \tag{7.49}$$

belong to different orbits when $N \in \mathbb{N}$ since their monodromy matrices coincide (their angles differ by $2\pi N$). This issue is settled by the winding number (7.32): the curve (7.27) associated with the solutions (7.47) is

$$\eta(\varphi) = \tan(\omega\varphi)\,, \tag{7.50}$$

which can be seen as a path on a circle written in terms of a stereographic coordinate $\eta = \tan(\theta/2)$, where the coordinate $\theta \in \mathbb{R}$ is identified as $\theta \sim \theta + 2\pi$. In terms of θ the path (7.50) is a rotation around the circle at constant velocity, $\theta(\varphi) = 2\omega\varphi$. The number of laps performed by this path around the circle when φ goes from zero to 2π is the winding number[2]

$$n_p = \lfloor 2\omega \rfloor \overset{(7.46)}{=} \left\lfloor \sqrt{\frac{24|p_0|}{c}} \right\rfloor \tag{7.51}$$

where $\lfloor\cdot\rfloor$ denotes the integer part. Thus the winding number associated with $p_0 < 0$ takes a definite value in each interval $]-\frac{(n+1)^2c}{24}, -\frac{n^2c}{24}[$, and jumps by one unit every time p_0 takes one of the exceptional values (7.9). For instance $n_p = 0$ when p_0 belongs to $]-c/24, 0[$, while $n_p = 1$ when $p_0 \in]-c/6, -c/24[$, and so on.

In conclusion, the orbits of two generic constants p_0 and $\tilde{p}_0$ are disjoint if and only if these constants differ. We have thus recovered the lower part ($p_0 < 0$) of Fig. 7.1. As a bonus we can now assign a monodromy matrix determined by (7.46), and a winding number (7.51), with each point on that part. In particular the integers n written on the left of the p_0 axis can be interpreted as winding numbers for constants p_0 located between $-(n+1)^2c/24$ and $-n^2c/24$.

Stabilizers

To conclude the description of orbits of generic constants $p_0 < 0$, it remains to find their stabilizer. As anticipated in (7.8), one shows that the stabilizer of p_0 is the group U(1) of rigid rotations $f(\varphi) = \varphi + \theta$ (or more precisely its universal cover $\mathbb{R}$ when dealing with $\widetilde{\text{Diff}}^+(S^1)$). The coadjoint orbit of (p_0, c) can thus be written as

$$\mathcal{W}_{(p_0,c)} \cong \widetilde{\text{Diff}}^+(S^1)/\mathbb{R} \cong \text{Diff}^+(S^1)/S^1\,, \tag{7.52}$$

which may be seen as an infinite-dimensional generalization of the orbit $\text{SL}(2,\mathbb{R})/S^1$ of $\text{SL}(2,\mathbb{R})$. The latter coincides with the momentum orbit (4.97) of a massive

[2] We denote the winding number by n_p instead of $n_{(p_0,c)}$ to reduce clutter.

Poincaré particle in three dimensions; in the same way, we shall see in Sect. 10.1 that (7.52) is the supermomentum orbit of a massive BMS_3 particle.

Remark The stabilizer $U(1)$ coincides with the stabilizer of the monodromy matrix (7.33), so the quotient (7.26) consists of a single point. This implies that all conformally inequivalent normalized solutions of the Hill equation associated with (p_0, c) can be obtained by acting with rotations on the solution (7.47).

7.2.3 Degenerate Parabolic Orbits

Orbit Representatives

We now turn our attention to coadjoint vectors (q, c) whose monodromy matrix is of the "degenerate" parabolic type (7.34), i.e. coincides with $\pm\mathbb{I}$. We proceed as in the elliptic case. In particular the monodromy matrix still ensures that (7.38) is a positive, 2π-periodic vector field belonging to the Lie algebra of the stabilizer of $q(\varphi)$. The negative number (7.40) is still well-defined and $\widetilde{\text{Diff}}^+(S^1)$-invariant, and formula (7.41) provides a diffeomorphism of S^1 such that Eq. (7.42) holds. Then (7.43) is a standard boost that maps the constant coadjoint vector p_0 on $q(\varphi)$; in particular the proposition surrounding (7.44) still holds up to the replacement of the word "elliptic" by "degenerate parabolic".

As in the elliptic case we can choose constant coadjoint vectors as orbit representatives. The corresponding Hill's equation then reads (7.45) and admits the normalized solutions (7.47), but the monodromy matrix is $\pm\mathbb{I}$. Such a monodromy matrix M only occurs when p_0 takes the exceptional form (7.9) for some strictly positive integer n, in which case

$$p_0 = -\frac{n^2 c}{24} \qquad \text{and} \qquad \mathsf{M} = (-1)^n \mathbb{I}. \tag{7.53}$$

By contrast, elliptic orbits *never* contain an exceptional constant; this is a sharp difference between elliptic and degenerate parabolic orbits.

The monodromy matrix (7.53) implies that two exceptional constants specified by integers n, n' can belong to the same orbit only if n and n' have the same parity; but at this stage we cannot tell if two orbits with the same parity are disjoint. As in the elliptic case we can address this question by studying the winding number of the curve (7.27) associated with the solutions (7.47). One can verify that the winding number coincides with the number n specified by $p_0 = -n^2 c/24$, which implies that any two orbits of exceptional constants specified by different values of $n > 0$ are disjoint. In conclusion, we have now recovered the dots in the lower part of Fig. 7.1, and the values of n displayed there coincide with winding numbers. In particular the orbit at $n = 1$ will be called the *vacuum orbit* from now on; in the context of Riemann surfaces, it is known as *universal Teichmüller space* [7, 11]. Note that the orbit of $p_0 = 0$ does *not* have degenerate parabolic monodromy, and so has not yet been accounted for by our classification of Hill's equations.

Stabilizers

We now study the stabilizers of orbits of exceptional constants $p_0 = -n^2c/24$. We saw below (7.8) that the stabilizer is three-dimensional for such values, and is generated by the vector fields

$$\frac{\partial}{\partial\varphi}\,, \qquad \sin(n\varphi)\frac{\partial}{\partial\varphi}\,, \qquad \cos(n\varphi)\frac{\partial}{\partial\varphi}\,. \tag{7.54}$$

The Lie algebra of the stabilizer is therefore isomorphic to $\mathfrak{sl}(2,\mathbb{R})$, but different values of n define non-conjugate embeddings of $\mathfrak{sl}(2,\mathbb{R})$ in $\mathrm{Vect}(S^1)$. In fact one can verify using (6.96) that the finite diffeomorphisms that span the stabilizer of $p_0 = -n^2c/24$ (and that reduce to (7.54) close to the identity) are projective transformations (6.95) spanning a group $\mathrm{PSL}^{(n)}(2,\mathbb{R})$ (the n-fold cover of $\mathrm{PSL}(2,\mathbb{R})$). In conclusion:

Lemma The stabilizer of $p_0 = -n^2c/24$ for the coadjoint action of $\widetilde{\mathrm{Diff}}^+(S^1)$ (resp. $\mathrm{Diff}^+(S^1)$) is the group $\widetilde{\mathrm{PSL}}^{(n)}(2,\mathbb{R})$ (resp. $\mathrm{PSL}^{(n)}(2,\mathbb{R})$) spanned by diffeomorphisms $f(\varphi)$ given by (6.95), where $\widetilde{\mathrm{PSL}}^{(n)}(2,\mathbb{R})$ is the universal cover of the n-fold cover of $\mathrm{PSL}(2,\mathbb{R}) = \mathrm{SL}(2,\mathbb{R})/\mathbb{Z}_2$. The coadjoint orbit of (p_0, c) can be written as

$$\mathcal{W}_{\left(-\frac{n^2c}{24},c\right)} \cong \widetilde{\mathrm{Diff}}^+(S^1)/\widetilde{\mathrm{PSL}}^{(n)}(2,\mathbb{R}) \cong \mathrm{Diff}^+(S^1)/\mathrm{PSL}^{(n)}(2,\mathbb{R}) \tag{7.55}$$

The Lie algebra of the stabilizer is generated by the vector fields (7.54).

In Sect. 9.3 we will interpret the orbit of $p_0 = -c/24$ as the set of gravitational perturbations around Minkowski space. In that context the little group $\mathrm{PSL}(2,\mathbb{R})$ will be seen as the Lorentz group in three dimensions. The remaining exceptional values $p_0 = -n^2c/24$ (with $n \geq 2$) will be interpreted as conical excesses where one turn around the origin of space spans an angle $2\pi n$.

Remark An important difference between elliptic and degenerate parabolic orbits is that, in the latter case, the stabilizer of the monodromy matrix (7.53) is the whole group $\mathrm{SL}(2,\mathbb{R})$, which does not leave the combination (7.38) invariant. Nevertheless, the integral (7.40) is still independent of the choice of the normalized solution vector Ψ because, in that specific case, any $\mathrm{SL}(2,\mathbb{R})$ transformation $\Psi \mapsto S\Psi$ is equivalent to the action of a diffeomorphism of the circle belonging to the stabilizer of p_0; since the integral (7.40) is invariant under diffeomorphisms, it follows that it is also invariant under $\Psi \mapsto S\Psi$ for any $S \in \mathrm{SL}(2,\mathbb{R})$.

7.2.4 Hyperbolic Orbits Without Winding

Consider a Virasoro coadjoint vector (q, c) whose monodromy matrix is of the hyperbolic type (7.37) with some $\omega > 0$. We shall see that hyperbolic orbits differ greatly depending on the winding number of the curve (7.27), so we focus here on the case of zero winding; the non-zero case will be treated in Sect. 7.2.5.

Finding Orbit Representatives

Let ψ_1 and ψ_2 be normalized solutions of Hill's equation associated with (q, c) and let $\eta = \psi_1/\psi_2$. Since the winding number of η is zero, we can choose our solution vector such that ψ_2 has no zeros on the real line. Then $\eta(\varphi)$ is smooth and, by virtue of (7.37), we have

$$\eta(\varphi + 2\pi) = e^{4\pi\omega}\eta(\varphi) \tag{7.56}$$

so $\eta(\varphi)$ is monotonically increasing on $\mathbb{R}$ (since $\omega > 0$). As in the case of elliptic orbits, our goal is to find a "standard boost" g_q whose inverse $g_q^{-1} \equiv f$ will map $q(\varphi)$ on a suitably chosen orbit representative. To do so we define

$$f(\varphi) \equiv \frac{1}{2\omega}\log(\eta(\varphi)) \overset{(7.27)}{=} \frac{1}{2\omega}\log\left(\frac{\psi_1(\varphi)}{\psi_2(\varphi)}\right) \tag{7.57}$$

which belongs to $\widetilde{\text{Diff}}^+(S^1)$ by virtue of (7.56). Now if we set

$$p_0 \equiv \frac{c\,\omega^2}{6}\,, \tag{7.58}$$

we can use (7.30) and the cocycle identity (6.77) to write

$$q(\varphi) = p_0(f'(\varphi))^2 - \frac{c}{12}\mathsf{S}[f]\,. \tag{7.59}$$

As in (7.42) we recognize the coadjoint action of $g_q = f^{-1}$ on the constant coadjoint vector $p(\varphi) = p_0 > 0$, and thus conclude:

Proposition Let (q, c) with $c > 0$ be a Virasoro coadjoint vector with hyperbolic monodromy and zero winding number. Then it belongs to the orbit of a constant coadjoint vector (p_0, c), where $p_0 > 0$ is determined by the monodromy matrix according to (7.58). In addition the diffeomorphism g_q defined as the inverse of (7.57) is a standard boost for the orbit of p in the sense (7.44).

Note that the definition (7.58) coincides with Eq. (7.46) for $p_0 > 0$. Roughly speaking, "hyperbolic orbits are an analytic continuation of elliptic orbits to imaginary values of the monodromy parameter ω". This is analogous to the fact that tachyonic momentum orbits may be seen as massive orbits with imaginary mass.

Stabilizers

At $p = p_0$, Hill's equation (7.12) reads $-\frac{c}{6}\psi'' + |p_0|\psi = 0$ where we stress that the sign of the potential term is opposite to the one in (7.45). A basis of solutions satisfying the Wronskian condition (7.16) is provided by

$$\psi_1^{\pm}(\varphi) = \pm \frac{1}{\sqrt{2\omega}}\, e^{\omega\varphi}, \qquad \psi_2^{\pm}(\varphi) = \pm \frac{1}{\sqrt{2\omega}}\, e^{-\omega\varphi} \tag{7.60}$$

where $\omega > 0$ is given by (7.46). The corresponding monodromy matrix is (7.37).

We have seen in (7.8) that the stabilizer is one-dimensional for $p_0 > 0$, and that it consists of rotations of the circle. Thus the stabilizer of p_0 is a group U(1) of rigid rotations (or more precisely its universal cover $\mathbb{R}$ when dealing with $\widetilde{\text{Diff}}^+(S^1)$). In particular the orbit of (p_0, c) for $p_0 > 0$ and $c > 0$ is diffeomorphic to

$$\mathcal{W}_{(p_0,c)} \cong \widetilde{\text{Diff}}^+(S^1)/\mathbb{R} \cong \text{Diff}^+(S^1)/S^1. \tag{7.61}$$

As in (7.52) this orbit may be seen as an infinite-dimensional generalization of $\text{SL}(2,\mathbb{R})$ orbits of the type $\text{SL}(2,\mathbb{R})/S^1$. However, the orbit differs from those of negative p_0's in that the two choices of signs in (7.60) are conformally inequivalent. Indeed, the stabilizer G_{M} of the matrix (7.37) under conjugation is isomorphic to $\mathbb{R} \times \mathbb{Z}_2$ while the universal cover of the little group of p_0 is just $\mathbb{R}$. Accordingly the quotient (7.26) contains two points, indicating that there are two inequivalent normalized families of solutions to Hill's equation at (p_0, c); these two families are labelled by the sign $\pm$ in (7.60).

In terms of Fig. 7.1, we have now completed our understanding of almost the whole real line $p_0 \in \mathbb{R}$, since we now know that the orbits that pass through $p_0 > 0$ are of hyperbolic type without winding. The only remaining mystery is the orbit of $p_0 = 0$, and of course all the orbits that do not contain constant representatives.

7.2.5 Hyperbolic Orbits with Winding

Building Orbit Representatives

We now consider a Virasoro coadjoint vector (q, c) with hyperbolic monodromy (7.37) but strictly positive winding number $n > 0$. The classification of orbits of such vectors is more involved than in the previously encountered cases, so we proceed in a "backwards" fashion. Namely, suppose we are given a pair of smooth real functions ψ_1, ψ_2 on $\mathbb{R}$, chosen in such a way that they satisfy the Wronskian condition (7.16). Then it is automatically true that the function $p(\varphi)$ defined by

$$p \equiv \frac{c}{6}\,\frac{\psi_1''}{\psi_1} = \frac{c}{6}\,\frac{\psi_2''}{\psi_2} \tag{7.62}$$

is smooth for any constant $c > 0$. If in addition there exists a monodromy matrix M such that (7.18) holds, then $p(\varphi)$ is 2π-periodic and ψ_1, ψ_2 are solutions of the corresponding Hill's equation. This procedure provides a way to build Virasoro coadjoint vectors out of functions ψ_i; in particular, in order to prove that there exist Virasoro orbits with hyperbolic monodromy and non-zero winding number, it suffices to find two normalized functions ψ_i satisfying these criteria, and the identification of the corresponding Virasoro coadjoint vectors will follow.

Thus, let $\omega > 0$ be a strictly positive real number and let $n > 0$ be a positive integer. Let us define the positive function

$$F_{n,\omega}(\varphi) \equiv \cos^2(n\varphi/2) + \left(\sin(n\varphi/2) + \frac{2\omega}{n}\cos(n\varphi/2)\right)^2 \tag{7.63}$$

as well as

$$\psi_1(\varphi) \equiv \frac{e^{\omega\varphi}}{\sqrt{F_{n,\omega}(\varphi)}}\sqrt{\frac{2}{n}}\left(\sin(n\varphi/2) + \frac{\omega}{n}\cos(n\varphi/2)\right), \tag{7.64}$$

$$\psi_2(\varphi) \equiv \frac{e^{-\omega\varphi}}{\sqrt{F_{n,\omega}(\varphi)}}\sqrt{\frac{2}{n}}\cos(n\varphi/2). \tag{7.65}$$

Since $F_{n,\omega}$ is strictly positive, the ψ_i's are smooth functions. They satisfy the Wronskian condition (7.16) and their monodromy matrix is (7.37). Their ratio is

$$\eta(\varphi) = e^{2\omega\varphi}\tan(n\varphi/2) + \frac{\omega}{n}$$

and describes a path on the circle with varying velocity and winding number n. It follows that the function $p(\varphi)$ defined by (7.62) is a Virasoro coadjoint vector with hyperbolic monodromy (7.37) and winding number $n > 0$. It is explicitly given by

$$p(\varphi) = \frac{c\,\omega^2}{6} + \frac{c}{12}\,\frac{n^2 + 4\omega^2}{F_{n,\omega}(\varphi)} - \frac{c}{8}\,\frac{n^2}{F^2_{n,\omega}(\varphi)} \tag{7.66}$$

in terms of the function (7.63). We have thus built explicit orbit representatives with hyperbolic monodromy and non-zero winding number.

It is worth spending some time to interpret formula (7.66). Let us take ω small and expand p around $\omega = 0$. To first order in ω, we get

$$p(\varphi) = -\frac{n^2 c}{24} + \omega\frac{nc}{3}\sin(n\varphi) + \mathcal{O}(\omega^2)\,. \tag{7.67}$$

The leading term in p is an exceptional constant $-n^2c/24$, so we can think of (7.66) as a deformation of that constant. The term of order one in ω in (7.67) is proportional to $\sin(n\varphi)$, which is one of the elements of the Lie algebra of the stabilizer of $-n^2c/24$. This ensures that the deformation does not belong to the orbit of $-n^2c/24$. Indeed, all deformations that *do* belong to that orbit take the form

$$\widehat{\mathrm{ad}}^*_X\left(-\frac{n^2c}{24}\right) \overset{(6.115)}{=} -\frac{c}{12}\left(n^2X' + X'''\right)$$

for some vector field X, where the term $n^2X' + X'''$ annihilates the contribution of the modes $\sin(n\varphi)$ or $\cos(n\varphi)$.

In Sect. 10.1 we will interpret (7.66) as the supermomentum of a BMS_3 tachyon with imaginary mass proportional to ω^2. Accordingly, from now on we refer to hyperbolic Virasoro orbits with non-zero monodromy as *tachyonic orbits*. They are our first example of orbits that do not admit any constant representative, so they are *not* accounted for by Fig. 7.1. In order to include them in our "map of coadjoint orbits", we think of them as orbits of deformations (7.67) of exceptional constants. With this viewpoint and the "tachyonic" terminology, it is natural to identify this kind of deformation with the horizontal line in Fig. 4.3b that represents tachyonic orbits of Poincaré. Accordingly we represent tachyonic Virasoro orbits by a horizontal line to the right of the point labelled "n" in Fig. 7.1. With this convention our schematic representation of Virasoro orbits becomes the one displayed in Fig. 7.2. It remains to understand which orbit contains the point $p_0 = 0$, and to find the remaining orbits that have no constant representative. Before doing so, we address a few minor points regarding tachyonic orbits:

- The construction that led from (7.63) to (7.66) did produce Virasoro coadjoint vectors with the desired monodromy and winding number, but it is not clear at this stage that *any* coadjoint vector satisfying these properties can be mapped on (7.66). However, this turns out to be the case; in this sense, the orbit representatives (7.66) exhaust all orbits with hyperbolic monodromy and non-zero winding. See [9] for the proof.
- The Lie algebra of the stabilizer of (7.66) is spanned by the periodic linear combinations of the functions (7.23). As it turns out, the only periodic combination

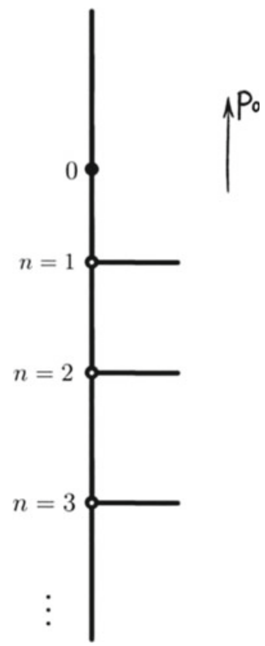

Fig. 7.2 A partial map of Virasoro orbits, including orbits of constant coadjoint vectors together with tachyonic orbits. Compare to Fig. 7.1

in this case is the product $\psi_1\psi_2$. The latter has $2n$ simple zeros inside $[0, 2\pi[$ and generates a non-compact group $\mathbb{R}$. In addition the function (7.66) is periodic with period $2\pi/n$, so the stabilizer must contain a group $\mathbb{Z}_n$ consisting of rotations by integer multiples of $2\pi/n$. In fact, one can show (see [9]) that the stabilizer of p in $\text{Diff}^+(S^1)$ is isomorphic to a product $\mathbb{R} \times \mathbb{Z}_n$, while its stabilizer in the universal cover $\widetilde{\text{Diff}}^+(S^1)$ is $\mathbb{R} \times T_{2\pi/n}$ where $T_{2\pi/n}$ is the group of translations of the real line by integer multiples of $2\pi/n$. We conclude that the orbit of (7.66) is diffeomorphic to

$$\mathcal{W}_{(p,c)} \cong \widetilde{\text{Diff}}^+(S^1)/(\mathbb{R} \times T_{2\pi/n}) \cong \text{Diff}^+(S^1)/(\mathbb{R} \times \mathbb{Z}_n). \tag{7.68}$$

7.2.6 Non-degenerate Parabolic Orbits

Here we include the last missing pieces of our description of Virasoro orbits. When the monodromy matrix is non-degenerate parabolic, it is conjugate to one of the four last elements in the list (7.34). As in the hyperbolic case we discuss zero and non-zero windings separately.

Zero Winding

At zero winding we proceed as in the elliptic and $n = 0$ hyperbolic cases, i.e. we look for standard boosts. Let therefore (q, c) be a Virasoro coadjoint vector such that a normalized solution vector $\Psi = (\psi_1\ \psi_2)^t$ associated with the corresponding Hill's equation has non-degenerate parabolic monodromy and zero winding number. The monodromy matrices in (7.34) imply that

$$\psi_1(\varphi + 2\pi) = \pm\big(\psi_1(\varphi) + \varepsilon\psi_2(\varphi)\big), \qquad \psi_2(\varphi + 2\pi) = \pm\psi_2(\varphi) \tag{7.69}$$

where ε is a priori $+1$ or -1. The corresponding curve (7.27) satisfies

$$\eta(\varphi + 2\pi) = \eta(\varphi) + \varepsilon \tag{7.70}$$

and (7.31) implies that ε must actually be equal to $+1$. The opposite sign corresponds to changing the orientation in the space of solutions of Hill's equation, so with our choice of orientation for ψ_1, ψ_2, only the value $\varepsilon = +1$ gives rise to an admissible monodromy matrix. Then the function

$$f(\varphi) \equiv 2\pi\,\eta(\varphi) \tag{7.71}$$

is a $2\pi\mathbb{Z}$-equivariant diffeomorphism of the real line, and property (7.30) implies that

$$q(\varphi) = -\frac{c}{12}\mathsf{S}[f](\varphi)\,.$$

As in Eqs. (7.42) and (7.59), we recognize the coadjoint action of $g_q \equiv f^{-1}$:

Proposition Let (q, c) with $c > 0$ be a Virasoro coadjoint vector with non-degenerate parabolic monodromy and vanishing winding number. Then it belongs to the orbit of $(0, c)$ and the inverse of the diffeomorphism (7.71) is a standard boost in the sense of Eq. (7.44).

Thus we have finally found the orbit of $p_0 = 0$! It was the only point of Fig. 7.1 that was still eluding us. We now know that its orbit has parabolic type. The corresponding stabilizer is the group U(1) of rigid rotations (as for all positive or generic constants p_0), and there are two conformally inequivalent solutions of Hill's equation at $p_0 = 0$, namely $\psi_1^\pm = \pm\varphi$, $\psi_2^\pm(\varphi) = \pm 1$. The orbit can be represented as a quotient space

$$\mathcal{W}_{(0,c)} \cong \widetilde{\text{Diff}}^+(S^1)/\mathbb{R} \cong \text{Diff}^+(S^1)/S^1$$

and is diffeomorphic to the orbits (7.52)–(7.61) of generic or positive constants.

Non-zero Winding

At non-zero winding our strategy will be similar to that used in the hyperbolic case with winding: we rely on the fact that formula (7.62) always defines a 2π-periodic function $p(\varphi)$ when ψ_1 and ψ_2 satisfy the Wronskian condition and admit a well-defined monodromy, which allows us to build orbit representatives.

Thus, pick a number $\varepsilon \in \{\pm 1\}$ and let $n \in \mathbb{N}^*$ be a non-zero winding number. Let us define the positive function

$$H_{n,\varepsilon}(\varphi) \equiv 1 + \frac{\varepsilon}{2\pi}\sin^2(n\varphi/2) \tag{7.72}$$

as well as

$$\psi_1(\varphi) \equiv \frac{1}{\sqrt{H_{n,\varepsilon}(\varphi)}}\left(\frac{\varepsilon\varphi}{2\pi}\sin(n\varphi/2) - \frac{2}{n}\cos(n\varphi/2)\right), \tag{7.73}$$

$$\psi_2(\varphi) \equiv \frac{1}{\sqrt{H_{n,\varepsilon}(\varphi)}}\sin(n\varphi/2)\,. \tag{7.74}$$

Since the function $H_{n,\varepsilon}$ is strictly positive, the ψ_i's are smooth functions. They satisfy the Wronskian condition (7.16) and their monodromy matrix is one of the four matrices on the right in the list (7.34), with the off-diagonal entry coinciding with ε and the overall $\pm 1 = (-1)^n$. The curve (7.27) corresponding to this basis of solutions is

$$\eta(\varphi) = \frac{\varepsilon\varphi}{2\pi} - \frac{2}{n}\cot(n\varphi/2)$$

and has winding number n. This is all as in the hyperbolic case below Eq. (7.65). It follows that the function $p(\varphi)$ defined by (7.62) is a Virasoro coadjoint vector with non-degenerate parabolic monodromy and winding number $n > 0$, explicitly given by

$$p(\varphi) = \frac{c}{12}\frac{n^2}{H_{n,\varepsilon}(\varphi)} - \frac{c}{8}\frac{n^2(1+\varepsilon/2\pi)}{H^2_{n,\varepsilon}(\varphi)}\,. \tag{7.75}$$

As in the hyperbolic case, one can think of (7.75) as a deformation of a suitable constant. However, in contrast ot (7.66), expression (7.75) seemingly contains no continuous parameter that one could tune to "small" values since ε is only allowed to take the values ± 1. In order to solve this problem, recall from (7.36) that the matrices

$$\begin{pmatrix} 1 & \varepsilon \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & \lambda\varepsilon \\ 0 & 1 \end{pmatrix} \tag{7.76}$$

are conjugate in $\text{SL}(2,\mathbb{R})$ for any positive real number λ. Accordingly we could just as well have chosen the representatives of non-degenerate parabolic conjugacy classes to involve an arbitrary positive parameter ε; the limit $\varepsilon \to 0$ then may be taken since it does not affect the conjugacy class of the monodromy matrix. The corresponding coadjoint vector is (7.75) and its expansion to first order in ε reads

$$p(\varphi) = -\frac{n^2c}{24}\left(1 + \frac{\varepsilon}{2\pi}(1 + 2\cos\varphi)\right) + \mathcal{O}(\varepsilon^2). \tag{7.77}$$

As in (7.67), the leading term is an exceptional constant (7.9) and we can think of (7.77) as a deformation thereof. The deformation is designed so that it does not belong to the orbit of $-n^2c/24$. When dealing with BMS_3 supermomentum orbits in Sect. 10.1, we will interpret (7.75) as the supermomentum of a massless BMS_3 particle. Accordingly, from now on we refer to non-degenerate parabolic orbits with non-zero winding as *massless orbits*. Note that the statement that the matrices (7.76) are conjugate is tantamount to saying that massless orbits are scale-invariant.

To conclude our analysis we state (without proof) a few features of massless orbits:

- One can show that the orbit representatives (7.75) are exhaustive in that any coadjoint vector belonging to a massless orbit can be brought in that form by a suitable diffeomorphism. See appendix C of [9].
- The Lie algebra of the stabilizer of (7.75) is generated by the vector field $X = \psi_2^2$, which has n double zeros. In fact, as in the hyperbolic case, the stabilizer is isomorphic to $\mathbb{R} \times \mathbb{Z}_n$, but the generator of the $\mathbb{R}$ part of that group is *not* the same as in the hyperbolic case. The orbit is diffeomorphic to a quotient of $\text{Diff}^+(S^1)$ by this stabilizer, or equivalently a quotient of $\widetilde{\text{Diff}}^+(S^1)$ by $\mathbb{R} \times T_{2\pi/n}$ where $T_{2\pi/n}$ is the same discrete translation group as in (7.68).
- Up to $\widetilde{\text{Diff}}^+(S^1)$ transformations, the solution (7.73)-(7.74) is the unique solution of Hill's equation with non-degenerate parabolic monodromy

$$(-1)^n \begin{pmatrix} 1 & \varepsilon \\ 0 & 1 \end{pmatrix} \tag{7.78}$$

  and winding number n.

7.2.7 *Summary: A Map of Virasoro Orbits*

The above analysis exhausts all coadjoint orbits of the Virasoro group. Since these orbits will play a key role in the remainder of this thesis, we now briefly summarize the salient features of the classification.

The schematic drawings of Figs. 7.1 and 7.2 represent Virasoro orbits. The only orbits which are not accounted for by these pictures are massless ones; in order to include them we use the same trick as in Fig. 4.3b, where massless orbits are represented by two dots near the origin (one with positive energy, the other with negative energy). We will use the same notation here, except that such a pair of massless orbits occurs for all positive integers $n \in \mathbb{N}^*$. With this convention, Fig. 7.2 turns into the complete map of Virasoro coadjoint orbits displayed in Fig. 7.3.

Each point in that map represents an orbit representative; different points correspond to different representatives and define disjoint orbits. All orbits are now

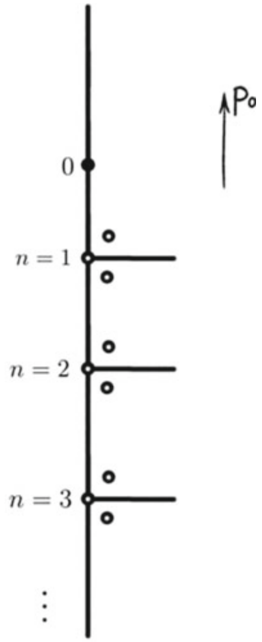

Fig. 7.3 The map of Virasoro coadjoint orbits at positive central charge. Note the similarity with Fig. 4.3b. Roughly speaking, the map consists of an infinity of copies of Poincaré momentum orbits glued together and labelled by the winding number n. Locally (near a node n), the two pictures look identical. This is not surprising given that Poincaré momentum orbits in three dimensions coincide with $SL(2, \mathbb{R})$ coadjoint orbits, which in turn are classified similarly to the conjugacy classes of $SL(2, \mathbb{R})$ that were instrumental for Virasoro coadjoint orbits. This hints that there exists a relation between Virasoro and Poincaré symmetry; we shall see in part III that this relation is embodied by the BMS_3 group

accounted for since the orbit representatives are exhaustive. The vertical line represents orbits that contain a constant orbit representative:

- For generic $p_0 < 0$ the orbit has elliptic monodromy determined by Eq. (7.46). Its winding number is given by (7.51), so the points of Fig. 7.3 located between n and $n+1$ have winding number n (while points such that $-c/24 < p_0 < 0$ have zero winding number).
- For exceptional values $p_0 = -n^2c/24$ with $n \in \mathbb{N}^*$, the orbit has degenerate parabolic monodromy determined by (7.53). Its winding number is n. In particular, the orbit at $n = 1$ is the *vacuum orbit*.
- For $p_0 > 0$, the orbit has hyperbolic monodromy with zero winding, and the conjugacy class of the monodromy matrix is determined by (7.46).
- The orbit of $p_0 = 0$ has non-degenerate parabolic monodromy with zero winding.

On the other hand, the points of Fig. 7.3 that do *not* belong to the vertical axis represent orbits that do *not* contain any constant representative:

- Each horizontal line starting at a point labelled by n represents a family of tachyonic orbits with winding number n. The orbit representatives are given by (7.66) and involve a continuous parameter $\omega > 0$ that determines the corresponding monodromy matrix (7.37).
- Each pair of dots surrounding a tachyonic line at n represents the two massless orbits with winding number n. The orbit representatives are given by (7.75) and involve a discrete parameter $\varepsilon = \pm 1$ that determines the corresponding monodromy matrix (7.78).

Focussing for definiteness on the multiply connected group $\text{Diff}^+(S^1)$, the stabilizers of Virasoro orbits are as in Table 7.1.

In the universal cover of the Virasoro group the first four entries of the right column would be replaced by their universal covers, while the two last ones would be replaced by $\mathbb{R} \times T_{2\pi/n}$. This should be compared with (and is very similar to) the list of Poincaré little groups in Table 4.1. Note that, at $n = 1$, the Virasoro stabilizers are quotients by $\mathbb{Z}_2$ of their Poincaré counterparts. This is because Table 4.1 lists the little groups given by the *double cover* (4.93) of the Poincaré group (Table 7.1).

Remark Figure 7.3 may be misleading since it suggests that all Virasoro orbits of constant coadjoint vectors are of a similar type, which is clearly not the case since

Table 7.1 Coadjoint orbits and their stabilizers for the Virasoro group

Orbit	Stabilizer
Vacuum-like $p_0 = -n^2c/24$, $n \geq 1$	$\text{PSL}^{(n)}(2, \mathbb{R})$
Elliptic	U(1)
Hyperbolic, zero winding	U(1)
Non-degenerate parabolic, zero winding	U(1)
Massless, winding $n \geq 1$	$\mathbb{R} \times \mathbb{Z}_n$
Tachyonic, winding $n \geq 1$	$\mathbb{R} \times \mathbb{Z}_n$

orbits of constants $p_0 > 0$ are hyperbolic while those of (generic) constants $p_0 < 0$ are elliptic. In this sense, the map of orbits would have been more accurate if we had represented the orbits of $p_0 > 0$ by a *horizontal* line to suggest that they have the same type of monodromy as the tachyonic orbits; see e.g. Fig. 1 of [9]. Our convention in Fig. 7.3 is motivated instead by the fact that the value of p_0 essentially measures energy (see below), so that higher points in Fig. 7.3 have higher energy.

7.3 Energy Positivity

In this section we investigate the boundedness properties of an energy functional on Virasoro orbits. This question is motivated both by its use in two-dimensional conformal field theory, and by its applications in three-dimensional gravity. We start by defining the Virasoro energy functional, before showing that the Schwarzian derivative satisfies an "average lemma" which will play a key role for this functional's boundedness. We then show that the only orbits with energy bounded from below are either orbits of constants $p_0 \geq -c/24$, or the massless orbit at winding $n = 1$ and monodromy $\varepsilon = -1$. To reduce clutter we return to our earlier abusive notation by writing as $\mathrm{Diff}(S^1)$ the universal cover of the group of orientation-preserving diffeomorphisms of the circle. Relevant references include [7, 9] as usual.

7.3.1 Energy Functional

The group $\mathrm{Diff}(S^1)$ can be interpreted as (part of) the symmetry group of a two-dimensional conformal field theory. In that context the quadratic density $p(\varphi)d\varphi^2$ is (a component of) the stress tensor of the theory, and its zero-mode

$$E[p] \equiv \frac{1}{2\pi} \int_0^{2\pi} d\varphi \, p(\varphi) \tag{7.79}$$

is the associated energy. We shall refer to this quantity as the *Virasoro energy functional* evaluated at p. If the theory admits a configuration whose stress tensor is $p(\varphi)$, then consistency with conformal symmetry requires that it also admits configurations with stress tensor $f \cdot p$, where $f \in \mathrm{Diff}(S^1)$ and the dot denotes the coadjoint action (6.114) for some definite value of the central charge. The energy functional varies under conformal transformations, since

$$E[f \cdot p] = \frac{1}{2\pi} \int_0^{2\pi} \frac{d\varphi}{f'(\varphi)} \left[p(\varphi) + \frac{c}{12} \mathsf{S}[f](\varphi) \right] \tag{7.80}$$

generally differs from (7.79).

Now consider a CFT with central charge $c > 0$ and let $\mathbf{\Omega}$ be the space of its stress tensors $p(\varphi)$; in general $\mathbf{\Omega}$ is a certain subset of the space $\mathcal{F}_2(S^1)$ of quadratic densities. Since any quantum system with a well-defined vacuum is expected to have energy bounded from below, the map

$$\mathbf{\Omega} \to \mathbb{R} : p \mapsto E[p] \tag{7.81}$$

should be bounded from below. In addition, consistency with conformal symmetry implies that $\mathbf{\Omega}$ is a union of Virasoro coadjoint orbits. One is thus led to the following question:

$$\begin{gathered} Which\, of\, the\, Virasoro\, coadjoint\, orbits\, of\, Fig.\, 7.3\, have \\ energy\, bounded\, from\, below\, under\, conformal\, transformations? \end{gathered} \tag{7.82}$$

In the sequel we will refer to orbits with energy bounded from below as orbits "with positive energy", although their energy (7.79) may actually be negative for some field configurations $p(\varphi)$.

Note that all orbits have energy unbounded from *above*. Indeed the term involving the Schwarzian derivative in (7.80) can be written as

$$\frac{c}{24\pi}\int_0^{2\pi}\frac{d\varphi}{f'(\varphi)}\mathsf{S}[f](\varphi) = -\frac{c}{24\pi}\int_0^{2\pi}d\varphi\,\mathsf{S}[f^{-1}](\varphi) \tag{7.83}$$

where we have renamed the integration variable from φ to $f^{-1}(\varphi)$, then used (6.16) and the cocycle identity (6.77). Since the Schwarzian derivative can be written as

$$\mathsf{S}[f](\varphi) = \left(\frac{f''}{f'}\right)' - \frac{1}{2}\left(\frac{f''}{f'}\right)^2, \tag{7.84}$$

we can also recast (7.83) in the form

$$\frac{c}{48\pi}\int_0^{2\pi}d\varphi\left(\frac{(f^{-1})''}{(f^{-1})'}\right)^2.$$

This can be made arbitrarily large for suitable choices of f, which proves that the energy functional E is unbounded from above on any Virasoro orbit.

7.3.2 The Average Lemma

As a first step towards the answer of the question (7.82), we focus on the piece of Eq. (7.80) that involves the Schwarzian derivative. The result that we shall describe was first derived in [12] and was based on projective geometry (see also [13, 14]). The elementary proof given here is borrowed from [9].

Average lemma Let $f \in \widetilde{\mathrm{Diff}}^+(S^1)$ and let $\mathsf{S}[f](\varphi)$ be its Schwarzian derivative (6.76) at φ. Then the average of the Schwarzian derivative satisfies the inequality

$$\int_0^{2\pi} d\varphi\, \mathsf{S}[f](\varphi) \;\leq\; \int_0^{2\pi} d\varphi\, \frac{1}{2}\left(1-(f'(\varphi))^2\right), \tag{7.85}$$

with equality if and only if $f(\varphi)$ is a projective transformation of the form (6.88).

Proof We consider the functional

$$I[f] \equiv -\int_0^{2\pi} d\varphi \left[\frac{1}{2}(f'(\varphi))^2 + \mathsf{S}[f](\varphi)\right]. \tag{7.86}$$

Our goal is to show that this quantity is bounded from below and that its minimum value is $-\pi$. By (7.84), it only depends on f' and f''. A convenient way to express this dependence is to define

$$Y(\varphi) \equiv f'(f^{-1}(\varphi)) \overset{(6.16)}{=} \frac{1}{(f^{-1})'(\varphi)}\,. \tag{7.87}$$

Since f is a $2\pi\mathbb{Z}$-equivariant, orientation-preserving diffeomorphism, $Y(\varphi)$ is strictly positive and 2π-periodic. In terms of Y we can rewrite (7.86) as

$$I[Y] = \frac{1}{2}\int_0^{2\pi} d\varphi \left[\frac{(Y'(\varphi))^2}{Y(\varphi)} - Y(\varphi)\right] \tag{7.88}$$

where the integrand is well-defined since $Y > 0$. Let us denote the minimum and maximum of $Y(\varphi)$ by

$$m \equiv \min_{\varphi\in[0,2\pi]} Y(\varphi)\,, \qquad M \equiv \max_{\varphi\in[0,2\pi]} Y(\varphi)\,. \tag{7.89}$$

With this notation the function

$$m + M - Y(\varphi) - \frac{mM}{Y(\varphi)} = \frac{1}{Y(\varphi)}\left[\left(\frac{M-m}{2}\right)^2 - \left(Y(\varphi) - \frac{M+m}{2}\right)^2\right]$$

is non-negative and vanishes only at the points where Y reaches its minimum or its maximum. Now consider the obvious inequality

$$\left(\frac{|Y'|}{\sqrt{Y}} - \sqrt{m+M-Y-\frac{mM}{Y}}\right)^2 \geq 0. \tag{7.90}$$

Integrating this over the circle and using (7.88), we obtain

$$I[Y] \geq -\pi(m + M - mM) + \int_0^{2\pi} d\varphi \frac{|Y'|}{Y} \sqrt{\left(\frac{M-m}{2}\right)^2 - \left(Y - \frac{M+m}{2}\right)^2}. \tag{7.91}$$

If there was no absolute value in the integrand on the right-hand side, we could just change the integration variable from φ to Y using $d\varphi\, Y'(\varphi) = dY$; the absolute value prevents us from doing this globally, but we can do it locally between two consecutive extrema of the function $Y(\varphi)$ (since the sign of Y' is constant in such an interval). We can then express the right-hand side of (7.91) in terms of the primitive function of the integrand,

$$\mathcal{F}(Y) = \int_m^Y \frac{dz}{z} \sqrt{\left(\frac{M-m}{2}\right)^2 - \left(z - \frac{M+m}{2}\right)^2} \equiv \int_m^Y dz\, \mathcal{G}(z), \tag{7.92}$$

where we have introduced the function $\mathcal{G}(z)$ to reduce clutter below. To see the use of this, consider a function $Y(\varphi)$ of the following shape (the general case follows straightforwardly):

This function has two local minima at φ_1 and φ_3 and two local maxima at φ_2 and φ_4 (the numbers of local minima and maxima coincide since $Y(\varphi)$ is smooth and 2π-periodic). Then the integral in (7.91) can be written as (Fig. 7.4)

$$\begin{aligned}
&\int_0^{2\pi} d\varphi \frac{|Y'|}{Y} \sqrt{\left(\frac{M-m}{2}\right)^2 - \left(Y - \frac{M+m}{2}\right)^2} = \\
&= \int_{Y_1}^{Y_2} dY\, \mathcal{G}(Y) - \int_{Y_2}^{Y_3} dY\, \mathcal{G}(Y) + \int_{Y_3}^{Y_4} dY\, \mathcal{G}(Y) - \int_{Y_4}^{Y_1} dY\, \mathcal{G}(Y) \\
&\overset{(7.92)}{=} 2\left[\mathcal{F}(Y_2) + \mathcal{F}(Y_4) - \mathcal{F}(Y_1) - \mathcal{F}(Y_3)\right]
\end{aligned}$$

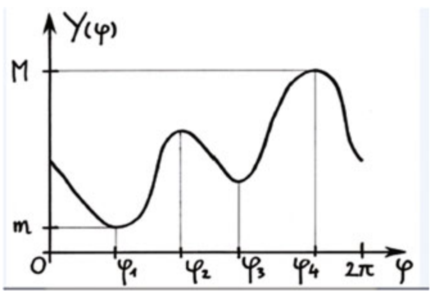

Fig. 7.4 The function $Y(\varphi)$ is 2π-periodic and strictly positive. Here we choose it with four local extrema, the global minimum being $Y(\varphi_1) = m$ and the global maximum $Y(\varphi_4) = M$

with the shorthand notation $Y(\varphi_i) \equiv Y_i$. The same computations would work for arbitrarily many minima and maxima of $Y(\varphi)$, with the same result: the integral is twice the sum of $\mathcal{F}$'s evaluated at the maxima minus twice the sum of $\mathcal{F}$'s evaluated at the minima. Thus the inequality (7.91) can be written as

$$\begin{aligned} I[Y] &\overset{(7.89)}{\geq} -\pi(m+M-mM)+2[\mathcal{F}(M)-\mathcal{F}(m)]+2[\mathcal{F}(Y_2)-\mathcal{F}(Y_3)] \\ &\geq -\pi(m+M-mM)+2\mathcal{F}(M) \end{aligned} \tag{7.93}$$

where we also used the fact that $\mathcal{F}(m)=0$ by virtue of the definition (7.92). Now it turns out that $\mathcal{F}(M)=\frac{\pi}{2}\left(\sqrt{M}-\sqrt{m}\right)^2$, which allows us to rewrite (7.93) as

$$I[Y] \geq -\pi(m+M-mM)+\pi\left(\sqrt{M}-\sqrt{m}\right)^2 \geq -\pi. \tag{7.94}$$

We conclude that $I[Y]$ is bounded from below by the value $-\pi$, which is exactly the inequality (7.85). It only remains to find the conditions under which (7.85) becomes an equality. For this to be the case, the inequalities (7.90), (7.93) and (7.94) must all be saturated; this occurs when $Y(\varphi)$ satisfies the following three conditions:

- In order to saturate (7.90), it satisfies the differential equation

$$Y'^2 = (m+M)Y - Y^2 - mM. \tag{7.95}$$

- In order to saturate (7.93), $Y(\varphi)$ has only one minimum and one maximum, where it takes the values m and M, respectively.
- In order to saturate the second inequality of (7.94), $M = 1/m$.

To solve (7.95) we use (7.87) and rewrite the equation in terms of f^{-1}. Using $M = 1/m$ the derivative of (7.95) becomes

$$Y'\left(1-((f^{-1})')^2-2\,\mathsf{S}[f^{-1}]\right)=0, \tag{7.96}$$

which is equivalent to (6.94). We have shown below (7.54) that the only f's satisfying this property are those that belong to the group of projective transformations (6.88), which concludes the proof. ■

7.3.3 *Orbits with Constant Representatives*

The average lemma allows us to investigate the boundedness properties of the energy functional (7.79) on Virasoro orbits. For now we limit ourselves to orbits that admit a constant representative.

Proposition The vacuum orbit, containing the point $p_{\text{vac}} = -c/24$, has energy bounded from below:

$$E[f \cdot p_{\text{vac}}] \geq E[p_{\text{vac}}] = -\frac{c}{24}. \tag{7.97}$$

The minimum of energy is located at p_{vac}.

Proof We consider formula (7.80) with $p(\varphi) = p_{\text{vac}} = -c/24$. Renaming the integration variable from φ to $f^{-1}(\varphi)$ and using Eqs. (6.16) and (6.77), we find

$$E[f \cdot p_{\text{vac}}] = \frac{c}{24\pi}\int_0^{2\pi} d\varphi \left[-\frac{1}{2}\big((f^{-1})'(\varphi)\big)^2 - \mathsf{S}[f^{-1}](\varphi)\right]$$

which we recognize as the functional (7.86) evaluated at f^{-1}. The average lemma (7.85) then implies that $E[f \cdot p_{\text{vac}}] \geq -c/24$, with equality if and only if f is a projective transformation (6.88). Our earlier result (7.55) ensures that such transformations precisely span the stabilizer of p_{vac}, so the minimum of energy is reached at p_{vac}. ■

Let us turn to other orbits containing a constant representative $p(\varphi) = p_0$. The key will be to rewrite their energy functional as the vacuum energy functional, plus another term. Starting from formula (7.80) we obtain

$$E[f \cdot p_0] = \frac{p_0 + c/24}{2\pi}\int_0^{2\pi} \frac{d\varphi}{f'(\varphi)} + E[f \cdot p_{\text{vac}}] \tag{7.98}$$

where the integral of $1/f'$ can be rewritten as

$$\frac{1}{2\pi}\int_0^{2\pi} \frac{d\varphi}{f'(\varphi)} \overset{(6.16)}{=} 1 + \frac{1}{2\pi}\int_0^{2\pi} d\varphi \left[(f^{-1})'(\varphi) - 1\right]^2$$

as follows from $\int_0^{2\pi} d\varphi (f^{-1})'(\varphi) = 2\pi$. Plugging this into (7.98) and using (7.97), we obtain

$$E[f \cdot p_0] \geq p_0 + \frac{p_0 + c/24}{2\pi}\int_0^{2\pi} d\varphi \left[(f^{-1})'(\varphi) - 1\right]^2.$$

The right-hand side here is the sum of p_0 and an integral whose integrand is manifestly non-negative. This implies the following result:

Proposition If $p_0 \geq -c/24$, then the orbit of (p_0, c) has energy bounded from below, with the energy minimum located at p_0:

$$p_0 \geq -\frac{c}{24} \qquad \Rightarrow \qquad E[f \cdot p_0] \geq E[p_0] = p_0 \,.$$

Now what happens when p_0 is *lower* than $-c/24$? In that case energy is *unbounded*, as can be shown by finding a family of diffeomorphisms that lower the energy indefinitely. Indeed, consider the matrix

$$\begin{pmatrix} \cosh(\gamma/2) & \sinh(\gamma/2) \\ \sinh(\gamma/2) & \cosh(\gamma/2) \end{pmatrix} \in \mathrm{SL}(2, \mathbb{R}) \tag{7.99}$$

where $\gamma \in \mathbb{R}$ (the normalization is chosen for later convenience). The corresponding projective transformation (6.88) is

$$e^{if(\varphi)} = \frac{e^{i\varphi}\cosh(\gamma/2) + \sinh(\gamma/2)}{-e^{i\varphi}\sinh(\gamma/2) + \cosh(\gamma/2)}\,, \tag{7.100}$$

and one verifies that

$$\frac{1}{f'(\varphi)} = |e^{i\varphi}\cosh(\gamma/2) + \sinh(\gamma/2)|^2 = \cosh\gamma + \sinh\gamma\,\cos\varphi\,. \tag{7.101}$$

The Schwarzian derivative of f is given by (6.94), so we find that

$$E[f \cdot p_0] \overset{(7.80)}{=} \frac{p_0 + c/24}{2\pi}\int_0^{2\pi}\frac{d\varphi}{f'(\varphi)} - \frac{c}{24}$$

where we have used the fact that the integral of f' over S^1 is normalized to 2π. The integral of (7.101) then yields

$$E[f \cdot p_0] = (p_0 + c/24)\cosh\gamma - \frac{c}{24}, \tag{7.102}$$

and this can become arbitrarily negative when $p_0 < -c/24$. In conclusion:

$$\begin{array}{c} \textit{The coadjoint orbit } \mathcal{W}_{(p_0,c)} \textit{ of a constant } p_0 \\ \textit{has energy bounded from below if and only if } p_0 \geq -c/24. \end{array} \tag{7.103}$$

Thus, when $p_0 < -c/24$, Fig. 7.5 is no longer valid because the energy functional can reach arbitrarily low values in certain directions. The orbit then looks like an infinite-dimensional saddle instead of the hyperboloid represented in Fig. 7.5.

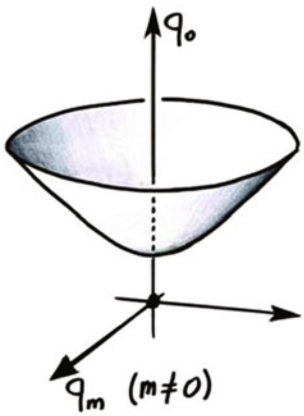

Fig. 7.5 Schematic representation of the Virasoro orbit of a constant p_0 located above the vacuum value $-c/24$, here understood to be the origin of the coordinate system. The coordinates $q_m, m \in \mathbb{Z}$ are the Fourier modes of coadjoint vectors $q(\varphi)$; in particular the zero-mode $q_0 = E[q]$ is their energy, which is bounded from below on the orbit. Compare to the massive Poincaré orbit with positive energy in Fig. 4.3a

Note that the matrix (7.99) can be interpreted as the $\mathrm{SL}(2, \mathbb{R})$ group element that represents a Lorentz boost with rapidity γ in three dimensions[3] thanks to the isomorphism (4.83), which also explains our choice of normalization. In that context, formula (7.102) is the transformation law of the energy of a particle with mass $p_0 + c/24$ under Lorentz boosts. We will return to this interpretation in part III.

7.3.4 Orbits Without Constant Representatives

We now describe the boundedness properties of the energy functional on Virasoro coadjoint orbits that do *not* admit a constant representative. As it turns out there is only one orbit with energy bounded from below, while all other ones have unbounded energy (Fig. 7.6).

Consider the non-degenerate parabolic orbit with winding number $n = 1$ and monodromy $\varepsilon = -1$; a typical orbit representative is given by (7.75). One can then prove the following result:

Proposition The energy functional on the massless orbit specified by $n = 1$ and $\varepsilon = -1$ is bounded from below by $-c/24$. There exist infinitely many points on the orbit whose energy is arbitrarily close to that value, but there is no orbit representative that realizes this value of energy.

We will not prove this proposition here and refer instead to [9]. Roughly speaking, the proof follows from a construction very similar to the one used in the proof of

[3] Rapidity is related to velocity v by $\gamma = \operatorname{arctanh}(v)$.

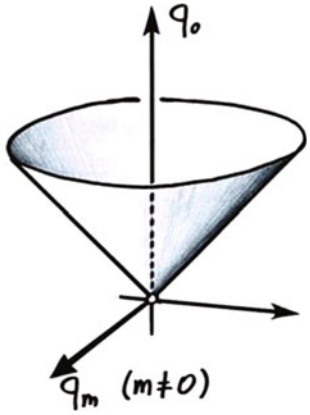

Fig. 7.6 Schematic representation of the Virasoro orbit with non-degenerate parabolic monodromy $\varepsilon = -1$ and winding number $n = 1$. The origin of the coordinate system is the vacuum coadjoint vector, $q_m = -(c/24)\delta_{m0}$, which does not belong to the orbit. Energy is bounded from below on the orbit but its infimum is never quite reached, in contrast to Fig. 7.5. Compare to the massless Poincaré orbit with positive energy in Fig. 4.3a

the average lemma (7.85), except that it crucially relies on the parameters $n = 1$, $\varepsilon = -1$. The proof of the fact that the infimum of energy is never reached on the orbit follows from the construction of a one-parameter family of points belonging to the orbit in such a way that they converge to the constant $-c/24$ without ever quite reaching it.

One might think that the other orbits without constant representatives behave in a similar way, i.e. that they also have energy bounded from below. However, for any such orbit, it is possible to build a one-parameter family of orbit elements whose energy can be arbitrarily low, similarly to constant representatives $p_0 < -c/24$. We refer again to [9] for explicit constructions. Thus one concludes that

$$\begin{gathered} all\,tachyonic\,or\,massless\,Virasoro\,orbits\,have\,unbounded\,energy,\\ except\,the\,one\,with\,non-degenerate\,parabolic\,monodromy\\ \varepsilon = -1\,and\,winding\,n = 1. \end{gathered} \tag{7.104}$$

7.3.5 Summary: A New Map of Virasoro Orbits

The considerations of the last few pages allow us to include more information in the map of Virasoro orbits of Fig. 7.3; see Fig. 7.7. Its two striking features are the occurrence of a single orbit without constant representatives and positive energy, and the fact that the lowest-lying orbit with positive energy is that of $p_{\text{vac}} = -c/24$. This

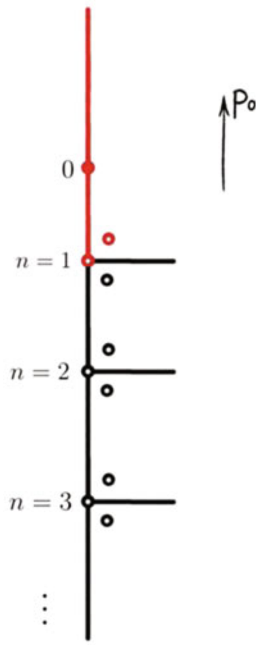

Fig. 7.7 The map of Virasoro coadjoint orbits at positive central charge. Orbits with energy bounded from below are coloured in *red*. Those are orbits of constants $p_0 \geq -c/24$, plus the unique massless orbit with monodromy (7.78) such that $\varepsilon = -1$ and winding number $n = 1$. All other orbits have energy unbounded from below

observation justifies referring to the latter orbit as the "vacuum orbit" and to p_{vac} as the "vacuum stress tensor". Note that the exact same situation occurs with relativistic particles, as the only ones with energy bounded from below are either massive (with non-negative mass) or massless.

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Chapter 8
Symmetries of Gravity in AdS_3

In this chapter we explore a physical model where the Virasoro group plays a key role, namely three-dimensional gravity on Anti-de Sitter (AdS) backgrounds and its putative dual two-dimensional conformal field theory (CFT). These considerations will be a basis and a guide for our study of asymptotically flat space-times in part III.

The plan is the following. Section 8.1 is a prelude where we recall a few basic facts about (three-dimensional) gravity, in particular regarding the notion of asymptotic symmetries. Section 8.2 is then devoted to three-dimensional space-times whose metric approaches that of Anti-de Sitter space at spatial infinity; this includes Brown–Henneaux boundary conditions and their asymptotic symmetries, which will turn out to consist of two copies of the Virasoro group. In Sect. 8.3 we describe the phase space of AdS_3 gravity as a hyperplane at fixed central charges in the space of the coadjoint representation of two Virasoro groups. Finally, in Sect. 8.4 we describe unitary highest-weight representations of the Virasoro algebra and relate them to the quantization of the AdS_3 phase space.

Bibliographical remarks. This chapter is based on several combined references. Perhaps the most important one is the original paper by Brown and Henneaux [1], which triggered the development of the field as a whole. In that paper the authors relied on the methods of [2–5] to build surface charges associated with asymptotic symmetries, but our approach will be led by their Lagrangian (or "covariant") reformulation [6–8]. In particular, our presentation of Brown–Henneaux boundary conditions and of the associated asymptotic Killing vector fields follows [9]. The general solution of the equations of motion first appeared in [10, 11]. It contains in particular the BTZ black hole, which was discovered and studied in [12, 13]. Finally, the group-theoretic approach to the gravitational phase space first appeared in [14–16], which is also where the AdS_3 positive energy theorem was derived. (See also [17–19] for earlier related considerations.)

B. Oblak, *BMS Particles in Three Dimensions*, Springer Theses,
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8.1 Generalities on Three-Dimensional Gravity

Here we recall a few basic facts about classical general relativity in three dimensions. We start by explaining that three-dimensional Einstein gravity has no local degrees of freedom, then turn to a discussion of boundary conditions and the ensuing boundary terms that one adds to the action in order to make the variational principle well-defined. This finally leads to the concept of asymptotic symmetries and the important observation that the Poisson brackets of surface charges that generate these symmetries generally contain central extensions.

8.1.1 Einstein Gravity in Three Dimensions

We consider an orientable three-dimensional space-time manifold $\mathcal{M}$ endowed with coordinates x^μ ($\mu = 0, 1, 2$) on which we put a metric $g_{\mu\nu}$ with signature $(- + +)$. The equations of motion are determined by the *Einstein–Hilbert action*,

$$S_{\text{EH}}[g_{\mu\nu}, \Phi] = \frac{1}{16\pi G}\int_{\mathcal{M}} d^3x\sqrt{-g}\,(R - 2\Lambda)\ . \tag{8.1}$$

Here G is the Newton constant in three dimensions, R is the Ricci scalar associated with $g_{\mu\nu}$ and $\Lambda \in \mathbb{R}$ is a cosmological constant. In three dimensions, and using units such that $c = \hbar = 1$, Newton's constant G is a length scale. Equivalently $1/G$ is an energy scale that coincides with the Planck mass.

Upon varying the action (8.1) and neglecting all boundary terms (which we shall talk about later), one obtains the vacuum Einstein's equations with a cosmological constant:

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = 0\,, \qquad \text{i.e.} \qquad R_{\mu\nu} = 2\Lambda g_{\mu\nu}\,. \tag{8.2}$$

What is special about three-dimensional manifolds is that their Ricci curvature wholly determines their Riemann tensor independently of the equations of motion:

$$R_{\lambda\mu\nu\rho} = g_{\lambda\nu}R_{\mu\rho} - g_{\lambda\rho}R_{\mu\nu} - g_{\mu\nu}R_{\lambda\rho} + g_{\mu\rho}R_{\lambda\nu} - \frac{1}{2}R(g_{\lambda\nu}g_{\mu\rho} - g_{\lambda\rho}g_{\mu\nu})\,. \tag{8.3}$$

Then the Einstein equations (8.2) imply that, at each point of space-time, the on-shell Riemann tensor is that of a maximally symmetric manifold with curvature determined by the cosmological constant:

$$R_{\lambda\mu\nu\rho} = \Lambda(g_{\lambda\nu}g_{\mu\rho} - g_{\lambda\rho}g_{\mu\nu})\,. \tag{8.4}$$

In other words, any solution of Einstein's equations in three dimensions is locally isometric to three-dimensional de Sitter, Minkowski or Anti-de Sitter space depending on whether Λ is positive, vanishing or negative respectively. This is strikingly different from higher-dimensional general relativity and relies on the relation (8.3) expressing Riemann in terms of Ricci, valid only in two and three dimensions.[1] In technical terms it is the statement that

there are no local degrees of freedom in three-dimensional Einstein gravity.

It follows in particular that there are no gravitational waves, hence no gravitons. Equivalently, all configurations of the metric are locally gauge-equivalent to empty space. Importantly, this is *not* to say that the only solution of three-dimensional gravity is empty space. For example, any quotient of Minkowski space by some discrete group solves Einstein's equations, but is not globally isometric to Minkowski. Thus global aspects are essential: even though all solutions of Einstein's equations are locally isometric, they are generally *not* globally isometric and therefore represent physically distinct field configurations. In this sense the absence of local degrees of freedom in three-dimensional gravity does not prevent the overall absence of degrees of freedom: it only means that the actual, physical degrees of freedom of the theory cannot be captured by a local analysis, but require instead a *global* one, taking into account topological properties of the space-time manifold. Field theories of this type, having no local degrees of freedom but still globally non-trivial, are called *topological field theories*.

Note that the absence of local degrees of freedom is confirmed by the Hamiltonian formalism [20]: picking a time direction in $\mathcal{M}$, one can split the metric field into a lapse N, a shift N^i and a spatial metric g_{ij} with conjugate momenta π^{ij}, the indices $i, j \in \{1, 2\}$ labelling spatial directions. The lapse and shift play the role of Lagrange multipliers enforcing the constraints that generate reparameterizations of time and spatial diffeomorphisms, respectively. One thus obtains three dynamical Lagrange variables g_{ij} with three conjugate momenta π^{ij}, subject to three first-class constraints. These constraints can be solved by choosing three gauge-fixing conditions (this is the statement that "first-class constraints count twice"), which reduces the number of physical degrees of freedom of three-dimensional Einstein gravity to $\frac{1}{2}(3 \times 2 - 3 - 3) = 0$, as expected.

Remark Since three-dimensional Einstein gravity has no local degrees of freedom, it is an unrealistic model of the world (where gravitational waves do exist [21]). This motivates the construction of alternative theories of three-dimensional gravity that do contain local degrees of freedom, such as topologically massive gravity [22] or new massive gravity [23]. In this thesis we shall be concerned only with Einstein gravity, although many of our considerations also apply to such modified theories.

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[1] In two dimensions one has in addition $R_{\mu\nu} = \frac{1}{2} R g_{\mu\nu}$.

8.1.2 Boundary Conditions and Boundary Terms

A field theory, as a Hamiltonian system, is defined by (i) its field content and Poisson brackets, and (ii) boundary conditions on fields and momenta. The second point is crucial for gauge theories such as gravity. Here we explain certain generalities on boundary conditions, leaving specific definitions in three-dimensional gravity for later. In general terms, given a set of fields living on a manifold $\mathcal{M}$, one chooses coordinates (r, x) on $\mathcal{M}$ and calls "infinity" the region where r goes to infinity while all other coordinates are kept finite. One then specifies certain fall-off conditions for fields and their derivatives on that region, typically of the form

$$\Phi(r, x) = \mathcal{O}(r^{\#}) \quad \text{as} \quad r \to +\infty$$

where Φ is some field and the coefficient # depends on the choice of fall-off conditions. In writing this it is understood that $\partial_r \Phi$ is of order $\mathcal{O}(r^{\#-1})$ at infinity.[2]

The influence of fall-offs is visible at the level of the action principle. Indeed, it is understood that the action of the theory should be plugged in an exponential e^{iS}, which is then to be integrated over field configurations in a path integral so as to produce quantum-mechanical transition amplitudes. In the classical limit, the leading contribution to the path integral should be due to on-shell field configurations; but for this to be true the integrand must be differentiable, which is to say that the functional derivative $\delta S/\delta\Phi(x)$ is a local quantity. This, in turn, is only true provided the variation of the action contains no boundary terms. For instance, the variation of the Einstein–Hilbert action (8.1) is given by

$$\begin{aligned}\delta S_{\text{EH}} = &\frac{1}{16\pi G}\int_{\mathcal{M}} d^3x\sqrt{-g}\left(R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu}\right)\delta g^{\mu\nu}\\ &+ \frac{1}{16\pi G}\int_{\mathcal{M}} d^3x\,\partial_\alpha\big(\sqrt{-g}g^{\mu\nu}\delta\Gamma^{\alpha}_{\mu\nu} - \sqrt{-g}g^{\mu\alpha}\delta\Gamma^{\lambda}_{\lambda\mu}\big)\,.\end{aligned} \tag{8.5}$$

The first term of this expression is the integral of the variation of the metric multiplying the vacuum Einstein equations, as expected. The second term is the integral of a total divergence and is therefore equal, by Stokes' theorem, to the flux of a vector field through the boundary $\partial\mathcal{M}$ of $\mathcal{M}$. Depending on one's choice of fall-off conditions for the metric, this boundary term may or may not vanish. If it does vanish, then the pure bulk action (8.1) can be legally plugged into a path integral. If it does not, then (8.1) is not differentiable and cannot be inserted as such in a path integral, which is to say that the semi-classical limit of a path integral involving only the action (8.1) is not given by on-shell field configurations. Accordingly, in order for the theory to have a well-defined semi-classical limit given by the equations of motion (8.2), one is generally forced to modify the pure bulk action (8.1) as

[2] This is not a trivial requirement; for instance the function $\sin(r^{42})/r$ is of order $\mathcal{O}(1/r)$ as $r \to +\infty$ but its derivative is not of order $\mathcal{O}(1/r^2)$.

$$S[g_{\mu\nu}] = S_{\text{EH}}[g_{\mu\nu}] + \int_{\partial\mathcal{M}} d^2x\, \mathcal{L}(g_{\mu\nu}, \partial g_{\mu\nu}, \ldots)\,. \tag{8.6}$$

Here $\mathcal{L}$ is a certain Lagrangian density on the boundary of $\mathcal{M}$, chosen so as to cancel the possibly non-vanishing boundary terms coming from the variation (8.5). Provided one can find a suitable $\mathcal{L}$, the variation of the improved action (8.6) only involves the first term of (8.5) and the theory is classically consistent.

This explains, in terms of the action, how boundary conditions affect the definition of the theory. A few remarks are in order:

- We have been sloppy in our discussion of the notion of "boundary". Indeed we claimed that the fields of our theory live on a manifold $\mathcal{M}$ and called $\partial\mathcal{M}$ its boundary, which we identified with the region $r \to +\infty$ in terms of some radial coordinate r. But typical space-time manifolds (such as $\mathbb{R}^3$) actually have no boundary in the strict sense, so we should have been more precise: when we say that the region $r \to +\infty$ is the boundary $\partial\mathcal{M}$ of $\mathcal{M}$, we really mean that we complete $\mathcal{M}$ into a larger manifold, say $\overline{\mathcal{M}}$, which now has a boundary, and in terms of the original coordinate r that boundary is located at $r = +\infty$. This completed manifold $\overline{\mathcal{M}}$ is known as a *conformal compactification* of $\mathcal{M}$ [24].
- Aside from fall-off conditions, there is a second reason for adding boundary terms to the Einstein–Hilbert action. Namely, the Ricci scalar contains second-order derivatives of the metric, so in order to insert legally the gravity action in a path integral when the metric satisfies Dirichlet boundary conditions, boundary terms must be added to the Einstein–Hilbert action to cancel these second-order terms. This is the origin of the Gibbons–Hawking–York boundary term [25, 26].
- The discussion of boundary terms clarifies in which sense a theory having no local degrees of freedom can still have non-trivial topological degrees of freedom: even though the bulk dynamics is trivial, that of the boundary is highly non-trivial! In particular topological field theories, such as three-dimensional gravity, are the simplest examples of holographic systems since all their physical degrees of freedom live on the boundary of space-time. In higher space-time dimensions, the discussion of boundary terms remains the same but it is complicated by the presence of local, bulk degrees of freedom.
- Three-dimensional Einstein gravity can be reformulated as a Chern–Simons theory whose gauge group is determined by the sign of the cosmological constant [27, 28] (see also [29–31]). This allows one to rewrite the Einstein–Hilbert action (plus boundary terms) as a purely two-dimensional action describing a field theory on the boundary of space-time, as follows from the relation between Chern–Simons theory, Wess–Zumino–Witten models and Liouville theory. It is often referred to as "dimensional reduction", and was first worked out in [32] for Brown–Henneaux boundary conditions, while flat boundary conditions were studied in [33].

8.1.3 Asymptotic Symmetries

Having justified the necessity of boundary terms for field theories, we now turn to gauge theories and explain qualitatively how one can find their global symmetries. These symmetries turn out to depend in a crucial way on the choice of fall-off conditions. We will first argue that the conserved charges associated with rigid global symmetries are strikingly different from those of gauge theories, then describe the ensuing notion of asymptotic symmetries. We conclude with the observation that the canonical generators of these symmetries generally satisfy a centrally extended Poisson algebra.

The Problem of Gauge Symmetries

Suppose we are given some gauge-invariant field theory living on a manifold $\mathcal{M}$, with some bulk action $S[\Phi]$. The system has gauge redundancies, i.e. gauge symmetries, and one expects that there exist corresponding conserved quantities. The question is: how to build such conserved charges? To answer this we follow [6].

A naive guess is to simply apply the Noether procedure. For a field theory which is left invariant by certain symmetry transformations generated by some parameters ϵ^a, $a = 1, \ldots, N$, with field and space-time transformations of the general form

$$x \mapsto x + \delta_\epsilon x, \qquad \Phi \mapsto \Phi + \delta_\epsilon \Phi\,,$$

the N Noether currents j^μ_a can be obtained by "gauging" the symmetry, that is, replacing the rigid parameters ϵ^a by arbitrary functions $\epsilon^a(x)$ on space-time. The variation of the action then takes the form

$$\delta S = -\int_{\mathcal{M}} d^D x\; j^\mu_a\,\partial_\mu \epsilon^a \tag{8.7}$$

from which one can read off the definition of the currents j^μ_a. Their conservation follows from the fact that $\delta S \approx 0$ on-shell, and the corresponding conserved Noether charges are the fluxes of these currents through a space-like slice Σ of space-time:

$$Q_a = \int_\Sigma (d^{D-1}x)_\mu j^\mu_a\,, \tag{8.8}$$

where $(d^{D-1}x)_\mu$ is proportional to $\epsilon_{\mu\alpha_1\ldots\alpha_{D-1}} dx^{\alpha_1}\ldots dx^{\alpha_{D-1}}$. Equivalently, $(d^{D-1}x)_\mu \propto d^{D-1}x \cdot n_\mu$ where n^μ is the future-pointing time-like unit vector field orthogonal to Σ, and indices are moved thanks to the space-time metric (Fig. 8.1).

The problem with gauge symmetries now becomes apparent. Indeed, in that case the symmetry parameters ϵ^a are *already* gauged, which is to say that the right-hand side of (8.7) vanishes. This in turn implies that the Noether currents associated with gauge transformations all vanish! In particular, there seems to be no way of defining conserved charges of the form (8.8) for a gauge symmetry; this problem is the key difference between *gauge* symmetries and rigid symmetries.

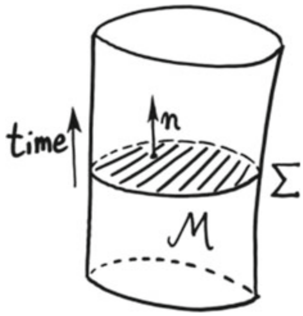

Fig. 8.1 A space-time manifold $\mathcal{M}$ with an embedded space-like slice Σ and future-pointing time-like normal vector n

The solution is provided by the following observation: the Noether current defined by (8.7) is not unique, as one can add to it the divergence of a two-form without affecting the left-hand side. In other words Eq. (8.7) does not specify the Noether current j_a^μ uniquely, since the modified current $\tilde{j}_a^\mu = j_a^\mu + \partial_\nu k_a^{\mu\nu}$, where $k_a^{\mu\nu} = -k_a^{\nu\mu}$, satisfies the same property provided the antisymmetric tensor k falls off fast enough at infinity. The corresponding Noether charge (8.8) is left unaffected by this modification provided the integral of k on the boundary of Σ vanishes; if that integral does *not* vanish, however, the charge receives an additional surface contribution of the form

$$Q_{\text{surface}} = \int_{\partial\Sigma} (d^{D-2}x)_{\mu\nu}\, k^{\mu\nu} \tag{8.9}$$

where $(d^{D-2}x)_{\mu\nu}$ is proportional to $\epsilon_{\mu\nu\alpha_1\ldots\alpha_{D-2}} dx^{\alpha_1} \ldots dx^{\alpha_{D-2}}$. As we have just argued, the would-be Noether charges of a gauge theory can *only* receive surface contributions such as (8.9) since the corresponding Noether current vanishes up to the divergence of a two-form.

At first sight this means that the situation is even worse than expected, since the Noether charges of gauge theories are apparently ill-defined: there is no a priori way to associate a $k^{\mu\nu}$ with a given symmetry generator, so the surface integral (8.9) can take any value. But in fact, this also suggests a solution to the problem: instead of trying to build a conserved current j^μ, one can associate, with a gauge symmetry, a $(D-2)$-form $k^{\mu\nu}$ and define the corresponding charge by (8.9). If $k^{\mu\nu}$ is conserved on-shell in the sense that $\nabla_\mu k^{\mu\nu} \approx 0$, then the corresponding charge (8.9) is conserved by time evolution. In that context, the field $k^{\mu\nu}$ is called a *superpotential* and its integral (8.9) over the boundary of Σ is known as the associated *surface charge*.[3] For example, in electrodynamics, the superpotential coincides with the strength tensor $F^{\mu\nu}$ and the corresponding surface charge is the flux of the electric field at infinity, that is, the total electric charge. Its conservation follows from the fact that $\partial_\mu F^{\mu\nu}$ vanishes on-shell by virtue of Maxwell's equations.

[3] The term "superpotential" here has nothing to do with supersymmetry.

Thus the computation of conserved charges for gauge symmetries boils down to the problem of associating a conserved superpotential with a given gauge transformation, and understanding to what extent that superpotential is unique.

Asymptotic Symmetries

While the definition (8.7) of the Noether current associated with a global symmetry transformation is straightforward, that of the superpotential associated with a gauge transformation is much more involved; see e.g. [6, 8, 34]. Here we simply summarize the main ideas so as to apply them later to the specific case of three-dimensional gravity. The construction consists of several steps:

1. Define the theory by choosing a bulk action, imposing certain fall-off conditions on the field content, and possibly adding a boundary term to the bulk action such that the full action is differentiable.
2. Find, among all possible gauge transformations, those that preserve the fall-off conditions. Such gauge transformations are said to be *allowed*, as opposed to the gauge transformations that spoil the fall-off conditions and are therefore "forbidden". Allowed gauge transformations should then be thought of as the symmetries (global or gauge) of the theory.
3. Associate, with each allowed gauge transformation, a conserved superpotential $k^{\mu\nu}$; the latter depends linearly on the gauge parameters, while its dependence on the field content depends on the model under study. We will not write down that dependence explicitly here and refer to [6, 34] for details.
4. For each superpotential $k^{\mu\nu}$, define a surface charge Q by (8.9). If all surface charges associated with allowed gauge transformations are finite, then the boundary conditions are consistent. The allowed gauge transformations whose surface charges vanish are said to be *trivial*, while those whose surface charges do not vanish are *non-trivial*.

This construction provides a distinction between three families of gauge transformations — forbidden, allowed and trivial — and is illustrated in Fig. 8.2. It is not just a matter of terminology; different classes of gauge transformations truly represent physically distinct notions of symmetries:

- Trivial gauge transformations are genuine (allowed) gauge transformations, that is, redundancies in the description of the theory.
- Non-trivial gauge transformations are global symmetries that map a field configuration on a physically different one. They fall off at infinity much slower than trivial gauge transformations and change the state of the system when acting on it. For example, in electrodynamics, non-trivial gauge transformations at spatial infinity take the form $\delta A_\mu(x) = \partial_\mu \epsilon(x)$ with $\epsilon(x) = \text{const.}$[4] This corresponds to a global U(1) symmetry and the associated charge is the electric charge.

[4]In practice, for constant ϵ this gives $\delta A_\mu = 0$, but for fields with non-zero electric charge the transformation given by constant ϵ's is non-trivial.

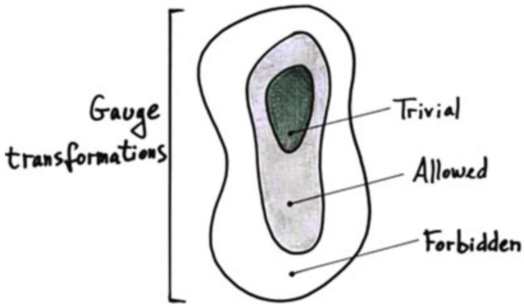

Fig. 8.2 Gauge transformations fall in three classes: forbidden transformations are those that do not preserve the fall-off conditions of the theory; allowed transformations are those that do, although they generally change the state of the system; trivial transformations are those that preserve the fall-off conditions *and* leave the state of the system unchanged. In this sense trivial gauge transformations are actual gauge redundancies, and the global symmetry group of the system is the quotient of the group of allowed transformations by its subgroup of trivial gauge transformations

- Forbidden gauge transformations are neither gauge transformations, nor even global symmetries: they are literally excluded from the theory since they do not leave its phase space invariant.

Note that infinitesimal gauge transformations are always endowed with a Lie bracket. Accordingly, they span a Lie algebra. The notions introduced above then lead to the following terminology:

Definition The *asymptotic symmetry algebra* of a theory is the quotient of the algebra of allowed gauge transformations by its ideal consisting of trivial transformations.

In the context of gravity, gauge transformations are diffeomorphisms of the space-time manifold, generated by certain vector fields. Allowed gauge transformations are generated by so-called *asymptotic Killing vector fields*. Their Lie bracket is the standard Lie bracket of vector fields and the asymptotic symmetry algebra coincides with the global symmetry algebra of the putative dual theory. In Sect. 8.2 we will illustrate these notions in the case of AdS_3 gravity with Brown–Henneaux boundary conditions, while Sect. 9.1 will be devoted to their asymptotically flat analogue.

Central Extensions in the Surface Charge Algebra

The surface charges associated with asymptotic symmetries are designed in such a way that they implement asymptotic symmetry transformations on the fields of the theory. Explicitly, if we call ξ some infinitesimal gauge parameter generating an allowed non-trivial gauge transformation and if we denote the associated surface charge by $Q[\xi]$, then the Poisson bracket of this charge with any field Φ takes the form

$$\{Q[\xi], \Phi\} = -\delta_\xi \Phi \tag{8.10}$$

where the right-hand side is (minus) the variation of Φ under the transformation generated by ξ. This is a restatement of Eq. (5.34), where we noted that Poisson brackets with momentum maps generate symmetry transformations.

Since Poisson brackets satisfy the Jacobi identity, Eq. (5.35) still holds: for any two infinitesimal gauge transformations ξ, ζ and any field configuration Φ, we have

$$\big\{\{Q[\xi], Q[\zeta]\}, \Phi\big\} = \big\{Q\big[[\xi,\zeta]\big], \Phi\big\}. \tag{8.11}$$

It is tempting to remove the Poisson brackets from both sides of this equality and conclude that surface charges provide an exact representation of the asymptotic symmetry algebra. However, this naive removal would overlook the crucial point (5.36) that surface charges generally close according to a (classical) *central extension* of the algebra of asymptotic symmetry generators:

$$\{Q[\xi], Q[\zeta]\} = Q\big[[\xi,\zeta]\big] + \mathsf{c}(\xi,\zeta). \tag{8.12}$$

Here $\mathsf{c}(\xi,\zeta)$ is a real-valued two-cocycle that acts trivially on any field and is therefore invisible in Eq. (8.11). The point of the seminal paper [1] was to show that such non-trivial central extensions do arise in asymptotic symmetries of gravitational systems.

8.2 Brown–Henneaux Metrics in AdS_3

In this section we analyse Brown–Henneaux boundary conditions for Einstein gravity in AdS_3. After recalling some elementary geometric aspects of three-dimensional Anti-de Sitter space, we introduce Brown–Henneaux fall-offs and work out the corresponding asymptotic Killing vector fields. We also display the general solution of Einstein's equations satisfying these boundary conditions and use it to derive the algebra of surface charges associated with asymptotic symmetries, resulting in a direct sum of two Virasoro algebras with non-zero central charges. We end by describing an important family of Brown–Henneaux metrics that includes BTZ black holes.

8.2.1 Geometry of AdS_3

Anti-de Sitter Space in Three Dimensions

Consider the space $\mathbb{R}^4 = \mathbb{R}^{2,2}$ endowed with coordinates (x, y, u, v) and the metric

$$ds^2 = dx^2 + dy^2 - du^2 - dv^2. \tag{8.13}$$

Then three-dimensional *Anti-de Sitter* space (or simply AdS_3) is the submanifold of $\mathbb{R}^{2,2}$ given by

$$AdS_3 \equiv \left\{(x, y, u, v) \in \mathbb{R}^{2,2} \middle| u^2 + v^2 = \ell^2 + x^2 + y^2\right\} \tag{8.14}$$

for some parameter $\ell^2 > 0$, equipped with the induced metric of $\mathbb{R}^{2,2}$. The parameter ℓ is called the *AdS radius*. The manifold (8.14) is diffeomorphic to a product $S^1 \times \mathbb{R}^2$ where the circle is time-like; in particular it contains closed time-like curves. Its isometry group is $O(2, 2)$ and acts transitively according to $x^\mu \mapsto \Lambda^\mu{}_\nu x^\nu$, where x^μ denotes the coordinates (x, y, u, v) and Λ is a 4×4 matrix that preserves the "Minkowski metric" (8.13). The stabilizer for this action is isomorphic to $O(2, 1)$, so there is a diffeomorphism (Fig. 8.3)

$$AdS_3 \cong O(2, 2)/O(2, 1) \cong SO(2, 2)/SO(2, 1).$$

In practice, physical models of space-time are manifolds without closed time-like curves. It is therefore customary to unwind AdS_3 into its universal cover, $\widetilde{AdS_3}$, which is diffeomorphic to $\mathbb{R}^3$ as a manifold. (Of course the metric on $\widetilde{AdS_3} \cong \mathbb{R}^3$ is not flat!) To describe $\widetilde{AdS_3}$, we introduce new coordinates (r, φ, t) given on (8.14) by

$$\begin{aligned} r &= \sqrt{x^2 + y^2} = \sqrt{u^2 + v^2 - \ell^2}, \\ \varphi &= \arctan(y/x), \\ t &= \ell \operatorname{arctanh}(v/u). \end{aligned} \tag{8.15}$$

On AdS_3 the coordinate $t \in \mathbb{R}$ is subject to the identification $t \sim t + 2\pi\ell$, while on the universal cover $\widetilde{AdS_3}$ it takes all real values, without identification; see Fig. 8.4. In terms of these coordinates the AdS_3 metric induced by (8.13) is

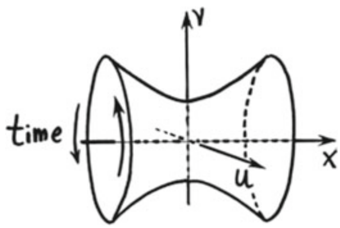

Fig. 8.3 Two-dimensional anti-de Sitter space-time embedded in $\mathbb{R}^3$ as the submanifold $u^2 + v^2 = x^2 + \ell^2$ in terms of coordinates u, v, x such that the mock-Minkowski metric of $\mathbb{R}^3$ reads $-du^2 - dv^2 + dx^2$. Circles at constant x are closed time-like curves in AdS_2. The spatial boundary of AdS_2 consists of two circles at $|x| \to +\infty$. For AdS_3, the boundary is a time-like torus $S^1 \times S^1$

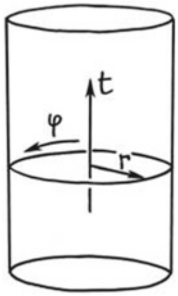

Fig. 8.4 The universal cover of three-dimensional anti-de Sitter space-time, diffeomorphic to $\mathbb{R}^3$. It is equivalent to the interior of a *solid cylinder*, which may be seen as the Penrose diagram of $\widetilde{\text{AdS}_3}$. The time coordinate t is directed along the axis of the cylinder while r is a radial coordinate, and φ is a 2π-periodic coordinate on the circle. The spatial boundary $r \to +\infty$ is a two-dimensional time-like cylinder spanned by the coordinates (φ, t), or equivalently by the light cone coordinates $x^{\pm}$

$$ds^2 = -(1 + r^2/\ell^2)dt^2 + \frac{dr^2}{1 + r^2/\ell^2} + r^2 d\varphi^2 \,. \tag{8.16}$$

From now on we always refer to the universal cover $\mathbb{R}^3$ of (8.14) simply as AdS_3, without tilde. With this notation the coordinates $t \in \mathbb{R}$, $r \in [0, +\infty[$, $\varphi \in \mathbb{R}$ with $\varphi \sim \varphi + 2\pi$, are global coordinates on AdS_3. In general-relativistic terms, AdS_3 is the (universal cover of the) maximally symmetric solution of Einstein's vacuum equations in three dimensions with a negative cosmological constant $\Lambda = -1/\ell^2$. Note at the outset that gravitation on an AdS_3 background is determined by two independent length scales ℓ and G. In particular the dimensionless coupling constant of the theory is G/ℓ, so that the semi-classical regime corresponds to $\ell/G \to +\infty$.

Killing Vectors

The Killing vectors that generate isometries of AdS_3 can be found thanks to the embedding (8.14), where "Lorentz" transformations are generated by the six independent vector fields

$$\begin{aligned} \xi_1 &= u\partial_v - v\partial_u, & \xi_2 &= x\partial_y - y\partial_x, & \xi_3 &= u\partial_y + y\partial_u, \\ \xi_4 &= v\partial_x + x\partial_v, & \xi_5 &= u\partial_x + x\partial_u, & \xi_6 &= v\partial_y + y\partial_v. \end{aligned}$$

The combinations of signs appearing here are due to the metric $(++--)$ in (8.13). Upon defining

$$\begin{aligned} \ell_0 &\equiv \tfrac{1}{2}(\xi_1 + \xi_2), & \bar{\ell}_0 &\equiv \tfrac{1}{2}(\xi_1 - \xi_2), \\ \ell_1 &\equiv \tfrac{1}{2}(\xi_3 + \xi_4 - i\xi_5 + i\xi_6), & \bar{\ell}_1 &\equiv \tfrac{1}{2}(-\xi_3 + \xi_4 - i\xi_5 - i\xi_6), \\ \ell_{-1} &\equiv \tfrac{1}{2}(\xi_3 + \xi_4 + i\xi_5 - i\xi_6), & \bar{\ell}_{-1} &\equiv \tfrac{1}{2}(-\xi_3 + \xi_4 + i\xi_5 + i\xi_6), \end{aligned} \tag{8.17}$$

one finds the following Lie brackets for $m, n = -1, 0, 1$:

$$i[\ell_m, \ell_n] = (m-n)\ell_{m+n}\,, \qquad i[\bar{\ell}_m, \bar{\ell}_n] = (m-n)\bar{\ell}_{m+n}\,, \qquad i[\ell_m, \bar{\ell}_n] = 0\,. \tag{8.18}$$

This exhibits the isomorphism $\mathfrak{so}(2,2) \cong \mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{sl}(2,\mathbb{R})$,[5] upon identifying the Lie brackets (5.90). Note that the generator of time translations is $\partial_t = \frac{1}{\ell}(\ell_0 + \bar{\ell}_0)$ while the generator of rotations is $\partial_\varphi = \ell_0 - \bar{\ell}_0$.

Spatial Infinity

The region $r \to +\infty$ is a cylinder spanned by coordinates (φ, t) at space-like infinity. It is the spatial boundary $\partial\mathcal{M}$ of AdS_3. In that region the metric (8.16) is

$$ds^2 \sim \frac{\ell^2}{r^2}dr^2 - r^2\left(\frac{dt^2}{\ell^2} - d\varphi^2\right) = \frac{\ell^2}{r^2}dr^2 - r^2 dx^+ dx^- \tag{8.19}$$

where we have introduced the light-cone coordinates

$$x^\pm \equiv \frac{t}{\ell} \pm \varphi\,. \tag{8.20}$$

For large r the Killing vector fields (8.17) are asymptotic to

$$\ell_m \sim e^{imx^+}\partial_+ - \frac{1}{2}ime^{imx^+}r\partial_r\,, \qquad \bar{\ell}_m \sim e^{imx^-}\partial_- - \frac{1}{2}ime^{imx^-}r\partial_r$$

where $m = -1, 0, 1$. They generate global conformal transformations of the cylinder at infinity, including time translations $\ell_0 + \bar{\ell}_0 = \ell\partial_t$ and rotations $\ell_0 - \bar{\ell}_0 = \partial_\varphi$. These expressions have the general form

$$\xi \sim X(x^+)\partial_+ - \frac{1}{2}\partial_+ X(x^+) r\partial_r\,, \qquad \bar{\xi} \sim \bar{X}(x^-)\partial_- - \frac{1}{2}\partial_- \bar{X}(x^-) r\partial_r \tag{8.21}$$

where the functions X and $\bar{X}$ are 2π-periodic. Brown–Henneaux boundary conditions will be such that vector fields of the form (8.21) are asymptotic symmetry generators for *arbitrary* functions X, $\bar{X}$.

8.2.2 Brown–Henneaux Boundary Conditions

We now wish to define a family of metrics on $\mathbb{R}^3$ that are "asymptotically Anti-de Sitter" in the sense that they take the form of a pure AdS_3 metric (8.16) at infinity.

[5] Strictly speaking we have displayed this isomorphism here for the complexification of $\mathfrak{so}(2,2)$, but it also holds for real Lie algebras.

As a starting point we ask what is the minimum amount of metrics that we wish to include. A natural choice is to take pure AdS$_3$ together with conical deficits, which are obtained by cutting out a wedge out of the middle of AdS$_3$ and identifying its two sides. Concretely, consider the manifold described by coordinates $r \in [0, +\infty[$, $\varphi \in \mathbb{R}$, $t \in \mathbb{R}$ subject to the identifications

$$(r, \varphi, t) \sim \big(r, \varphi + 4\pi\omega, t - 2\pi A\big) \tag{8.22}$$

for some $A \in \mathbb{R}$ and $\omega > 0$. (The normalization of ω is chosen for later convenience.) For $\omega = 1/2$ and $A = 0$ this reduces to the identifications that define pure AdS$_3$. For $0 < \omega < 1/2$ it is a conical deficit; for $\omega > 1/2$ it is a conical excess. Since this is a global (topological) identification, the resulting pseudo-Riemannian manifold still solves Einstein's vacuum equations everywhere, except at the origin. In fact, the metric (8.16) with identifications (8.22) is the solution of Einstein's equations coupled to the stress tensor of a point mass at the origin. Using the change of coordinates

$$t' \equiv t + \frac{A}{2\omega}\varphi\,, \qquad r' \equiv r\,, \qquad \varphi' \equiv \frac{\varphi}{2\omega}\,, \tag{8.23}$$

the space-time metric can be rewritten as

$$ds^2 = -\left(1 + \frac{r'^2}{\ell^2}\right)(dt' - Ad\varphi')^2 + \frac{dr'^2}{1 + r'^2/\ell^2} + 4\omega^2 r'^2 d\varphi'^2 \tag{8.24}$$

where now there are no identifications on t', while φ' is 2π-periodic. The term $A\,dt'd\varphi'$ suggests that A is proportional to angular momentum, as will indeed be the case below. Note that the integral curves of $\partial_{\varphi'}$ contain closed time-like curves unless

$$|A| \leq 2\omega\ell \qquad \text{and} \qquad r'^2 \geq \frac{A^2\ell^2}{4\omega^2\ell^2 - A^2}\,. \tag{8.25}$$

Thus, the space-time manifold has no pathologies only in the region where r' is large enough (and in particular in the asymptotic region $r' \to +\infty$), and provided the parameter A is not too large compared to $\omega\ell$. Accordingly, from now on we refer to the solutions (8.24) with $0 < \omega < 1/2$ and $|A| = 2\omega\ell$ as *extreme conical deficits*.

In order to find boundary conditions that genuinely describe AdS$_3$ space-times, one would like the asymptotic symmetry algebra to at least include $\mathfrak{so}(2, 2)$. If in addition the phase space is to contain conical deficits (8.24), one is led to act with $\mathfrak{so}(2, 2)$ transformations on such conical deficit metrics so that, if ξ is an AdS$_3$ Killing vector and $g_{\mu\nu}$ is the metric of a conical deficit, the fall-off conditions are satisfied by the infinitesimally transformed metric

$$g_{\mu\nu} + \mathcal{L}_\xi g_{\mu\nu}\,. \tag{8.26}$$

Here $\mathcal{L}_\xi g_{\mu\nu}$ generally does *not* vanish because ξ may not be a Killing vector for the conical deficit. In terms of cylindrical coordinates (r, φ, t), one thus obtains metrics that satisfy the fall-off conditions [1]

$$(g_{\mu\nu}) = \begin{pmatrix} g_{rr} & g_{r\varphi} & g_{rt} \\ g_{\varphi r} & g_{\varphi\varphi} & g_{\varphi t} \\ g_{tr} & g_{t\varphi} & g_{tt} \end{pmatrix} \sim \begin{pmatrix} \frac{\ell^2}{r^2} + \mathcal{O}(r^{-4}) & \mathcal{O}(r^{-3}) & \mathcal{O}(r^{-3}) \\ \mathcal{O}(r^{-3}) & r^2 + \mathcal{O}(1) & \mathcal{O}(1) \\ \mathcal{O}(r^{-3}) & \mathcal{O}(1) & -\frac{r^2}{\ell^2} + \mathcal{O}(1) \end{pmatrix}. \quad (8.27)$$

In practice, we will impose an extra gauge-fixing condition that simplifies the computation of asymptotic symmetries. Namely, it turns out that the mixed components $g_{r\varphi}$ and g_{rt} can always be set to zero (identically) by applying a trivial diffeomorphism — one whose surface charges all vanish. The subleading corrections to $g_{rr} = \frac{\ell^2}{r^2} + \mathcal{O}(r^{-4})$ can similarly be set to zero. We refer to this gauge choice as the *Fefferman–Graham gauge*. It leads to the following definition [9]:

Definition Let $\mathcal{M}$ be a three-dimensional manifold with a pseudo-Riemannian metric ds^2. Suppose there exist local coordinates (r, x^a) on $\mathcal{M}$ (with $a = 0, 1$), defined for r larger than some lower limit, such that the region $r \to +\infty$ is a time-like cylinder at spatial infinity where the asymptotic behaviour of the metric is

$$ds^2 \overset{r\to+\infty}{\sim} \frac{\ell^2}{r^2}dr^2 + \big(r^2\eta_{ab} + \mathcal{O}(1)\big)dx^a dx^b \quad (8.28)$$

with $\eta_{ab}dx^a dx^b$ the two-dimensional Minkowski metric on the cylinder. Then we say that $(\mathcal{M}, ds^2)$ is *asymptotically Anti-de Sitter* in the sense of Brown–Henneaux (in the Fefferman–Graham gauge), with a cosmological constant $\Lambda = -1/\ell^2$.

From now on, when dealing with AdS_3 gravity, we always restrict our attention to metrics satisfying the Brown–Henneaux boundary conditions (8.28). For practical purposes we will mostly describe the time-like cylinder in terms of light-cone coordinates $x^\pm$, in which case the label a in (8.28) takes the values $\pm$ and the Minkowski metric on the cylinder is $\eta_{ab}dx^a dx^b = -dx^+dx^-$. Note that asymptotically AdS_3 space-times need not be (and generally are not) globally diffeomorphic to AdS_3; in particular there may be singularities in the bulk, as the definition (8.28) only requires r to be larger than some lower limiting value. In the following pages we establish the main properties of this family of metrics, including their asymptotic symmetry algebra.

Remark The fact that one is allowed to choose the Fefferman–Graham gauge without losing any information is a general property of locally asymptotically Anti-de Sitter space-times [35]. It is related to the Fefferman–Graham expansion of AdS metrics and the ambient construction of conformal structures [36], where conformal manifolds are built as boundaries, or celestial spheres, of higher-dimensional bulk manifolds.

8.2.3 Asymptotic Killing Vector Fields

The asymptotic Killing vector fields associated with Brown–Henneaux boundary conditions are vector fields that generate diffeomorphisms which preserve the fall-offs (8.28). If $g_{\mu\nu}$ is a Brown–Henneaux metric and if ξ is such a vector field, this is to say that

$$\mathcal{L}_\xi g_{rr} = \mathcal{L}_\xi g_{r\pm} = 0, \qquad \mathcal{L}_\xi g_{ab} = \mathcal{O}(1) \quad (a, b = \pm) \tag{8.29}$$

in terms of light-cone coordinates (8.20). Here the first condition follows from the fact that the components $g_{rr} = \ell^2/r^2$ and $g_{r\pm} = 0$ are fixed, while the components g_{ab} are allowed to fluctuate by terms of order r^0 at infinity.

Lemma Let $g_{\mu\nu}$ be a metric that is asymptotically AdS$_3$ in the sense (8.28) and let ξ be a vector field that satisfies the properties (8.29). Then

$$\xi = X(x^+)\partial_+ + \bar{X}(x^-)\partial_- - \frac{1}{2}\big(\partial_+ X(x^+) + \partial_- \bar{X}(x^-)\big) r\partial_r + \text{(subleading)} \tag{8.30}$$

where $X(x^+)$ and $\bar{X}(x^-)$ are two arbitrary (smooth) 2π-periodic functions while the subleading terms take the form

$$\begin{aligned} &-\frac{\ell^2}{2}\partial_a(\partial_+ X + \partial_- \bar{X}) \int_r^{+\infty} \frac{dr'}{r'} g^{ab}(r', x^\pm)\partial_b = \\ &= \frac{\ell^2}{2r^2}\big[\partial_-(\partial_+ X + \partial_- \bar{X})\partial_+ + \partial_+(\partial_+ X + \partial_- \bar{X})\partial_-\big] + \mathcal{O}(r^{-4}) \,. \end{aligned} \tag{8.31}$$

These formulas associate an asymptotic Killing vector ξ with an asymptotically AdS$_3$ metric $g_{\mu\nu}$ and a vector field $X(x^+)\partial_+ + \bar{X}(x^-)\partial_-$ on the cylinder; the dependence of ξ on the latter is linear.

Proof Let $g_{\mu\nu}$ be an asymptotically AdS$_3$ metric (8.28). We first note that the requirement $\mathcal{L}_\xi g_{rr} = 0$ imposes $\partial_r \xi^r = \xi^r/r$, whose solution is

$$\xi^r(r, x^\pm) = r\mathcal{F}(x^\pm) \tag{8.32}$$

for some function $\mathcal{F}$ on the cylinder. On the other hand the condition $\mathcal{L}_\xi g_{r\pm} = 0$ yields $\partial_r \xi^c = -g^{ca}\frac{\ell^2}{r}\partial_a \mathcal{F}$, which is solved by

$$\xi^a = X^a(x^\pm) + \ell^2 \partial_b \mathcal{F}(x^\pm) \int_r^{+\infty} \frac{dr'}{r'} g^{ab}(r', x^\pm) \tag{8.33}$$

where $X^a\partial_a$ is an arbitrary vector field on the cylinder. Note that the integral over r' converges since $g_{ab}(r, x^\pm) = r^2\eta_{ab} + \mathcal{O}(1)$ by virtue of (8.28), so that the inverse is $g^{ab} = \frac{\eta^{ab}}{r^2} + \mathcal{O}(r^{-4})$. Plugging this in the integral of (8.33) we find explicitly

$$\xi^a = X^a(x^\pm) + \frac{\ell^2}{2r^2}\eta^{ab}\partial_b\mathcal{F}(x^\pm) + \mathcal{O}(r^{-4})\,. \tag{8.34}$$

In light-cone coordinates (8.20), the two-dimensional Minkowski metric reads

$$(\eta_{ab}) = \begin{pmatrix}\eta_{++} & \eta_{+-}\\ \eta_{-+} & \eta_{--}\end{pmatrix} = \begin{pmatrix}0 & -1/2\\ -1/2 & 0\end{pmatrix} \quad \text{and} \quad (\eta^{ab}) = \begin{pmatrix}0 & -2\\ -2 & 0\end{pmatrix}$$

so that (8.34) becomes

$$\xi^\pm = X^\pm - \frac{\ell^2}{r^2}\partial_\mp\mathcal{F} + \mathcal{O}(r^{-4})\,. \tag{8.35}$$

Finally one finds $\mathcal{L}_\xi g_{ab} = r^2\,(2\mathcal{F}\eta_{ab} + \mathcal{L}_X\eta_{ab}) + \mathcal{O}(1)$ where $\mathcal{L}_X$ denotes the Lie derivative on the cylinder with respect to the vector field $X^a\partial_a$. The requirement that this expression be of order one yields the conformal Killing equation for X, $\mathcal{L}_X\eta_{ab} = -2\mathcal{F}\eta_{ab}$. Contracting this with η^{ab} one finds $\mathcal{F} = -\frac{1}{2}(\partial_+X^+ + \partial_-X^-)$ and the remaining constraints set $\partial_-X^+ = \partial_+X^-$, which implies $X^+ = X(x^+)$ and $X^- = \bar{X}(x^-)$. Formula (8.30) follows, while the subleading terms (8.31) are produced by (8.33). ■

Note that the asymptotic Killing vector (8.30) takes the anticipated form (8.21) and thus provides the generalization we were hoping to find. We will denote by $\xi_{(X,\bar{X})}$ the asymptotic Killing vector field determined by the functions $X(x^+)$ and $\bar{X}(x^-)$. One can decompose these functions in Fourier modes and define the vector fields

$$\ell_m \equiv \xi_{(e^{imx^+},0)}\,, \qquad \bar{\ell}_m \equiv \xi_{(0,e^{imx^-})}\,, \tag{8.36}$$

whose Lie brackets take the form (8.18) up to subleading corrections, with indices m, n ranging over all integer values. Thus, asymptotically, the finite-dimensional isometry algebra $\mathfrak{so}(2,2)$ of AdS$_3$ is enhanced to two commuting copies of the infinite-dimensional Witt algebra (6.24). In fact we can already anticipate the result:

Theorem The asymptotic symmetry group of AdS$_3$ gravity with Brown–Henneaux boundary conditions is a direct product $\widetilde{\text{Diff}}^+(S^1) \times \widetilde{\text{Diff}}^+(S^1)$ whose elements are diffeomorphisms

$$(x^+, x^-) \mapsto \big(f(x^+), \bar{f}(x^-)\big) \tag{8.37}$$

acting as conformal transformations on the cylinder at spatial infinity.

At this stage, we have not yet proven this claim since we do not know whether all asymptotic Killing vector fields (8.30) have non-vanishing surface charges on the phase space; this will be done in the following pages. Also note that we are being slightly sloppy in (8.37), since the diffeomorphisms generated by (8.30) affect the radial coordinate. Hence formula (8.37) only holds up to $1/r$ corrections; it is accompanied by transformations of the radial coordinate that we do not bother writing down, but that do preserve the limit $r \to +\infty$ in that they map r on a positive $\mathcal{O}(1)$ multiple of itself.

Remark In our description of asymptotic symmetries we mentioned that the algebra of vector fields (8.36) is a direct sum of two Witt algebras *up to subleading corrections* which we did not take into account. This is because these corrections are unimportant: starting from the standard Lie bracket of vector fields, one can define a "modified bracket" that coincides with the standard one at infinity but ensures that the asymptotic symmetry algebra is satisfied everywhere in the bulk; see e.g. [6, 37].

8.2.4 On-Shell Brown–Henneaux Metrics

In order for the equations of motion to provide a true extremum of the action functional, the latter must be differentiable in the space of fields subject to the chosen boundary conditions. In the case of Brown–Henneaux fall-offs, one can show that the improved action

$$S[g_{\mu\nu}] \equiv S_{\text{EH}}[g_{\mu\nu}] - \frac{1}{8\pi G}\int_{\partial\mathcal{M}} d^2x\sqrt{-\det(g_{ab})}\left(K + \frac{1}{\ell}\right)$$

is differentiable in the space of metrics satisfying Brown–Henneaux boundary conditions, where S_{EH} is the Einstein–Hilbert action (8.1) while K is the trace of the extrinsic curvature at the boundary [25, 26] (see also [38–41]).

With this improved action it makes sense to solve Einstein's equations in the space of metrics (8.28). We will not review this computation here and refer to [9–11] for details. The bottom line is that the general solution of the equations of motion with Brown–Henneaux boundary conditions in the Fefferman–Graham gauge reads

$$ds^2 = \frac{\ell^2}{r^2}dr^2 - \left(rdx^+ - \frac{4G\ell}{r}\bar{p}(x^-)dx^-\right)\left(rdx^- - \frac{4G\ell}{r}p(x^+)dx^+\right) \quad (8.38)$$

where $p(x^+)$ and $\bar{p}(x^-)$ are arbitrary, 2π-periodic functions of their arguments. The factors of $4G\ell$ are introduced for later convenience. We will study this space of solutions in greater detail below. For now, we only note that it is endowed with a well-defined action of asymptotic symmetry transformations. Indeed, we define the variation of p and $\bar{p}$ under the action of an asymptotic Killing vector (8.30) by

$$\mathcal{L}_{\xi_{(X,\bar{X})}} ds^2 \equiv 4G\ell\ \delta_X p(x^+)\,(dx^+)^2 + 4G\ell\ \delta_{\bar{X}}\bar{p}(x^-)\,(dx^-)^2 + \text{(subleading)},$$

and this variation preserves the structure of the solution (8.38). In particular, observe that $\xi_{(X,\bar{X})}$ is an exact Killing vector if the variations $\delta_X p$ and $\delta_{\bar{X}}\bar{p}$ vanish. Using (8.30) one finds

$$\delta_X p = X\partial_+ p + 2p\partial_+ X - \frac{c}{12}\partial_+^3 X\,, \qquad \delta_{\bar{X}}\bar{p} = \bar{X}\partial_-\bar{p} + 2\bar{p}\partial_-\bar{X} - \frac{\bar{c}}{12}\partial_-^3\bar{X} \tag{8.39}$$

where $c = \bar{c}$ is the *Brown–Henneaux central charge*

$$\boxed{c = \bar{c} = \frac{3\ell}{2G}\,.} \tag{8.40}$$

The transformations (8.39) are exactly those of the components of a CFT stress tensor under conformal transformations; they coincide with the coadjoint representation (6.115) of the Virasoro algebra when seeing $p(x^+)$ and $\bar{p}(x^-)$ as Virasoro coadjoint vectors. In that context the condition for $\xi_{(X,\bar{X})}$ to be an exact Killing vector is equivalent to the statement that $(X, \bar{X})$ belongs to the stabilizer of $(p, \bar{p})$. We refrain from interpreting these results any further for now; we will return to them in Sect. 8.3. Note that at this stage there is actually no reason to call (8.40) a central charge: even though it does appear in (8.39) exactly as the inhomogeneous term of the coadjoint representation (6.115), the specific value (8.40) is irrelevant since changing the normalization of p or $\bar{p}$ would change the value of c and $\bar{c}$. The importance of the parameter (8.40) will become apparent only from the algebra of surface charges.

8.2.5 Surface Charges and Virasoro Algebra

Surface Charges

Take an asymptotic Killing vector field (8.30) specified by the functions $X(x^+)$, $\bar{X}(x^-)$, and choose an on-shell metric (8.38) specified by $p(x^+)$, $\bar{p}(x^-)$. We wish to evaluate the surface charge associated with the symmetry transformation generated by $\xi_{(X,\bar{X})}$ on the background specified by p, $\bar{p}$. As explained around Eq. (8.9), this charge depends linearly on the components of $\xi_{(X,\bar{X})}$. In addition we need to choose a "background" solution for which all surface charges vanish, which we take to be the degenerate conical deficit at $p = \bar{p} = 0$,

$$\bar{g} = \frac{\ell^2}{r^2}dr^2 - r^2 dx^+ dx^-. \tag{8.41}$$

With this normalization one can show that the conserved superpotentials corresponding to Brown–Henneaux asymptotic symmetries are such that the surface charge (8.9) associated with the vector field $\xi_{(X,\bar{X})}$ on the solution $(p, \bar{p})$ is

$$Q_{(X,\bar{X})}[p, \bar{p}] = \frac{1}{2\pi}\int_0^{2\pi} d\varphi\big[p(x^+)X(x^+) + \bar{p}(x^-)\bar{X}(x^-)\big] \tag{8.42}$$

where $\varphi = \frac{1}{2}(x^+ - x^-)$. (See e.g. [9] for an explicit computation.)

This charge can be interpreted in two ways: first, as the Noether charge associated with a conformal transformation $(X, \bar{X})$ in a two-dimensional CFT on the cylinder with stress tensor $(p, \bar{p})$; second, as the pairing (6.111) between the direct sum of two Virasoro algebras and its dual. This is consistent with the fact that the transformation law (8.39) coincides with the coadjoint representation of Virasoro. In particular, the charge associated with time translations corresponds to the asymptotic Killing vector $\partial_t = (\partial_+ + \partial_-)/\ell$; it is the ADM mass of the system, or equivalently the Hamiltonian

$$M[p, \bar{p}] = \frac{1}{2\pi\ell} \int_0^{2\pi} d\varphi \left[p(x^+) + \bar{p}(x^-)\right] \tag{8.43}$$

and it coincides (up to a factor $1/\ell$) with the sum of two Virasoro energy functionals (7.79). Similarly the charge associated with rotations, generated by the asymptotic Killing vector $\partial_\varphi = \partial_+ - \partial_-$, is the angular momentum

$$J = \frac{1}{2\pi} \int_0^{2\pi} d\varphi \left[p(x^+) - \bar{p}(x^-)\right] \tag{8.44}$$

and coincides with the difference of two Virasoro energy functionals. With this normalization, pure AdS_3 (8.47) has mass $M = -\frac{1}{8G}$; all its other surface charges vanish.

Surface Charge Algebra

We now compute the algebra satisfied by the surface charges (8.42) under Poisson brackets. Recall that these brackets are such that they generate symmetry transformations according to (8.10). We can apply this property here: if we let $(p, \bar{p})$ be an on-shell metric (8.38), then the bracket of two charges $Q_{(X,0)}[p, \bar{p}]$ and $Q_{(Y,0)}[p, \bar{p}]$ is

$$\begin{aligned}\left\{Q_{(X,0)}[p, \bar{p}], Q_{(Y,0)}[p, \bar{p}]\right\} &\overset{(8.39)}{=} -\tfrac{1}{2\pi} \textstyle\int_0^{2\pi} d\varphi \left(X\partial_+ p + 2p\partial_+ X - \tfrac{c}{12}\partial_+^3 X\right) Y(x^+) \\ &= Q_{([X,Y],0)}[p, \bar{p}] + c\, \mathsf{c}(X, Y)\,.\end{aligned} \tag{8.45}$$

In the last line we have introduced the bracket $[X, Y]$ defined as the usual Lie bracket of vector fields on the line, while $\mathsf{c}(X, Y)$ is the Gelfand–Fuks cocycle (6.43) expressed in the coordinate x^+. This is a Virasoro algebra (6.108), with a classical central extension! The same computation would hold in the barred (antichiral) sector, while chiral and antichiral charges commute. Thus we conclude:

Theorem The algebra of surface charges associated with asymptotic symmetries of AdS_3 space-times in the sense of Brown–Henneaux is the direct sum of two Virasoro algebras with central charges (8.40).

The Poisson bracket algebra (8.45) can also be rewritten in terms of more conventional Virasoro generators. If we define the charges

$$\mathcal{L}_m \equiv Q_{(e^{imx^+},0)}[p, \bar{p}]\,, \qquad \bar{\mathcal{L}}_m \equiv Q_{(0,e^{imx^-})}[p, \bar{p}]\,, \tag{8.46}$$

their Poisson brackets close according to two copies of the Virasoro algebra (6.118), up to the renaming $p \to \mathcal{L}$. The central charges take the definite value $c = \bar{c} = 3\ell/2G$. In particular, the normalization of the homogeneous term of the bracket fixes the normalization of the Brown–Henneaux central charge, confirming the fact that it is an unambiguous parameter specifying the phase space. In this language the mass (8.43) and the angular momentum (8.44) are

$$M = \frac{1}{\ell}(\mathcal{L}_0 + \bar{\mathcal{L}}_0)\,, \qquad J = \mathcal{L}_0 - \bar{\mathcal{L}}_0\,,$$

as in a two-dimensional conformal field theory. In particular, pure AdS_3 has $\mathcal{L}_0 = \bar{\mathcal{L}}_0 = -c/24$, as does a CFT vacuum on the cylinder. Note that the Brown–Henneaux central charge is essentially the Planck mass measured in units of the inverse of the AdS_3 radius. Equivalently, it is the inverse of the coupling constant of the system, so the semi-classical limit corresponds to $c \to +\infty$.

Remark Brown–Henneaux boundary conditions are the "standard" boundary conditions for gravity on AdS_3 but other fall-off conditions exist as well, both in pure Einstein gravity and in modified theories of gravity. For instance, in the Einstein case, free boundary conditions [42] extend those of Brown–Henneaux by allowing the conformal factor of the metric on the boundary to fluctuate, resulting in an even larger asymptotic symmetry algebra. Many other families of boundary conditions exist, such as the chiral boundary conditions of [43] or the AdS_3 boundary conditions of topologically massive gravity [35, 44, 45] and new massive gravity [46, 47], but we will have very little to say about these alternative possibilities.

8.2.6 Zero-Mode Solutions

In order to interpret the metrics (8.38), let us study zero-mode solutions, where $p(x^+) = p_0$ and $\bar{p}(x^-) = \bar{p}_0$ are constants. In that case the only non-zero surface charges (8.42) are the Virasoro zero-modes $\mathcal{L}_0 = p_0$ and $\bar{\mathcal{L}}_0 = \bar{p}_0$.

At $p_0 = \bar{p}_0 = -c/24 \overset{(8.40)}{=} -\ell/16G$ the space-time metric is that of pure AdS_3,

$$ds^2_{\text{AdS}} = \frac{\ell^2}{r^2}dr^2 - \left(r dx^+ + \frac{\ell^2}{4r}dx^-\right)\left(r dx^- + \frac{\ell^2}{4r}dx^+\right). \tag{8.47}$$

To verify that this is indeed pure AdS_3, note that the change of coordinates

$$r = \frac{\ell}{2}e^{\text{arcsinh}(\bar{r}/\ell)} \tag{8.48}$$

brings this metric into the manifest AdS_3 form (8.16) (up to the bar on the coordinate $\bar{r}$) by virtue of the identity

$$\frac{dr}{r} = \frac{d\bar{r}}{\sqrt{\ell^2 + \bar{r}^2}}\,.$$

The angular momentum vanishes while the ADM mass of the solution is

$$M_{\text{vac}} = \frac{1}{\ell}(\mathcal{L}_0 + \bar{\mathcal{L}}_0) = -\frac{c}{12\ell} = -\frac{1}{8G}\,. \tag{8.49}$$

This is the energy of the vacuum state of a two-dimensional CFT on the cylinder.

Recall that Brown–Henneaux boundary conditions were designed so as to include conical deficits. One can show that the zero-mode solution specified by

$$\mathcal{L}_0 = p_0 = -\frac{\ell}{16G}\left(2\omega - \frac{A}{\ell}\right)^2, \qquad \bar{\mathcal{L}}_0 = \bar{p}_0 = -\frac{\ell}{16G}\left(2\omega + \frac{A}{\ell}\right)^2 \tag{8.50}$$

is precisely a conical deficit (8.24) written in Fefferman-Graham coordinates provided $|A|/\ell < 2\omega < 1$. In terms of Virasoro charges (8.46), conical deficits have $-\frac{c}{24} < \mathcal{L}_0, \bar{\mathcal{L}}_0 \leq 0$. The angular momentum is $J = p_0 - \bar{p}_0 = \omega A/2G$, while the ADM mass is

$$M = \frac{p_0 + \bar{p}_0}{\ell} = -\frac{1}{8G}\left(4\omega^2 + \frac{A^2}{\ell^2}\right).$$

Extreme conical deficits are solutions of this type for which either p_0 or $\bar{p}_0$ vanishes, or equivalently for which $|A| = 2\omega\ell$. Conical excesses are solutions for which $p_0, \bar{p}_0$ are of the form (8.50) with $|A| \leq 2\omega$ but $\omega > 1/2$, and the line separating deficits from excesses is a section of parabola

$$\ell M = -\frac{\ell}{8G} - \frac{2G}{\ell}J^2, \qquad |J| \leq \ell/4G \tag{8.51}$$

whose endpoints are tangent to the lines $\ell M = |J|$. The solution at $p_0 = \bar{p}_0 = 0$ is the degenerate conical deficit (8.41) that we used to normalize charges. Note that conical excesses with an angle of $2\pi n$ around the origin correspond to $\omega = n/2$; for fixed n, the set of such excesses is again a section of parabola in the $(J, \ell M)$ plane specified by

$$\ell M = -\frac{\ell}{8G}n^2 - \frac{2G}{\ell}\frac{J^2}{n^2}\,,$$

which generalizes (8.51). Note that, for vanishing angular momentum ($A = 0$), Eq. (8.50) yields $p_0 = \bar{p}_0 = -c\,\omega^2/6$ in terms of the Brown–Henneaux central charge. This is precisely the relation (7.46) between constant elliptic Virasoro coadjoint vectors and their monodromy matrix.

When p_0 and $\bar{p}_0$ are positive constants, the metric (8.38) turns out to be that of a *BTZ black hole* with mass $M = (p_0 + \bar{p}_0)/\ell$ and angular momentum $J = p_0 - \bar{p}_0$

written in Fefferman-Graham coordinates [10]:

$$ds^2_{\text{BTZ}} = \frac{\ell^2}{r^2}dr^2 - \left(r dx^+ - \frac{2G\ell}{r}(\ell M - J)dx^-\right)\left(r dx^- - \frac{2G\ell}{r}(\ell M + J)dx^+\right). \tag{8.52}$$

In that context the requirement $p_0, \bar{p}_0 \geq 0$ is interpreted as a cosmic censorship condition $|J| \leq \ell M$, which is saturated by extremal black holes. Beyond that barrier, all zero-mode metrics for which $|J| > \ell|M|$ contain closed time-like curves at arbitrarily large r.

The lightest BTZ black hole at $M = J = 0$ is the degenerate conical deficit (8.41). This is strikingly different from four-dimensional black holes: in the latter case, the lightest black hole is typically empty space, whereas in three dimensions the lightest black hole is separated from AdS$_3$ by a classical mass gap. The metrics that fill this gap are conical deficits, i.e. metrics of point particles, so one can loosely say that a particle turns into a black hole when its mass is higher than the threshold $c/24\ell$, which is essentially the Planck mass. We will encounter a similar phenomenon in flat space, though in that case black holes will be replaced by cosmological space-times.

Remark Since three-dimensional gravity has no local degrees of freedom, all solutions of Einstein's equations in three dimensions are locally isometric to AdS$_3$ and can therefore be realized as quotients of AdS$_3$. In particular, the BTZ metric (8.52) has no curvature singularity at $r = 0$, where, as everywhere else, it is locally isometric to AdS$_3$. So how can it be a black hole? The answer to this question was clarified in [13], where it was noted that the point $r = 0$ is a singularity in the causal sense even though it is a regular point in the metric sense. Note that black holes obtained as regular identifications of AdS also exist in higher dimensions [48, 49].

8.3 The Phase Space of AdS$_3$ Gravity

From a Hamiltonian perspective, a phase space is a manifold consisting of "positions and momenta" endowed with a Poisson structure, and time evolution is generated by Poisson brackets with a Hamiltonian function $\mathcal{H}$. This time evolution is in fact the one-parameter group of diffeomorphisms generated by a Hamiltonian vector field (5.12); it follows that phase space trajectories corresponding to different initial conditions never cross, so one is free to think of phase space as the set of possible initial conditions of the equations of motion. In other words, one can identify the phase space of a system with the space of solutions of its equations of motion [50]. This reformulation is at the core of the covariant approach to Hamiltonian mechanics, which is sometimes stressed by referring to the phase space as being *covariant*. (In contrast to the standard Hamiltonian approach, the covariant one treats space and time coordinates on an equal footing.)

According to this viewpoint, the space of solutions (8.38) is really the phase space of AdS$_3$ gravity with Brown–Henneaux boundary conditions. The purpose of this section is to analyse some of its properties and to relate them with holography. Thus, we interpret points of phase space as CFT stress tensors, describe and interpret their transformation law under Brown–Henneaux transformations, and derive a positive energy theorem for AdS_3 gravity. Quantization is relegated to Sect. 8.4.

8.3.1 AdS_3 Metrics as CFT_2 Stress Tensors

According to (8.39), the functions p, $\bar{p}$ that specify an on-shell metric transform under asymptotic symmetry transformations as the components of the stress tensor of a two-dimensional CFT with central charges (8.40). The corresponding surface charges (8.42) generate two Virasoro algebras (8.45). Accordingly, from now on we interpret Brown–Henneaux asymptotic symmetries as the global conformal symmetries of a two-dimensional CFT "dual" to AdS_3 gravity. One can think of that theory as living on the cylindrical boundary of AdS_3. Its central charges are (8.40) and the components of its stress tensor should be operators whose one-point functions are the functions $p(x^+)$ and $\bar{p}(x^-)$ appearing in the metric (8.38), which is in fact a general feature of the AdS/CFT correspondence [40, 51, 52]. Thus the covariant phase space of AdS_3 gravity coincides with the space of CFT stress tensors on the cylinder at fixed central charges. The finite transformation laws of these stress tensors under conformal transformations are given by the coadjoint representation of the Virasoro group, Eq. (6.114).

The Poisson structure on the phase space of AdS_3 gravity is determined by the requirement (8.10) ensuring that surface charges generate the correct transformation laws when acting on the fields of the theory. For Brown–Henneaux boundary conditions this leads to the Poisson brackets of charges (8.45), which coincides with the Kirillov–Kostant Poisson bracket (6.118). Hence we conclude:

Theorem The covariant phase space of AdS_3 gravity with Brown-Henneaux boundary conditions is a hyperplane at fixed central charges (8.40) embedded in the space of the coadjoint representation of the direct product of two Virasoro groups, and endowed with the corresponding Kirillov–Kostant Poisson structure.

A loose way to interpret this theorem is to say that AdS_3 gravity *is* group theory: the whole phase space of the system is determined by the structure of the Virasoro group, save for the fact that the value of the central charge is fixed by the coupling constant. We will encounter a similar phenomenon in the next chapter when dealing with asymptotically flat gravity. This being said, the occurrence of a Virasoro coadjoint representation should not come as too big a surprise. Indeed, it is always true that the charges associated with certain symmetries transform under the coadjoint representation of the symmetry group (since these charges are nothing but momentum maps). Accordingly, the surface charges of AdS_3 gravity were bound to involve the coadjoint representation of the Virasoro group. The only surprise is that Virasoro coadjoint vectors exactly coincide with the functions specifying the metric, instead of being some complicated non-linear combinations of the entries of the metric and their derivatives. In particular, note that the set of on-shell Brown–Henneaux metrics (8.38) is a vector space.

Remark It is not strictly true that the whole phase space of AdS_3 gravity coincides with the dual of two Virasoro algebras. Indeed, this conclusion is entirely based on the asymptotic solutions (8.38), but completely overlooks the fact that some of these solutions cannot be extended arbitrarily far into the bulk. This subtlety leads to (finitely many) additional directions in the complete phase space of the theory, as discussed in [53]. We will ignore this detail since it plays a minor role for our purposes.

8.3.2 Boundary Gravitons and Virasoro Orbits

The covariant phase space (8.38) is spanned by pairs of functions $\big(p(x^+), \bar{p}(x^-)\big)$. The zero-mode solutions were described in Sect. 8.2.6, but a generic par $(p, \bar{p})$ is definitely *not* a zero-mode since p and $\bar{p}$ may have some non-trivial profile on the circle. If we pick one such solution at random, we can generate infinitely many other ones by acting on it with asymptotic symmetry transformations. The resulting manifold is the product of two Virasoro coadjoint orbits at central charges (8.40),

$$\mathcal{W}_{(p,c)} \times \mathcal{W}_{(\bar{p},\bar{c})} \,. \tag{8.53}$$

If we think of the asymptotic symmetry group as a generalization of the space-time isometry group O(2, 2), and of $(p, \bar{p})$ as an infinite-dimensional generalization of space-time momentum, then the orbit (8.53) is an infinite-dimensional generalization of the standard orbits of momenta under, say, Lorentz transformations. In particular the metrics spanning the orbit (8.53) should be seen as boosts of the metric $(p, \bar{p})$.

This is a good point to introduce a terminology which has come to be more or less standard [14, 54, 55]: a metric $(p, \bar{p})$ obtained by acting on pure AdS_3 with a certain asymptotic symmetry transformation is known as a (classical) *boundary graviton*. This nomenclature is then extended to any metric obtained from a zero-mode solution by an asymptotic symmetry transformation. The name is justified by the fact that three-dimensional gravity has no local (bulk) degrees of freedom, but does have non-trivial topological (boundary) degrees of freedom visible in the arbitrariness of the pair $(p, \bar{p})$ that specifies a solution of the equations of motion.

If our goal is to classify all solutions (8.38), then orbits provides a natural organizing criterion: rather than classifying the solutions, we can classify their orbits under asymptotic symmetries. Since we know the classification of Virasoro coadjoint orbits, we may claim to control the full covariant phase space of AdS_3 gravity. In particular, the classification of zero-mode solutions in Fig. 8.5 is a first step towards the full classification: each point in the plane $(J, \ell M)$ defines the orbit of the corresponding zero-mode solution, and different points define distinct orbits. However, as we have seen in Sect. 7.2, not all orbits have constant representatives: there exist

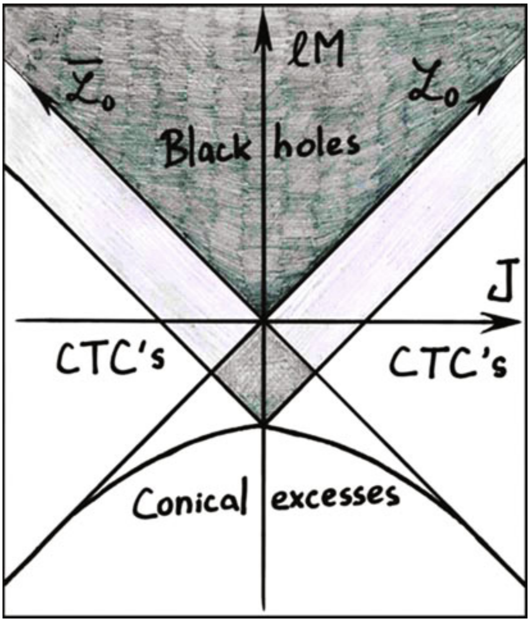

Fig. 8.5 The zero-mode solutions of AdS_3 gravity with Brown–Henneaux boundary conditions. The origin of the coordinate system $(J, \ell M)$ is the degenerate conical deficit (8.41); the AdS_3 metric is located below, on the ℓM axis, at the lower tip of the *shaded square*. BTZ black holes are located in the wedge $|J| \leq \ell M$. Conical deficits and excesses are located in the lower wedge $|J| \leq -\ell M$, respectively above and below the parabola (8.51). All metrics such that $|J| > \ell |M|$ contain closed time-like curves at arbitrarily large radius. Anticipating the results of Sect. 8.3.3, we have shaded the solutions whose orbit has energy bounded from below under Brown–Henneaux transformations; those are all BTZ black holes, the AdS_3 metric, and all conical deficits such that $p_0, \bar{p}_0 \geq -c/24$. Certain solutions with energy bounded from below are pathological in that they contain closed time-like curves at infinity — those are the two diagonal strips surrounding the BTZ wedge

infinitely many conformally inequivalent on-shell metrics that cannot be brought into zero-mode solutions by asymptotic symmetry transformations. Thus the complete classification of AdS_3 metrics is essentially a product of two copies of Fig. 7.3, where zero-mode solutions are those where both p and $\bar{p}$ belong to the vertical axis of the figure. This classification foliates the covariant phase space of AdS_3 gravity into disjoint orbits of the asymptotic symmetry group (Fig. 8.6).

Remark The relation between AdS_3 gravity and orbits of the Virasoro group has recently been the object of renewed interest, as it was realized that a similar structure arises in many other contexts. To the author's knowledge, the first explicit mention of that relation appears in [17, 18, 56]; it is also hidden between the lines in [19, 57]. The relation was later studied in [14, 15] due to its implications for positive energy theorems, while [58] (see also [59]) is devoted to the geometric properties of metrics corresponding to non-constant pairs $(p, \bar{p})$.

Fig. 8.6 A schematic representation of the AdS$_3$ phase space foliated into orbits of the asymptotic symmetry group. Solutions belonging to the same symplectic leaf are related to one another by asymptotic symmetry transformations, i.e. "boosts", but there are no boosts that connect different leaves

8.3.3 Positive Energy Theorems

As in Sect. 7.3 one may ask which solutions of AdS$_3$ gravity belong to Virasoro orbits on which the energy functional (8.43) is bounded from below. These solutions can then be considered as "physical", in contrast to the pathological solutions whose energy can be made arbitrarily low by suitable asymptotic symmetry transformations. Since the transformation law of the components $(p, \bar{p})$ is the coadjoint representation of the Virasoro group, the results of Sect. 7.3 are directly applicable to the problem at hand. Thus the only solutions $(p, \bar{p})$ with energy bounded from below are those in which both p and $\bar{p}$ belong to one of the orbits highlighted in red in Fig. 7.7. More explicitly, zero-mode solutions $(p_0, \bar{p}_0)$ belong to orbits with energy bounded from below if and only if both p_0 and $\bar{p}_0$ are larger than (or equal to) the vacuum value $-c/24$. For solutions that do not admit a rest frame, either p or $\bar{p}$ (or both) must belong to the unique massless orbit with energy bounded from below.

These arguments can be interpreted as a positive energy theorem for AdS$_3$ gravity [14, 15]. They imply in particular that, in the diagram of zero-mode solutions of Fig. 8.5, all BTZ black holes belong to orbits with energy bounded from below, while all conical excesses belong to orbits with unbounded energy. The pure AdS$_3$ metric also belongs to an orbit with energy bounded from below, while the only conical deficits whose energy is bounded from below under asymptotic symmetries are those located in the square $-c/24 \geq p_0, \bar{p}_0 \geq 0$. The absolute minimum of energy among all solutions with energy bounded from below is realized by AdS$_3$ space-time.

Remark Positive energy theorems in general relativity have a long history; in short, the problem is to show that energy is bounded from below in a suitably defined phase space of metrics. This problem is classically addressed in four-dimensional asymptotically flat space-times, where positivity of energy was first proved in [60]. A supersymmetry-based proof can also be found in [61], while the case of the Bondi mass was settled shortly thereafter by various authors — see e.g. [62] for a list of references. Note that a relation between positive energy theorems [63] and Virasoro orbits was already suggested in footnote 8 of [64], albeit in a very different context. In Sect. 9.3 we will encounter a positive-energy theorem in three-dimensional asymptotically flat space-times.

8.4 Quantization and Virasoro Representations

Recall from Sect. 5.2 that the quantization of coadjoint orbits yields unitary group representations. Assuming that this applies to the Virasoro group, unitary representations can be interpreted as quantized orbits of Brown–Henneaux metrics under asymptotic symmetry transformations. Accordingly, we now investigate the relation between unitary representations of the Virasoro algebra and AdS$_3$ quantum gravity. We start with an overview of $\mathfrak{sl}(2,\mathbb{R})$ and Virasoro highest-weight representations, which we interpret as particles dressed with gravitational degrees of freedom in AdS$_3$. We then conclude with the observation that Virasoro characters coincide with quantum gravity partition functions in AdS$_3$.

Remark This section is our first encounter with representations of Lie *algebras* in this thesis, so our language and notations will be somewhat different from those of part I. The link between the language of part I and Lie algebra representations will be established through induced modules, in Sect. 10.2.

8.4.1 Highest-Weight Representations of $\mathfrak{sl}(2,\mathbb{R})$

Here we describe highest-weight unitary representations of the Lie algebra $\mathfrak{sl}(2,\mathbb{R})$, which will be useful guides for studying Virasoro representations. From a space-time perspective, $\mathfrak{sl}(2,\mathbb{R})$ is half of the isometry algebra of AdS$_3$, so tensor products of highest-weight representations of $\mathfrak{sl}(2,\mathbb{R})$ are particles propagating in AdS$_3$. Their Minkowskian analogue are the Poincaré representations of Sect. 4.3.

Highest Weights and Descendants

The Lie algebra $\mathfrak{sl}(2,\mathbb{R})$ of $\mathrm{SL}(2,\mathbb{R})$ consists of traceless real 2×2 matrices; its basis can be chosen as in (4.87), but for the purpose of describing unitary representations it is convenient to use the complex basis

$$L_0 \equiv -it_0\,, \qquad L_1 \equiv t_1 + it_2\,, \qquad L_{-1} \equiv -t_1 + it_2\,. \tag{8.54}$$

Equivalently, $L_m = i\ell_m$ in terms of the basis (5.89) and the brackets (5.90) become

$$[L_m, L_n] = (m-n)L_{m+n} \tag{8.55}$$

with $m, n = -1, 0, 1$. In any unitary representation $\mathscr{T}$ of $\mathfrak{sl}(2, \mathbb{R})$, the real generators t_μ are represented by anti-Hermitian operators acting in a suitable Hilbert space. In terms of the generators (8.54), this is to say that the Hermiticity conditions

$$\mathscr{T}[L_m]^\dagger = \mathscr{T}[L_{-m}] \tag{8.56}$$

hold in a unitary representation. From now on we will abuse notation and neglect writing the representation $\mathscr{T}$, so that $\mathscr{T}[L_m] \equiv L_m$. This abuse is common in physics, so it should not lead to any misunderstanding. Until the end of this chapter we also use the Dirac notation instead of the less standard notation of part I.

Since the group $\mathrm{SL}(2, \mathbb{R})$ is simple but non-compact, all its non-trivial unitary representations are infinite-dimensional. Fortunately, the complexification of $\mathfrak{sl}(2, \mathbb{R})$ coincides with that of $\mathfrak{su}(2)$, and we definitely know how to build unitary highest-weight representations of the latter. Let us therefore use the same approach for $\mathfrak{sl}(2, \mathbb{R})$: we start from a (normalized) *highest-weight state* $|h\rangle$ belonging to the Hilbert space of the representation, such that

$$L_0|h\rangle = h|h\rangle, \qquad L_1|h\rangle = 0\,, \tag{8.57}$$

where h is the *highest weight*. The conditions (8.56) imply that the operator representing L_0 is Hermitian, so h must be real. The interpretation of (8.57) is that $|h\rangle$ has energy h if we think of L_0 as the Hamiltonian, while L_1 is an annihilation operator.[6]

In order to produce a representation we must also act with operators L_{-1} on the highest-weight state. This leads to *descendant states* of the form

$$(L_{-1})^N|h\rangle, \tag{8.58}$$

where the non-negative integer N is the *level* of the descendant. Each descendant state is an eigenstate of L_0 with eigenvalue $h + N$, so L_{-1} is analogous to a creation operator. We then declare that the carrier space $\mathscr{H}$ of the representation is spanned by all linear combinations of descendant states. Since we want $\mathscr{H}$ to be a Hilbert space, descendant states with different levels must be orthogonal because their eigenvalues under L_0 differ. Furthermore, all descendant states must have non-negative norm squared. Using the Hermiticity conditions (8.56), this amounts to the requirement

$$0 \leq \left\|(L_{-1})^N|h\rangle\right\|^2 = \left\langle h\middle|[(L_{-1})^N]^\dagger(L_{-1})^N\middle|h\right\rangle \overset{(8.56)}{=} \left\langle h\middle|(L_1)^N(L_{-1})^N\middle|h\right\rangle.$$

To ensure that this condition holds, we evaluate scalar products of descendant states. Thanks to the commutation relations (8.55) one finds

[6]The terminology is somewhat backwards, since the highest weight h is actually the *lowest* eigenvalue of L_0 in the space of the representation; this terminological clash is standard.

$$\left\langle h\middle|(L_1)^N(L_{-1})^N\middle|h\right\rangle = N!\prod_{k=0}^{N-1}(2h+k) \tag{8.59}$$

where we have used $\langle h|h\rangle = 1$. Thus, all descendant states have strictly positive norm squared if and only if $h > 0$. If $h = 0$, then the representation is trivial.

Note that here the only restriction imposed on h by unitarity is $h \geq 0$. However, if we were to integrate a highest-weight representation of $\mathfrak{sl}(2,\mathbb{R})$ into an *exact* unitary representation of the group $SL(2,\mathbb{R})$, then h would have to be an integer in order to ensure that a rotation by 2π is represented by the identity. On the other hand, *projective* representations of $SL(2,\mathbb{R})$ allow h to be an arbitrary positive real number since the fundamental group of $SL(2,\mathbb{R})$ is isomorphic to $\mathbb{Z}$.

Representations by Quantization

The representations just described can be identified with representations obtained by quantizing suitable coadjoint orbits of $SL(2,\mathbb{R})$. Indeed, note that the Lie algebra (8.55) admits a quadratic Casimir operator

$$\mathcal{C} = L_0^2 - \frac{1}{2}(L_1L_{-1} + L_{-1}L_1) \overset{(8.54)}{=} -t_0^2 + t_1^2 + t_2^2\,. \tag{8.60}$$

This operator is proportional to the identity in any irreducible representation of $\mathfrak{sl}(2,\mathbb{R})$. In the highest-weight representation (8.58) it takes the value

$$\mathcal{C} = h(h-1), \tag{8.61}$$

which proves by the way that the representation is irreducible. The fact that (8.60) takes a constant value is reminiscent of the "mass shell" condition defining the coadjoint orbit (5.93), and indeed the highest-weight representation just displayed is the quantization of such an orbit for $h > 0$. The only subtlety is that the value of the Casimir (8.61) is not quite h^2, but $h(h-1)$; the two numbers coincide in the semi-classical limit $h \to +\infty$, and the $h-1$ of (8.61) may be seen as a quantum correction of the classical result.

This identification is confirmed by the computation of $SL(2,\mathbb{R})$ characters. Indeed, the counting argument of the end of Sect. 5.3 can now be applied to the space $\mathscr{H}$ spanned by the highest-weight state $|h\rangle$ and its descendants. As a result one finds that $\mathrm{Tr}(q^{L_0})$ is precisely given by formula (5.102) up to the replacement of $h+1/2$ by h. From a space-time perspective, the product of two such characters is the character of an irreducible representation of the isometry algebra $\mathfrak{so}(2,2)$ of AdS_3. If the two representations have weights h and $\bar{h}$ say, the product of their characters can be interpreted as the partition function of a particle with mass $(h+\bar{h})/\ell$ and spin $h-\bar{h}$ propagating in AdS_3.

8.4.2 *Virasoro Modules*

We now generalize the representation theory of $\mathfrak{sl}(2,\mathbb{R})$ to the Virasoro algebra. Before doing so, a word of caution is in order: in the case of $\mathrm{SL}(2,\mathbb{R})$, we were able to interpret highest-weight representations as quantized coadjoint orbits of the type (5.93); this identification was supported by the matching of the Casimir operator (8.61) with the definition of the orbit. In the case of the Virasoro algebra the situation is complicated by the fact that all coadjoint orbits (at non-zero central charge) are infinite-dimensional; in addition, the only Casimir operators of the Virasoro algebra are functions of its central charge [65]. Accordingly, we start with a few comments regarding Virasoro geometric quantization, before turning to the construction of its highest-weight representations and the evaluation of the associated characters.

Semi-classical Regime

If one believes in the orbit method, geometric quantization applied to the coadjoint orbits of the Virasoro algebra should produce unitary Virasoro representations. This viewpoint was adopted in [64, 66, 67], with the conclusion that the quantization of orbits with positive energy and constant representatives indeed provides highest-weight representations in the large c limit. By contrast, the limit of small c is much more elusive, and at present it is not known if the discrete series of Virasoro representations at $c \leq 1$ can be obtained by geometric quantization (see e.g. [68]). From a gravitational point of view, the Virasoro central charge (8.40) is the AdS radius in units of the Planck length, so large c corresponds to the semi-classical regime. This is confirmed by symplectic geometry: the Kirillov–Kostant symplectic form (5.29) evaluated at a constant Virasoro coadjoint vector (p_0, c) is

$$\omega_{(p_0,c)}\big((\xi_m)_{p_0}, (\xi_n)_{p_0}\big) \overset{(6.111)}{=} -im\left(2p_0 + \frac{c}{12}m^2\right)\delta_{m+n,0} \tag{8.62}$$

where $\xi_m = \mathrm{ad}^*_{\mathcal{L}_m}$ is the vector field on $\mathcal{W}_{(p_0,c)}$ that generates the coadjoint action of the Virasoro generator $\mathcal{L}_m$ given by (6.109). The occurrence of c confirms that the regime of large c is semi-classical in the sense that a large volume is assigned to any portion of phase space. Conversely, small c corresponds to the non-perturbative regime, where quantum corrections may alter classical results in a radical way.

This heuristic argument is consistent with the fact that geometric quantization is relatively well established at large c, but poorly understood at small c. Since applications to three-dimensional gravity rely on the semi-classical limit anyway, from now on we restrict ourselves to the regime of large c. This assumption turns out to greatly simplify representation theory, and allows us to think of highest-weight Virasoro representations as quantizations of Virasoro orbits with constant representatives.

This being said, to our knowledge there is as yet no strict mathematical proof of the fact that geometric quantization of Virasoro orbits produces highest-weight representations, despite numerous attempts in the literature (see e.g. [69, 70]). Our viewpoint here will be pragmatic, and we shall assume that the representations obtained by quantizing such orbits are indeed highest-weight representations. This assumption will be supported, among other observations, by the fact that Virasoro characters match suitable gravitational partition functions (see Sect. 8.4.4).

Highest-Weight Representations

The basis of the Virasoro algebra given by (6.109) is such that, in any unitary representation, the operators representing the generators $\mathcal{L}_m + \mathcal{L}_{-m}$, $i(\mathcal{L}_m - \mathcal{L}_{-m})$ and $\mathcal{Z}$ are *anti*-Hermitian. A more standard basis is given by

$$L_m \equiv i\mathcal{L}_m + i\frac{\mathcal{Z}}{12}\delta_{m,0}, \qquad Z \equiv i\mathcal{Z}, \tag{8.63}$$

where the constant shift in L_0 ensures that the vacuum state has zero eigenvalue under L_0. According to this definition the operators representing L_m and Z in a unitary representation satisfy the Hermiticity conditions

$$L_m^\dagger = L_{-m}, \qquad Z^\dagger = Z \tag{8.64}$$

where we abuse notation by denoting the basis element L_m and the operator that represents it with the same symbol. In any irreducible representation the Hermitian central operator Z is proportional to the identity with a coefficient $c \in \mathbb{R}$, so we may write the commutation relations of the operators representing the generators (8.63) as

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m+n,0}\,. \tag{8.65}$$

Here the existence of the $\mathfrak{sl}(2,\mathbb{R})$ subalgebra (8.55) is manifest, since the contribution of the central extension vanishes for $m = -1, 0, 1$.

Highest-weight representations of the Virasoro algebra (8.65) seem to have first appeared in [71–73], and were then further studied in [74–76]. They are built in direct analogy to the highest-weight representations of $\mathfrak{sl}(2,\mathbb{R})$. In accordance with geometric quantization, the parameters that specify these representations coincide with those that determine the corresponding coadjoint orbits. In the present case, taking the orbit of a constant coadjoint vector (p_0, c), one defines a real number h by

$$p_0 = h - \frac{c}{24}\,. \tag{8.66}$$

This ensures that $h = 0$ for the vacuum configuration, while $h \geq 0$ for orbits with energy bounded from below. With this notation, the representation obtained by quantizing the orbit of (p_0, c) is obtained as follows.

To begin, in analogy with (8.57), one defines the *highest-weight state* of the representation to be a normalized state $|h\rangle$ such that[7]

$$L_0|h\rangle = h|h\rangle, \qquad L_m|h\rangle = 0 \quad \text{for } m > 0. \tag{8.67}$$

The state $|h\rangle$ is also called a *primary state*. Its definition ensures that it has energy h under the Hamiltonian L_0, while the operators L_m with $m > 0$ are annihilation operators. In analogy with (8.58) one also defines *descendant states*

$$L_{-k_1} \dots L_{-k_n}|h\rangle, \qquad 1 \leq k_1 \leq k_2 \leq \dots \leq k_n\,. \tag{8.68}$$

Thus one can interpret the operators L_{-m} with $m > 0$ as creation operators. We will discuss the gravitational interpretation of this representation in Sect. 8.4.4.

Using the commutation relations (8.65) of the Virasoro algebra, one verifies that each descendant (8.68) is an eigenstate of L_0 with eigenvalue

$$h + \sum_{i=1}^{n} k_i \equiv h + N$$

where the non-negative integer N is the *level* of the descendant. One then declares that the space $\mathscr{H}$ of the Virasoro representation is the *Verma module* spanned by all linear combinations of descendant states. As in the case of $\mathfrak{sl}(2,\mathbb{R})$, descendants with different levels have different eigenvalues under L_0. According to (8.64) the latter must be Hermitian if the representation is to be unitary, so scalar products of descendants with different levels vanish. However, in contrast to $\mathfrak{sl}(2,\mathbb{R})$, there are in general many different descendant states with the same level. More precisely, at large central charge c, the number of different descendants at level N is the number $p(N)$ of partitions of N in distinct positive integers (e.g. $p(0) = p(1) = 1$, $p(2) = 2$, $p(3) = 3$, $p(4) = 5$, etc.).

Note that the representation whose carrier space is spanned by the descendant states (8.68) is an induced representation of the Virasoro algebra, i.e. an *induced module*. Indeed, the conditions (8.67) define a one-dimensional representation of the subalgebra generated by L_0 and the L_m's with $m > 0$, and the prescription (8.68) is the algebraic analogue of the statement that wavefunctions live on a quotient space G/H (recall Sect. 3.3). By the way, a similar interpretation holds for the highest-weight representations of $\mathfrak{sl}(2,\mathbb{R})$ displayed in Sect. 8.4.1. We will return to this observation in Sect. 10.2.

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[7]The actual value of the quantum weight h may differ from the classical parameter defined in (8.66) by corrections of order $\mathcal{O}(1/c)$, so from now on it is understood that h refers to the *quantum* value. This subtlety will have very little effect on our discussion.

Unitarity for Virasoro Representations

We now ask whether the vector space spanned by $|h\rangle$ and its descendants is a Hilbert space, given that Hermitian conjugation is defined by (8.64). Working only with low-level descendant states, one can easily derive some basic necessary conditions for unitarity. For instance, the only descendant at level one is $L_{-1}|h\rangle$ and its norm squared is $\langle h|L_1 L_{-1}|h\rangle = 2h$, so a necessary condition for unitarity is $h \geq 0$. Similarly, at level N there is a state $L_{-N}|h\rangle$ with norm squared

$$\langle h|L_N L_{-N}|h\rangle \overset{(8.65)}{=} 2Nh + \frac{c}{12}N(N^2-1)$$

whose positivity for large N requires $c \geq 0$. Thus, the only values of c and h that give rise to unitary representations are positive. In terms of coadjoint orbits of the Virasoro group, this is to say that only orbits with positive energy can produce unitary representations under quantization. In order to go further one generally relies on the so-called *Gram matrix* of the module, whose entries are the scalar products of descendants. Demanding unitarity then boils down to the requirement that the Gram matrix be positive-definite. One can show that this condition is always verified by Virasoro highest-weight representations at large c. Since this is a standard result in two-dimensional conformal field theory (see e.g. [77]), we simply state it here without proof:

Proposition Highest-weight representations of the Virasoro algebra with $c > 1$ and $h > 0$ are unitary and irreducible in the sense that all descendant states have strictly positive norm squared.

Note that, by contrast, unitary highest-weight representations at central charge $c \leq 1$ have a very intricate structure due to null states, i.e. states with vanishing norm that are modded out of the Hilbert space as in the definition (3.5) of L^2 spaces. In that case, not all descendant states (8.68) are linearly independent, since some of them are effectively set to zero — in this sense the Verma module is reducible. We will not take such subtleties into account here because we are interested only in the limit of large central charge, where null states are absent.

Remark It was recently shown [78] that all irreducible unitary representations of the Virasoro group with a spectrum of L_0 bounded from below are highest-weight representations of the type described here. In this sense, highest-weight representations exhaust all unitary representations of the Virasoro algebra.

Vacuum Representation

In $\mathfrak{sl}(2,\mathbb{R})$, the representation with highest weight $h = 0$ is trivial since all descendant states are null by virtue of Eq. (8.59). We now describe the *Virasoro* representation obtained by setting $h = 0$, to which we refer as the *vacuum representation*. It is obtained by quantizing the vacuum Virasoro orbit, containing the point $p_{\text{vac}} = -c/24$.

The highest weight state $|0\rangle$ of that representation satisfies the properties

$$L_0|0\rangle = L_{-1}|0\rangle = L_m|0\rangle = 0 \quad \text{for all } m > 0, \tag{8.69}$$

which ensures that the vacuum state $|0\rangle$ is invariant under the $\mathfrak{sl}(2, \mathbb{R})$ subalgebra generated by L_{-1}, L_0 and L_1; it is the quantum counterpart of the statement that the stabilizer of the vacuum orbit is the group PSL$(2, \mathbb{R})$. Crucially, the vacuum is *not* invariant under the higher-mode generators L_{-2}, L_{-3} etc. so that the representation whose carrier space is spanned by all descendant states

$$L_{-k_1} \dots L_{-k_n}|0\rangle, \qquad 2 \leq k_1 \leq \dots \leq k_n \tag{8.70}$$

is non-trivial. This representation is unitary for all $c > 0$, and it is free of null states (i.e. irreducible) whenever $c > 1$. It is in many ways analogous to the standard highest-weight representation generated by the descendant states (8.68), but the condition $L_1|0\rangle = 0$ makes it slightly smaller than generic highest-weight representations.

It may seem puzzling that the vacuum state is *not* left invariant by all Virasoro generators but only by a subset thereof as in (8.69). The reason is that, at non-zero central charge, it is impossible to define a non-zero state that is annihilated by all Virasoro generators. Indeed, if there was such a state $|\tilde{0}\rangle$, then we would have

$$0 = \langle\tilde{0}|L_n L_{-n}|\tilde{0}\rangle \overset{(8.65)}{=} \langle\tilde{0}| \left(2nL_0 + \frac{c}{12}n(n^2-1)\right)|\tilde{0}\rangle = \frac{c}{12}n(n^2-1)\left\langle\tilde{0}|\tilde{0}\right\rangle,$$

which is a contradiction when $c \neq 0$ and $n \neq -1, 0, 1$. Hence the vacuum state of a Virasoro-invariant theory is always $\mathfrak{sl}(2, \mathbb{R})$-invariant but *never* Virasoro-invariant. One should appreciate the counter-intuitive nature of this phenomenon: it means that even the absolute simplest Virasoro-invariant quantum theory is described by an infinite-dimensional Hilbert space of vacuum descendants. In Sect. 10.1 we will encounter a similar non-trivial vacuum representation for the BMS_3 group.

8.4.3 Virasoro Characters

The relation between AdS_3 quantum gravity and Virasoro representations is most easily expressed in terms of partition functions, so as a preliminary we now evaluate characters of highest-weight representations of the Virasoro algebra at $c > 1$.

Let $\mathscr{H}$ be the Hilbert space of an irreducible, unitary representations of the Virasoro algebra with central charge c and highest-weight h (if $h = 0$ we take the vacuum representation). We define the *character* of the representation as

$$\chi(\tau) \equiv \text{Tr}_{\mathscr{H}}\left[q^{L_0 - c/24}\right], \qquad q \equiv e^{2\pi i\tau} \tag{8.71}$$

where the notation is almost the same as in Eq. (5.102). The parameter τ is a complex number with positive imaginary part so that, if L_0 is interpreted as the Hamiltonian, then $\text{Im}(\tau)$ is the inverse temperature and the character itself is a canonical partition function. The normalization factor $-c/24$ is conventional.

Generic Highest-Weight Representations

Let $h > 0$ be a highest weight and $c > 1$ a large central charge. Consider the Verma module spanned by all descendant states (8.68). Then there are no null states and distinct descendants are linearly independent, so the spectrum of L_0 consists of all values $h + N$, with multiplicity $p(N)$. Accordingly, the character (8.71) reads

$$\chi(\tau) = \sum_{N=0} p(N) q^{h+N-c/24} = q^{h-c/24} \sum_{N=0}^{+\infty} p(N) q^N . \tag{8.72}$$

We now rewrite this in a more convenient way thanks to the following result:

Lemma The series (8.72) can be rewritten as an infinite product

$$\sum_{N=0}^{+\infty} p(N) q^N = \prod_{n=1}^{+\infty} \frac{1}{1-q^n} . \tag{8.73}$$

Proof We follow [79], to which we refer for a careful treatment of the convergence issues that will not be addressed here. To prove (8.73) we consider its right-hand side and expand each individual term of the infinite product as a geometric series:

$$\begin{aligned}\prod_{n=1}^{+\infty} \frac{1}{1-q^n} &= \prod_{n=1}^{+\infty} \sum_{k=0}^{+\infty} q^{nk} = \left(1 + q + q^2 + q^3 + \cdots\right)\left(1 + q^2 + q^4 + \cdots\right)\cdots \\ &= 1 + \sum_{N=1}^{+\infty} \sum_{n=1}^{N} \underbrace{\sum_{\substack{k_1, k_2, \ldots, k_n \\ k_1 + k_2 + \cdots + k_n = N \\ 1 \le k_1 \le k_2 \le \cdots \le k_n}}}_{p(N)} \underbrace{q^{k_1 + k_2 + \cdots + k_n}}_{q^N} = \sum_{N=0}^{+\infty} p(N) q^N .\end{aligned}$$

This concludes the argument. ■

Thus the character (8.72) can be rewritten as

$$\chi(\tau) = q^{h-c/24} \prod_{n=1}^{+\infty} \frac{1}{1-q^n} = \frac{q^{h-(c-1)/24}}{\eta(\tau)} \tag{8.74}$$

where in the second equality we have introduced the *Dedekind eta function*

$$\eta(\tau) \equiv q^{1/24} \prod_{n=1}^{+\infty} (1 - q^n). \tag{8.75}$$

The result (8.74) can be seen as an infinite product of $\mathfrak{sl}(2, \mathbb{R})$ characters (5.102) with parameters $n\tau$. Equivalently, since $\mathfrak{sl}(2, \mathbb{R})$ representations coincide with harmonic oscillators whose partition functions were written in (5.102), one can think of a Virasoro representation as an infinite collection of harmonic oscillators. This suggests that Virasoro characters can be interpreted as quantum field theory partition functions, which will be confirmed in Sect. 8.4.4 below. Note also the presence of the ubiquitous factor $1 - q^n$ in the denominator.

Remark Our derivation of (8.74) relied on the fact that the eigenvalue $h + N$ of L_0 has degeneracy $p(N)$. This is only true provided there are no null states, i.e. provided $c > 1$. By contrast, for $c \leq 1$, null states generally do exist and are modded out of the Hilbert space of the representation. This leads to a smaller degeneracy of eigenvalues of L_0, hence to a character that is strikingly different from (8.74). We will not display characters at $c \leq 1$ here; see e.g. [80, 81] for explicit formulas.

Vacuum Representation

The character of the vacuum Virasoro representation can be evaluated in the same way as for generic highest-weight representations. The only subtlety is that the vacuum is $\mathfrak{sl}(2, \mathbb{R})$-invariant, leading to a reduced number of descendant states (8.70). Explicitly, let $\Delta(N)$ denote the degeneracy of the eigenvalue N in the space spanned by the vacuum descendants. For $N \geq 2$, this degeneracy is the number of partitions of N in positive integers which are strictly greater than one (thus $\Delta(0) = 1$ by convention but $\Delta(1) = 0$, $\Delta(2) = \Delta(3) = \Delta(4) = \Delta(5) = 1$, $\Delta(6) = 2$, etc.). Then the vacuum character is

$$\chi(\tau) = \sum_{N=0}^{+\infty} \Delta(N) q^{N - c/24} = q^{-c/24} \sum_{N=0}^{+\infty} \Delta(N) q^N. \tag{8.76}$$

In order to relate $\Delta(N)$ to the usual partition $p(N)$, we note that

$$\Delta(N) = p(N) - \begin{pmatrix} \text{number of partitions of } N \\ \text{containing at least one "1"} \end{pmatrix} = p(N) - p(N-1)$$

which allows us to rewrite the character (8.76) as

$$\chi_{\text{vac},c}(\tau) = q^{-c/24} \sum_{N=0}^{+\infty} p(N) q^N (1 - q) \overset{(8.73)}{=} q^{-c/24} \prod_{n=2}^{+\infty} \frac{1}{1 - q^n}. \tag{8.77}$$

Note how the product in the denominator of (8.77) is truncated (no term $n = 1$) owing to the $\mathfrak{sl}(2, \mathbb{R})$ symmetry of the vacuum state. In particular the vacuum character (8.77) is *not* just the limit $h \to 0$ of the generic character (8.74).

8.4.4 Dressed Particles and Quantization

Since AdS_3 gravity has Virasoro symmetry, its quantization is expected to produce unitary representations of the direct sum of two Virasoro algebras. We now investigate to what extent this is the case. For notational simplicity we denote the Virasoro algebra by $\mathfrak{vir}$ (instead of $\widehat{\text{Vect}}(S^1)$), so that the asymptotic symmetry algebra of AdS_3 gravity with Brown–Henneaux boundary conditions is $\mathfrak{vir} \oplus \mathfrak{vir}$.

First let us make the proposal more precise: the orbit of a metric $(p, \bar{p})$ is a coadjoint orbit (8.53) of the product of two Virasoro groups with central charges (8.40). For simplicity let us assume that the metric is a zero-mode and that its energy is bounded from below under asymptotic symmetry transformations, so $p(x^+) = p_0 \geq -c/24$ and $\bar{p}(x^-) = \bar{p}_0 \geq -\bar{c}/24$. The non-zero modes belonging to the orbit (8.53) may be seen as classical analogues of the descendant states (8.68). Upon defining $h \equiv p_0 + c/4$ and $\bar{h} \equiv \bar{p}_0 + \bar{c}/24$, one expects that the geometric quantization of the orbit (8.53) produces the tensor product of two highest-weight representations of the Virasoro algebra labelled by (h, c) and $(\bar{h}, \bar{c})$.[8] The same would be true by quantizing the orbit of AdS_3, except that the result would be the tensor product of two vacuum representations.

It is worth comparing these representations to those of $\mathfrak{so}(2, 2) \cong \mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{sl}(2, \mathbb{R})$, the isometry algebra of AdS_3. Since the latter is a subalgebra of $\mathfrak{vir} \oplus \mathfrak{vir}$, any Virasoro representation with highest weights $(h, \bar{h})$ contains many $\mathfrak{so}(2, 2)$ subrepresentations with weights increasing from $(h, \bar{h})$ to infinity. Thus a Virasoro representation is an $\mathfrak{so}(2, 2)$ representation dressed with (infinitely many) extra directions in the Hilbert space obtained by acting with the Virasoro generators L_{-2}, L_{-3}, etc.

Now recall that a particle propagating in AdS_3 is an irreducible unitary representation of $\mathfrak{so}(2, 2)$, while asymptotic symmetries generalize isometries by including gravitational fluctuations. Accordingly one is led to interpret Virasoro representations as particles in AdS_3 dressed with some extra gravitational degrees of freedom accounted for by the modes L_n that do not appear in the isometry algebra. These are quantum analogues of the classical "boundary gravitons" described in Sect. 8.3.2. Thus a Virasoro representation is a particle in AdS_3 dressed with boundary gravitons. In particular the vacuum representation of $\mathfrak{vir} \oplus \mathfrak{vir}$ is identified with the Hilbert space of quantum boundary gravitons around pure AdS_3.

[8] As before, the quantum values of $(h, \bar{h})$ may differ from their classical counterparts by $1/c$ corrections.

As a verification of the fact that Virasoro symmetry is realized in AdS_3 quantum gravity, one may wonder whether the quantum partition function of gravity reproduces a (combination of) Virasoro character(s). This computation was carried out in [82], where the authors evaluated the one-loop partition function of gravity on AdS_3 at finite temperature $1/\beta$ and angular potential θ (both taken to be real). Upon combining these numbers into a modular parameter

$$\tau \equiv \frac{1}{2\pi}\left(\theta + i\frac{\beta}{\ell}\right), \tag{8.78}$$

it was found that the gravitational one-loop partition function reads

$$Z_{\text{grav}}(\beta, \theta) = \text{Tr}\left(q^{L_0 - c/24}\bar{q}^{\bar{L}_0 - c/24}\right) = |q|^{c/12}\prod_{n=2}^{+\infty}\frac{1}{|1-q^n|^2} \tag{8.79}$$

where $q \equiv e^{2\pi i\tau}$ and $\bar{q}$ is its complex conjugate. This is precisely the character of the tensor product of two Virasoro vacuum representations, which confirms that quantized boundary gravitons around AdS_3 span an irreducible highest-weight representation of $\mathfrak{vir} \oplus \mathfrak{vir}$. The same computation can be performed for orbifolds of AdS_3 obtained by imposing identifications of the form $\varphi \sim \varphi + 2\pi/N$ in terms of cylindrical coordinates, where $N \in \mathbb{N}^*$. The corresponding metric is that of a conical deficit labelled by the Virasoro coadjoint vectors $p_0 = \bar{p}_0 = -c/(24N^2)$. For $N \geq 2$ the resulting one-loop partition function is found to be

$$Z_{\text{grav,N}}(\beta, \theta) = |q|^{2h}\prod_{n=1}^{+\infty}\frac{1}{|1-q^n|^2} \tag{8.80}$$

where $h = \frac{c}{24}(1 - 1/N^2)$. This is again the character of the tensor product of two highest-weight representations of the Virasoro algebra with weights $h = \bar{h}$, which confirms the interpretation of Virasoro modules as particles dressed with boundary gravitons.

In Chap. 10 we will develop a similar interpretation for BMS_3 particles, which will then be confirmed in Chap. 11 by the matching of BMS_3 characters with one-loop partition functions for asymptotically flat gravity and higher-spin theories.

Remark In [19] it was conjectured that the one-loop partition function (8.79) is *exact* because it is the only expression compatible with Virasoro symmetry. Higher loop corrections would then renormalize the Brown–Henneaux central charge but would leave the q-dependent one-loop determinant unaffected. This conjecture was used to evaluate a Farey tail sum representing the putative full, non-perturbative, partition function of AdS_3 gravity (see also [83]). To our knowledge the one-loop exactness of (8.79) is still an unproven statement.

A Note on the Fabri-Picasso Theorem

Quantizing the orbit of the AdS$_3$ metric under Brown–Henneaux transformations yields the vacuum representation of $\mathfrak{vir} \oplus \mathfrak{vir}$, whose highest-weight state $|0\rangle$ is annihilated by all $\mathfrak{so}(2,2)$ generators. But the Virasoro generators L_m, $\bar{L}_m$ with $m \leq -2$ do *not* leave the vacuum invariant. Thus Virasoro symmetry is spontaneously broken in AdS$_3$ gravity, and the descendant states

$$L_{-k_1} \dots L_{-k_n} \bar{L}_{-\ell_1} \dots \bar{L}_{-\ell_m} |0\rangle \tag{8.81}$$

can be loosely interpreted as Goldstone bosons obtained by acting on the vacuum with broken symmetry generators. This interpretation has come to be standard in the realm of asymptotic symmetries; in four dimensions it leads to the identification of soft graviton states with Goldstone bosons for spontaneously broken BMS symmetry [84].

This being said, one should keep in mind that the comparison to Goldstone bosons should be handled with care. Indeed, spontaneously broken *internal* symmetries are always such that states obtained by acting with broken symmetry generators on the vacuum do *not* belong to the Hilbert space. This statement is the Fabri-Picasso theorem (see e.g. [85]), and it follows from the fact that the norm of the state $Q|0\rangle$ has an infrared (volume) divergence whenever Q generates a broken global internal symmetry. If the Fabri-Picasso theorem was to hold for asymptotic symmetries, then the descendant states (8.81) would make no sense. Fortunately the situation of asymptotic symmetries is different because the charges that generate them are *surface* charges (8.9) rather than Noether charges. As a result, when Q is a broken asymptotic symmetry generator, the norm squared of $Q|0\rangle$ is an integral over a compact manifold, and is therefore finite. In this sense spontaneously broken *asymptotic* symmetries behave in a way radically different from spontaneously broken *internal* symmetries.

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Part III
BMS_3 Symmetry and Gravity in Flat Space

This part contains the original contributions of the thesis and is devoted to Bondi–Metzner–Sachs (BMS) symmetry in three dimensions. It starts with an introductory chapter where the definition of the BMS_3 group is motivated by asymptotic symmetry considerations. We then move on to the quantization of BMS_3 symmetry and show that irreducible unitary representations of the BMS_3 group, i.e. *BMS_3 particles*, are classified by supermomentum orbits that coincide with coadjoint orbits of the Virasoro group. We also evaluate the associated characters and show that they coincide with one-loop partition functions of the gravitational field at finite temperature and angular potential. Finally, we extend this matching to higher spin theories and supergravity in three dimensions.

Chapter 9
Classical BMS_3 Symmetry

The Bondi–Metzner–Sachs (BMS) group is an infinite-dimensional symmetry group of asymptotically flat gravity at null infinity, that extends Poincaré symmetry. It was originally discovered in four space-time dimensions in the seminal work of Bondi, Van der Burg, Metzner [1, 2] and Sachs [3, 4]. In this chapter we introduce BMS symmetry in three dimensions [5] and describe its classical aspects, i.e. those that do not rely on its realization as the quantum symmetry group of a Hilbert space. We will show in particular that the phase space of asymptotically flat gravity coincides with (a hyperplane in) the coadjoint representation of the centrally extended BMS_3 group.

The structure is as follows. In Sect. 9.1 we show how BMS_3 symmetry emerges from an asymptotic symmetry analysis. Section 9.2 is devoted to the abstract mathematical definition of the BMS_3 group and its central extension, including their adjoint and coadjoint representations. In Sect. 9.3 we describe the phase space of three-dimensional asymptotically flat gravity embedded in the space of the coadjoint representation of BMS_3. Finally, in Sect. 9.4 we show how BMS_3 symmetry can be seen as a flat limit of Virasoro symmetry.

This chapter is mostly based on [6–8], although the first section follows the earlier references [9, 10]. As usual, more specialized references will be cited in due time.

9.1 BMS Metrics in Three Dimensions

The purpose of this section is to explain how the BMS_3 group (and its central extension) emerges as an asymptotic symmetry of three-dimensional Minkowskian space-times at null infinity. In particular we describe the embedding of Poincaré transformations and the action of BMS_3 on the covariant phase space of the system, and observe the appearance of a classical central extension. We refer to Sect. 8.1 for some background on three-dimensional gravity and asymptotic symmetries in general.

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9.1.1 Three-Dimensional Minkowski Space

Minkowski space in three dimensions is the manifold $\mathbb{R}^3$ endowed with a metric whose expression in inertial coordinates $(t, x, y) = (x^0, x^1, x^2)$ is

$$ds^2 = -dt^2 + dx^2 + dy^2 = \eta_{\mu\nu} dx^\mu dx^\nu \tag{9.1}$$

where $(\eta_{\mu\nu})$ is the Minkowski metric (4.35) in $D = 3$ dimensions. In general-relativistic terms Minkowski space-time is the maximally symmetric solution of Einstein's equations with vanishing cosmological constant.

The isometry group of Minkowski space is the Poincaré group (4.39) with $D = 3$: $\mathrm{IO}(2,1) = \mathrm{O}(2,1) \ltimes \mathbb{R}^3$. Its elements are pairs (f, α) acting transitively on $\mathbb{R}^3$,

$$x^\mu \mapsto f^\mu{}_\nu x^\nu + \alpha^\mu\,, \tag{9.2}$$

where $(f^\mu{}_\nu)$ is a Lorentz transformation while α^μ is a space-time translation. The stabilizer of the origin $x^\mu = 0$ is the Lorentz group, confirming the obvious diffeomorphism $\mathbb{R}^3 \cong \mathrm{IO}(2,1)/\mathrm{O}(2,1)$.

While inertial coordinates are the most common in Minkowski space, a different set of coordinates will be more convenient for the description of BMS_3 symmetry. Namely, as in (1.4), we define *retarded Bondi coordinates* (r, φ, u) by

$$r \equiv \sqrt{x^2 + y^2}\,, \qquad e^{i\varphi} \equiv \frac{x + iy}{r}\,, \qquad u \equiv t - r\,, \tag{9.3}$$

whose range is $r \in [0, +\infty[$, $u \in \mathbb{R}$, and $\varphi \in \mathbb{R}$ with the identification $\varphi \sim \varphi + 2\pi$. In that context the coordinate u is known as *retarded time*. We will also refer to the coordinates (r, φ, t) as *cylindrical coordinates*; they are analogous to (8.15) in AdS_3. In terms of cylindrical and Bondi coordinates, the Minkowski metric (9.1) reads

$$ds^2 = -dt^2 + dr^2 + r^2 d\varphi^2 = -du^2 - 2du dr + r^2 d\varphi^2 \tag{9.4}$$

which is the three-dimensional analogue of (1.5). Bondi coordinates are represented on the Penrose diagram of Minkowski space in Fig. 1.1. Note that parity acts as $\varphi \mapsto -\varphi$.

Killing Vectors

The Killing vector fields that generate Poincaré transformations (9.2) are simplest to write down in inertial coordinates, where they have the general form

$$\xi(x) = \big(\alpha^\rho + X^\mu x^\nu \epsilon_{\mu\nu}{}^\rho\big)\partial_\rho\,. \tag{9.5}$$

Here α^μ and X^μ are two arbitrary, constant vectors generating translations and Lorentz transformations, respectively, while $\epsilon_{\mu\nu\rho}$ is the completely antisymmetric tensor such that $\epsilon_{012} = 1$ (indices are raised and lowered with the Minkowski metric).

In particular, the component α^0 is responsible for time translations, while α^1 and α^2 generate translations in the directions $x = x^1$ and $y = x^2$, respectively. The component X^0 is responsible for spatial rotations while X^1 and X^2 give rise to boosts in the directions x^1 and x^2, respectively.

For later comparison with asympotic symmetries it is convenient to rewrite the Killing vectors (9.5) in Bondi coordinates (9.3). For pure translations we find

$$\xi_{\text{Translation}} = \alpha(\varphi)\partial_u - \frac{\alpha'(\varphi)}{r}\partial_\varphi + \alpha''(\varphi)\partial_r \tag{9.6}$$

where the function $\alpha(\varphi)$ is related to the translation vector α^μ by

$$\alpha(\varphi) = \alpha^0 - \alpha^1 \cos\varphi - \alpha^2 \sin\varphi\,. \tag{9.7}$$

For pure Lorentz transformations we similarly obtain

$$\xi_{\text{Lorentz}} = \big(X(\varphi) - \frac{u}{r}X''(\varphi)\big)\partial_\varphi + uX'(\varphi)\partial_u - \big(rX'(\varphi) - uX'''(\varphi)\big)\partial_r \tag{9.8}$$

where the function $X(\varphi)$ is related to the boost vector X^μ by

$$X(\varphi) = X^0 - X^1 \cos\varphi - X^2 \sin\varphi\,. \tag{9.9}$$

Note that both (9.6) and (9.8) depend on functions on the circle; already at this stage it is tempting to speculate that there exist boundary conditions such that asymptotic symmetry generators take that form with *arbitrary* functions (X, α) on the circle. The BMS boundary conditions below will do just that.[1]

The structure of the algebra spanned by the vector fields (9.6) and (9.8) can be made more transparent by a suitable choice of basis. Thus we define the complexified Poincaré generators

$$j_m \equiv \xi_{\text{Lorentz}}\Big|_{X(\varphi)=e^{im\varphi}}\,, \qquad p_m \equiv \xi_{\text{Translation}}\Big|_{\alpha(\varphi)=e^{im\varphi}}$$

where $m, n = -1, 0, 1$. The resulting Lie brackets read

$$i[j_m, j_n] = (m-n)\,j_{m+n}\,, \qquad i[j_m, p_n] = (m-n)\,p_{m+n}\,, \qquad i[p_m, p_n] = 0\,, \tag{9.10}$$

with $m, n = -1, 0, 1$. The Lie algebra of the BMS_3 group will extend these brackets by allowing arbitrary integer values of m, n, in the same way that the Witt algebra (6.24) extends $\mathfrak{sl}(2, \mathbb{R})$. Note that the structure $G \ltimes \mathfrak{g}$ of the Poincaré group (4.93) is manifest in these relations, as the bracket of j's with p's takes exactly the same form as the bracket of j's with themselves.

[1]We are cheating in (9.8), since for now there is no way to distinguish $X'''(\varphi)$ from $-X'(\varphi)$; the justification for this combination of derivatives will be provided by asymptotic symmetries.

9.1.2 Poincaré Symmetry at Null Infinity

Null Infinities and Celestial Circles

In preparation for the asymptotic analysis to come, note that in Bondi coordinates the region $r \to +\infty$ at finite φ and u is a cylinder spanned by coordinates (φ, u) at future null infinity. It is the upper cone on the boundary of the Penrose diagram of Fig. 1.1. This is due to our choice of coordinates: instead of (9.3) we could have defined *advanced* Bondi coordinates, with advanced time given by $v = t + r$ instead of $u = t - r$. As a result we would have found that the region $r \to +\infty$ is *past* null infinity, but up to this difference the whole construction would have been the same. In this thesis we use retarded Bondi coordinates throughout, but it is always understood that a parallel construction exists in terms of advanced Bondi coordinates. In particular the region $r \to +\infty$ will always be *future* null infinity, denoted $\mathscr{I}^+$.

Future null infinity is the region of space-time where all light rays eventually escape; similarly past null infinity is the origin of all incoming light rays. In optical terms, if we were living in a three-dimensional space-time, the region that we would see around us would be a circle on our past light-cone. As the distance from us to the circle increases, the latter approaches past null infinity. A similar (time-reversed) interpretation holds for future null infinity, and justifies the following terminology:

Definition The *future celestial circle* at retarded time u associated with the Bondi coordinates (r, φ, u) is the circle spanned by the coordinate φ on future null infinity, and at fixed time u. Similarly the *past celestial circle* at advanced time v is the circle at fixed time v on past null infinity.

This definition is illustrated in Fig. 1.2. From now on the words "celestial circle" always refer to a *future* celestial circle. As we shall see, BMS_3 symmetry will reformulate and generalize the action of Poincaré transformations on celestial circles, and more generally on null infinity.

Poincaré Transformations on $\mathscr{I}^+$

Since we have rewritten Minkowski Killing vectors in Bondi coordinates, it is worth asking whether one can write *finite* Poincaré diffeomorphisms (9.2) (as opposed to infinitesimal vector fields) in Bondi coordinates. The answer is obviously yes, but the result is not particularly illuminating because the linear nature of the transformations is hidden when writing them in terms of (r, φ, u). Fortunately, Bondi coordinates are designed so that things simplify at null infinity; in particular it turns out that Poincaré transformations preserve the limit $r \to +\infty$ in the sense that (i) they map r on a positive multiple of itself and (ii) they affect φ and u but leave them finite. Accordingly Poincaré transformations are well-defined at null infinity and one may ask how they act on the coordinates (φ, u) spanning $\mathscr{I}^+$. The procedure for finding this action is explained in greater detail in [11].

For definiteness we focus on the connected Poincaré group (4.40). We can use the isomorphism (4.83) to describe Lorentz transformations in terms of $SL(2, \mathbb{R})$ matrices, the correspondence being given explicitly by (4.91). One then finds that a pure space-time translation $x^\mu \mapsto x^\mu + \alpha^\mu$ acts on null infinity according to

$$(u, \varphi) \mapsto \big(u + \alpha(\varphi), \varphi\big) \qquad \text{(translation)} \tag{9.11}$$

where the function $\alpha(\varphi)$ is related to the components α^μ by (9.7). In particular, pure time translations act on Bondi coordinates as $u \mapsto u + \alpha^0$, without affecting the other coordinates. Similarly, a Lorentz transformation specified by an $\mathrm{SL}(2, \mathbb{R})$ matrix (6.86) acts on $\mathscr{I}^+$ according to $(u, \varphi) \mapsto \big(f'(\varphi)u, f(\varphi)\big)$ where $f(\varphi)$ is a projective transformation (6.88) of the celestial circle, with parameters A, B given by (6.89). For instance, spatial rotations act as $\varphi \mapsto \varphi + \theta$, leaving all other coordinates untouched. This is analogous to the four-dimensional situation described in Eqs. (1.6) and (1.7). Upon performing simultaneously a translation α and a Lorentz transformation f, the transformation of (u, φ) reads

$$(u, \varphi) \mapsto \Big(f'(\varphi)u + \alpha(f(\varphi))\,,\; f(\varphi)\Big) \tag{9.12}$$

where f takes the form (6.88) while α is given by (9.7). As we shall see below, the BMS_3 group acts on $\mathscr{I}^+$ in the same way, except that $f(\varphi)$ will be an arbitrary diffeomorphism of the circle and that $\alpha(\varphi)$ will be an arbitrary function on the circle. Analogous results hold at past null infinity.

Note that in (9.12) we are abusing notation slightly. Indeed, a Poincaré transformation is a diffeomorphism of the whole space-time (not just null infinity) and acts on all three coordinates r, φ, u. In particular the transformation law (9.12) only holds up to corrections of order $1/r$. These corrections vanish in the limit $r \to +\infty$ and leave out only the leading piece displayed in (9.12), but they matter for the extension of Poincaré (or BMS) transformations from the boundary into the bulk.

9.1.3 BMS$_3$ *Fall-Offs and Asymptotic Symmetries*

We now wish to define a family of metrics on $\mathbb{R}^3$ that are "asymptotically flat" at future null infinity in the sense that they approach the Minkowski metric (9.4) near the boundary of space-time. As in the AdS_3 case above, a good starting point is to ask what is the minimum amount of metrics that one wants to include; clearly, pure Minkowski space should be there, but in addition one may include conical deficits. These are defined by cutting out a wedge of angular opening $2\pi(1 - 2\omega)$ out of the middle of space and quotienting Minkowski space-time with identifications of the type (8.22) in terms of cylindrical coordinates. The change of coordinates (8.23) then turns the metric of (quotiented) Minkowski space into

$$ds^2 = (dt' - Ad\varphi')^2 + dr'^2 + 4\,\omega^2 r'^2 d\varphi'^2, \tag{9.13}$$

which is the flat limit ($\ell \to +\infty$) of Eq. (8.24). In these terms there are no identifications on t', and φ' is 2π-periodic. As in the AdS_3 case the cross-term $Adt'd\varphi'$ suggests that A is proportional to angular momentum, as will indeed be the case below. In contrast to AdS$_3$, however, the region of large r' is always free of closed time-like curves since the condition for the integral curves of $\partial_{\varphi'}$ to be space-like now simply yields $r'^2 > A^2/(4\omega^2)$, without condition on the ratio of A and ω. This is a flat limit of the more stringent conditions (8.25) encountered in AdS$_3$. We refer for instance to [12, 13] for a more thorough study of conical deficits.

Now suppose we wish to find boundary conditions that include such conical deficits. If we want the asymptotic symmetry group to contain the Poincaré group, we are forced to include in the phase space all metrics obtained by performing rotations, translations and boosts of conical deficits. This is the same argument as in Sect. 8.2, where we derived Brown–Henneaux boundary conditions. It leads to a class of metrics with prescribed asymptotic behaviour at null infinity, analogous to Eq. (8.27) in the AdS$_3$ case. Some of the subleading components of the metric can then be set to zero identically as a gauge choice, which leads to the following definition:

Definition Let $\mathcal{M}$ be a three-dimensional manifold with a pseudo-Riemannian metric ds^2. Suppose there exist local Bondi coordinates (r, φ, u) on $\mathcal{M}$, defined for r larger than some lower limit, such that the region $r \to +\infty$ be a two-dimensional cylinder at future null infinity and such that the asymptotic behaviour of the metric be

$$ds^2 \overset{r\to+\infty}{\sim} \mathcal{O}(1)du^2 - \big(2 + \mathcal{O}(1/r)\big)dudr + r^2 d\varphi^2 + \mathcal{O}(1)dud\varphi\,. \tag{9.14}$$

Then we say that $(\mathcal{M}, ds^2)$ is *asymptotically flat* at future null infinity in the BMS gauge. A parallel construction exists at past null infinity.

The BMS gauge condition is the flat space analogue of the Fefferman–Graham gauge used in (8.28). We stress that it is truly a *gauge* condition in the sense of asymptotic symmetries: the diffeomorphism used to bring a metric from a general asymptotically flat form into the BMS gauge is trivial, as it does not affect the surface charges of the metric. By contrast, the diffeomorphisms that change the physical state of the system are generated by non-zero surface charges and span the asymptotic symmetry group of the system, which will turn out to be the BMS$_3$ group. From now on, when dealing with asymptotically flat gravity, we always restrict our attention to metrics satisfying the BMS boundary conditions (9.14). Note that asymptotically flat space-times need not be (and generally are not) globally diffeomorphic to Minkowski space; the definition (9.14) only requires r to be larger than some lower limiting value. Note also that there is no restriction on the sign of the fluctuating components in the metric (9.14); in particular the term of order r^0 multiplying du^2 may be positive.

Asymptotic Killing Vectors

The asymptotic Killing vector fields associated with flat boundary conditions (in BMS gauge) are vector fields that generate diffeomorphisms which preserve the fall-off conditions (9.14). This is to say that, if $g_{\mu\nu}$ is an asymptotically flat metric, its Lie derivative under such a vector field ξ must satisfy

$$\mathcal{L}_\xi g_{rr} = \mathcal{L}_\xi g_{r\varphi} = \mathcal{L}_\xi g_{\varphi\varphi} = 0 \tag{9.15}$$

together with

$$\mathcal{L}_\xi g_{uu} = \mathcal{O}(1), \qquad \mathcal{L}_\xi g_{u\varphi} = \mathcal{O}(1), \qquad \mathcal{L}_\xi g_{ur} = \mathcal{O}(1/r) \tag{9.16}$$

in terms of retarded Bondi coordinates. Here (9.15) follows from the fact that the components $g_{rr} = g_{r\varphi} = 0$ and $g_{\varphi\varphi} = r^2$ are fixed in the BMS gauge (9.14); by contrast the components g_{uu}, $g_{u\varphi}$ and g_{ur} are allowed to fluctuate by terms of order r^0, r^0 and r^{-1} respectively.

Lemma Let $g_{\mu\nu}$ be an asymptotically flat metric in the sense (9.14) and let ξ be a vector field that satisfies (9.15) and (9.16). Then

$$\xi = X(\varphi)\partial_\varphi + \big(\alpha(\varphi) + uX'(\varphi)\big)\partial_u - rX'(\varphi)\partial_r + (\text{subleading}) \tag{9.17}$$

where $X(\varphi)$ and $\alpha(\varphi)$ are two arbitrary (smooth) 2π-periodic functions, while the subleading terms take the form

$$\begin{aligned}&\left[(\alpha' + uX'')\int_r^{+\infty}\frac{dr'}{r'^2}g_{ur}\right]\partial_\varphi\\&+\left[\partial_\varphi\left((\alpha' + uX'')\int_r^{+\infty}\frac{dr'}{r'^2}g_{ur}\right) + \frac{1}{r^2}(\alpha' + uX'')g_{u\varphi}\right]\partial_r =\\&= \frac{1}{r}(\alpha' + uX'')\partial_\varphi + \frac{1}{r}(\alpha'' + uX''')\partial_r + \mathcal{O}(r^{-2})\,.\end{aligned} \tag{9.18}$$

These formulas uniquely associate an asymptotic Killing vector field ξ with an asymptotically flat metric $g_{\mu\nu}$ and two functions $\big(X(\varphi), \alpha(\varphi)\big)$ on the celestial circle; the dependence of ξ on these functions is linear.

Proof Let $g_{\mu\nu}$ be an asymptotically flat metric (9.14). First note that the condition $\mathcal{L}_\xi g_{rr} = 0$ yields $\partial_r\xi^u = 0$, so ξ^u is r-independent. On the other hand the condition $\mathcal{L}_\xi g_{r\varphi} = 0$ gives a differential equation $\partial_r\xi^\varphi = -\frac{1}{r^2}g_{ru}\partial_\varphi\xi^u$, which is solved by

$$\xi^\varphi = X(u,\varphi) + \partial_\varphi\xi^u\int_r^{+\infty}\frac{dr'}{r'^2}g_{r'u} \tag{9.19}$$

where $X(u,\varphi)$ is an arbitrary function on the cylinder at null infinity. The integral over r' converges since $g_{ru} = -1 + \mathcal{O}(1/r)$ by virtue of (9.14), so that $\xi^\varphi = X(u,\varphi) + \frac{1}{r}\partial_\varphi\xi^u + \mathcal{O}(1/r^2)$. At this point we introduce a function $\alpha(u,\varphi)$ defined by

$$\xi^u = \alpha(u,\varphi) + uX'(u,\varphi) \tag{9.20}$$

(prime denotes partial differentiation with respect to φ), which is allowed by virtue of the fact that ξ^u is r-independent. In these terms the condition $\mathcal{L}_\xi g_{\varphi\varphi} = 0$ gives

$$\xi^r = -r\partial_\varphi\xi^\varphi - \frac{1}{r}g_{u\varphi}(\alpha' + uX'') \tag{9.21}$$

where ξ^φ is given by (9.19). Since we now know that the most general solution ξ of (9.15) is determined by two functions $X(u,\varphi)$ and $\alpha(u,\varphi)$ on null infinity, we can use the remaining conditions (9.16) to constrain these functions. Using first $\mathcal{L}_\xi g_{ur} = \mathcal{O}(1/r)$, we find

$$\partial_u \xi^u = X', \tag{9.22}$$

which upon rewriting ξ^u as (9.20) says that the combination $\partial_u\alpha + u\partial_u X'$ vanishes. The requirement $\mathcal{L}_\xi g_{u\varphi} = \mathcal{O}(1)$ then yields $\partial_u X = 0$, which is to say that $X(u,\varphi) = X(\varphi)$ only depends on the coordinate φ on the celestial circle. Plugging this back into (9.22) then yields $\partial_u\alpha = 0$ as well. Formula (9.17) follows, while the subleading terms (9.18) are produced by (9.19) and (9.21). ■

Note that the asymptotic Killing vectors (9.17) precisely take the anticipated form (9.6)–(9.8) and generalize Poincaré transformations in an infinite-dimensional way. In particular the asymptotic symmetry group contains all space-time translations (corresponding to $\alpha(\varphi)$ of the form (9.7)) and all Lorentz transformations (corresponding to $X(\varphi)$ of the form (9.9)). We shall denote by $\xi_{(X,\alpha)}$ the asymptotic Killing vector determined by the functions $X(\varphi)$ and $\alpha(\varphi)$. One verifies that the Lie brackets of such vector fields read

$$\left[\xi_{(X,\alpha)}, \xi_{(Y,\beta)}\right] = \xi_{([X,Y],[X,\beta]-[Y,\alpha])} + \text{(subleading)} \tag{9.23}$$

where the brackets in the subscript on the right-hand side are understood to be standard Lie brackets on the circle, e.g. $[X,\alpha] \equiv X\alpha' - \alpha X'$. The subleading terms can be neglected because they will turn out not to contribute to the surface charges; alternatively, as in the AdS$_3$ case, they can be absorbed by a redefinition of the Lie bracket such that the algebra is realized everywhere in the bulk [14, 15].

The structure of the algebra (9.23) can be made more transparent by decomposing the functions $\big(X(\varphi), \alpha(\varphi)\big)$ in Fourier modes and defining the vector fields

$$j_m \equiv \xi_{(e^{im\varphi},0)}, \qquad p_m \equiv \xi_{(0,e^{im\varphi})}. \tag{9.24}$$

As one can verify, formula (9.23) implies that their Lie brackets take the form (9.10) with *arbitrary integer labels* m, n, up to subleading corrections.

Thus we now know that the asymptotic symmetries of three-dimensional Minkowskian space-times span an algebra that contains the Witt algebra (extending the Lorentz algebra) and an infinite-dimensional Abelian algebra (extending space-time translations). The corresponding asymptotic symmetry transformations are referred to as *superrotations* and *supertranslations*, respectively[2]; they span an infinite-dimensional algebra known as the *BMS algebra in three dimensions*, that we shall denote as $\mathfrak{bms}_3$. Finite BMS$_3$ transformations act on null infinity according to formula (9.12), where $f(\varphi)$ is an arbitrary diffeomorphism of the celestial circle while $\alpha(\varphi)$ is an arbitrary function on the circle.

[2]The prefix "super" has nothing to do with supersymmetry, but stresses the fact that special-relativistic quantities are extended in an infinite-dimensional way.

We refrain from analysing the group-theoretic aspects of these symmetries at this point — this will be the subject of all later sections in this chapter. Instead, we now keep going in our study of asymptotically flat gravity; in particular we actually still have to confirm that superrotations and supertranslations are indeed non-trivial asymptotic symmetries, i.e. that the associated surface charges do not vanish.

Remark One should keep in mind that Bondi coordinates are *global*, since the definition (9.3) covers all points of Minkoswki space-time. Thus the fact that Bondi coordinates allow one to describe either only future or only past null infinity (and not both) does not mean that they cover only "half" of the space-time. A similar comment applies to BMS symmetry, whose definition in terms of space-time relies on a choice of coordinates that favours future over past null infinity (or vice-versa). Despite this asymmetry, it was recently realized that (for well-behaved asymptotically flat space-times [16]) the two definitions of BMS can be related by an "antipodal identification", which leads to the application of BMS symmetry to scattering phenomena [17–32]. A related question (as yet unsolved) is whether BMS symmetry can be defined at spatial infinity [33].

9.1.4 *On-Shell* BMS$_3$ *Metrics*

In order for the equations of motion to provide an extremum of the action functional, the latter must be differentiable in the space of fields satisfying certain fall-off conditions. In the case of asymptotically flat three-dimensional gravity, it was shown in [34], using the Chern–Simons formalism, that there exists a well-defined variational principle. The same conclusion was obtained more recently in [35] in the metric formalism, with the observation that the pure Einstein-Hilbert action (8.1), without any extra boundary term, is differentiable in the space of asymptotically flat metrics.

Accordingly, it makes sense to ask about the general solution of Einstein's vacuum equations in the BMS gauge. It was shown in [10] that this solution reads

$$ds^2 = 8G\, p(\varphi)du^2 - 2dudr + 8G\big(j(\varphi) + up'(\varphi)\big)dud\varphi + r^2 d\varphi^2 \tag{9.25}$$

where $p(\varphi)$ and $j(\varphi)$ are arbitrary, 2π-periodic functions of φ. Upon evaluating surface charges we will see that $p(\varphi)$ and $j(\varphi)$ are densities of energy and angular momentum at null infinity, respectively. As in the earlier AdS$_3$ case (8.38), the normalization factors involving Newton's constant G are included for later convenience.

The transformation law of the solution (9.25) under the action of asymptotic Killing vectors follows from the definition

$$\mathcal{L}_{\xi_{(X,\alpha)}} ds^2 \equiv 8G\, \delta_{(X,\alpha)} p(\varphi)\, du^2 + 8G\, \big(\delta_{(X,\alpha)} j(\varphi) + u\, \delta_{(X,\alpha)} p'(\varphi)\big)\, dud\varphi \tag{9.26}$$

where the functions $X(\varphi)$ and $\alpha(\varphi)$ determine the vector field (9.17). Evaluating the Lie derivative (9.26) one finds

$$\delta_{(X,\alpha)} j = X j' + 2X' j + \alpha p' + 2\alpha' p - \frac{c_2}{12}\alpha''', \tag{9.27}$$

$$\delta_{(X,\alpha)} p = X p' + 2X' p - \frac{c_2}{12} X''' \tag{9.28}$$

where c_2 is a dimensionful central charge proportional to the Planck mass [9]:

$$\boxed{c_2 = \frac{3}{G}\,.} \tag{9.29}$$

In this language the asymptotic vector field $\xi_{(X,\alpha)}$ is an *exact* Killing vector field for the metric (j, p) if both variations (9.27) and (9.28) vanish. The subscript "2" in (9.29) will be justified below.

The transformation law of p in (9.28) coincides with that of a CFT stress tensor under a conformal transformation generated by X; it is the coadjoint representation (6.115) of the Virasoro algebra. The transformation (9.27) of j is somewhat more involved. We refrain from interpreting these results for now, as we will return to them in much greater detail in the upcoming sections. Note that at this stage all normalizations are arbitrary, and in particular the central charge (9.29) would take another value if we chose to change the normalization of p.

9.1.5 Surface Charges and BMS$_3$ *Algebra*

Surface Charges

Take an asymptotic Killing vector field (9.17) specified by the functions $\big(X(\varphi), \alpha(\varphi)\big)$, and choose an on-shell metric (9.25) specified by $\big(j(\varphi), p(\varphi)\big)$. We wish to evaluate the surface charge associated with the symmetry transformation generated by $\xi_{(X,\alpha)}$ on the background specified by (j, p). This charge depends linearly on the components of $\xi_{(X,\alpha)}$, as explained around Eq. (8.9). In addition we must choose a normalization, that is, a "background" solution for which we declare that all surface charges vanish. Here we take it to be the null orbifold at $j = p = 0$,

$$\bar{g} = -2du dr + r^2 d\varphi^2. \tag{9.30}$$

With this normalization one can show that the surface charge (8.9) associated with the vector field $\xi_{(X,\alpha)}$ on the solution (j, p) is [10]

$$Q_{(X,\alpha)}[j, p] = \frac{1}{2\pi}\int_0^{2\pi} d\varphi\big[j(\varphi)X(\varphi) + p(\varphi)\alpha(\varphi)\big]. \tag{9.31}$$

It can be interpreted as the pairing of the $\mathfrak{bms}_3$ algebra, consisting of pairs (X, α), with its dual consisting of pairs (j, p). In particular, even though we haven't defined the BMS$_3$ group at this stage, we already know that the space of solutions (9.25) belongs to its coadjoint representation. The charge associated with time translations corresponds to the asymptotic Killing vector ∂_u; it is the Hamiltonian of the system,

$$M = \mathcal{P}_0 = \frac{1}{2\pi}\int_0^{2\pi} d\varphi\, p(\varphi)\,, \tag{9.32}$$

and it allows us to interpret $p(\varphi)$ as the energy density carried by the gravitational field at (future) null infinity. Thus $p(\varphi)$ is the *Bondi mass aspect* associated with the metric (9.25) and its zero-mode (9.32) is the Bondi mass. More generally the charges associated with supertranslations ($X = 0$) take the form

$$Q_{(0,\alpha)}[j, p] = \frac{1}{2\pi}\int_0^{2\pi} d\varphi\, p(\varphi)\alpha(\varphi)\,. \tag{9.33}$$

In the same way, the charge associated with rotations corresponds to the asymptotic Killing vector ∂_φ; it is the angular momentum

$$J = \mathcal{J}_0 = \frac{1}{2\pi}\int_0^{2\pi} d\varphi\, j(\varphi)\,. \tag{9.34}$$

We can interpret $j(\varphi)$ as the density of angular momentum carried by the gravitational field at null infinity; it is the *angular momentum aspect* associated with the metric (9.25). More generally all superrotation charges take the form

$$Q_{(X,0)}[j, p] = \frac{1}{2\pi}\int_0^{2\pi} d\varphi\, j(\varphi)X(\varphi)$$

and generalize centre of mass charges. With this normalization Minkowski space (9.4) has energy $M = -1/8G$ and all its other surface charges vanish.

Surface Charge Algebra

We now compute the Poisson brackets of surface charges for asymptotically flat space-times. Recall that these brackets generate symmetry transformations (8.10), on account of the fact that conserved charges are momentum maps (5.34). We can apply this property here to deduce the Poisson brackets of charges: if we let (j, p) be an on-shell metric (9.25), then the bracket of charges is

$$\begin{aligned}&\big\{Q_{(X,\alpha)}[j, p], Q_{(Y,\beta)}[j, p]\big\} = \\ &\overset{(9.31)}{=} -\frac{1}{2\pi}\int_0^{2\pi} d\varphi\left[\delta_{(X,\alpha)}j(\varphi)Y(\varphi) + \delta_{(X,\alpha)}p(\varphi)\beta(\varphi)\right].\end{aligned} \tag{9.35}$$

Using the infinitesimal transformation laws (9.27) and (9.28) and integrating by parts one can then show that

$$\left\{Q_{(X,\alpha)}[j,p],\,Q_{(Y,\beta)}[j,p]\right\}=Q_{([X,Y],[X,\beta]-[Y,\alpha])}[j,p]+c_2\left[\mathsf{c}(X,\beta)-\mathsf{c}(Y,\alpha)\right], \tag{9.36}$$

where as in (9.23) we denote by $[X, Y] \equiv XY' - YX'$ the standard Lie bracket of vector fields on the circle, while $\mathsf{c}(X, Y)$ is the Gelfand–Fuks cocycle (6.43). Thus, the surface charges of asymptotically flat space-times close under the Poisson bracket according to a central extension of the BMS$_3$ Lie algebra displayed in (9.23). Furthermore the central extension is remarkably similar to that of the Virasoro algebra (6.108). Again, we refrain from interpreting this result any further at this point, since we haven't truly defined the BMS$_3$ group yet. For future reference we simply note that the Poisson brackets (9.36) can be rewritten in terms of a discrete set of generators analogous to (9.24). Namely, let us define the charges

$$\mathcal{J}_m \equiv Q_{(e^{im\varphi},0)}[j,p], \qquad \mathcal{P}_m \equiv Q_{(0,e^{im\varphi})}[j,p]$$

for all $m \in \mathbb{Z}$, generalizing the Hamiltonian (9.32) and angular momentum (9.34). Then the bracket (9.36) yields the algebra

$$\begin{aligned} i\{\mathcal{J}_m,\mathcal{J}_n\} &= (m-n)\mathcal{J}_{m+n}\,,\\ i\{\mathcal{J}_m,\mathcal{P}_n\} &= (m-n)\mathcal{P}_{m+n}+\frac{c_2}{12}m^3\delta_{m+n,0}\,,\\ i\{\mathcal{P}_m,\mathcal{P}_n\} &= 0\,. \end{aligned} \tag{9.37}$$

This is an infinite-dimensional central extension of (9.10), with $m, n \in \mathbb{Z}$.

Note that the central extension proportional to c_2 in (9.37) pairs superrotation generators $\mathcal{J}_m$ with supertranslation generators $\mathcal{P}_m$. By contrast the Witt algebra spanned by superrotations receives no central extension. This is why we wrote the central charge (9.29) with an index "2": the notation c_1 will be kept for the central charge pairing superrotation generators with themselves. Despite many similarities, we stress that c_2 is *not* a Virasoro central charge; in particular it is a dimensionful quantity. This is consistent with the fact that the value of c_2 varies when changing the normalization of the charges $\mathcal{P}_m$: if we were to define $\tilde{\mathcal{P}}_m \equiv \lambda\mathcal{P}_m$ with some non-zero real number λ, the Poisson brackets of $\mathcal{J}$'s and $\tilde{\mathcal{P}}$'s would take the form (9.37) with the central charge c_2 replaced by λc_2. Nevertheless, the value displayed in (9.29) is canonical in the sense that it is the one provided by the normalization of the Hamiltonian (9.32), which in turn is the surface charge associated with the vector field ∂_u in terms of Bondi coordinates. In essence the central charge c_2 is analogous to that of the Bargmann group (4.103), which as we saw in (4.108) is also a mass scale. This is radically different from the Virasoro algebra, where the value of the central charge c in (6.118) is unambiguously fixed by the condition that the homogeneous structure constants take the form $(m - n)$.

Remark The BMS boundary conditions given here are the flat analogue of Brown–Henneaux boundary conditions. In this sense they are the "standard" fall-offs for three-dimensional asymptotically flat gravity. However, it is likely that other consistent boundary conditions exist in Einstein gravity — for instance adapting to flat space the free AdS_3 boundary conditions of [36]. In addition one can devise BMS-like boundary conditions for other theories of gravity, such as topologically massive gravity [37, 38], bigravity [39], conformal gravity [40] or new massive gravity [41, 42]. In particular, in parity-breaking theories such as TMG, one typically finds that the Virasoro algebra spanned by superrotations $\mathcal{J}_m$ develops a non-zero central charge c_1. Aside from this comment we will have very little to say about these alternative possibilities.

9.1.6 Zero-Mode Solutions

We focus here on zero-mode metrics, with constant $(j, p) = (j_0, p_0)$ in Eq. (9.25). The only non-vanishing surface charges for such metrics are the Bondi mass (9.32) and the angular momentum (9.34), which coincide with p_0 and j_0 respectively.

At $j_0 = 0$, $p_0 = -c_2/24 \overset{(9.29)}{=} -1/8G$, the metric is that of pure Minkowski space-time (9.4). Solutions having $p_0 = -c_2/24$ but non-zero j_0 corresponding to "spinning Minkowski space-time". Note that, while the normalization of p_0 and c_2 is arbitrary, the relation

$$p_{\text{vac}} = -\frac{c_2}{24}$$

is a normalization-independent statement.[3] It suggests that Minkowski space plays the role of a classical vacuum for a putative dual theory; we will return to this later.

Solutions having $0 > p_0 > -c_2/24$ are conical deficits for all values of j_0, with a deficit angle $2\pi(1 - 2\omega)$ given by (7.46). In particular, solutions with $p_0 = 0$ are degenerate conical deficits, and the solution $p_0 = j_0 = 0$ is the null orbifold (9.30) that we used to normalize charges. Solutions having $p_0 < -c_2/24$ are conical excesses with an excess angle $2\pi(2\omega - 1)$ given again by (7.46). For $p_0 = -c_2 n^2/24$ the excess angle is $2\pi(n - 1)$.

Zero-mode solutions with positive p turn out to describe flat space cosmologies, sometimes also called shifted boost orbifolds [43, 44]. They represent a $(2 + 1)$-dimensional universe that undergoes a big crunch followed by a big bang, where the transition between the contracting and expanding phases is smooth only if $j \neq 0$. When $j = 0$ these solutions can be thought of as a compactification of the three-dimensional Milne universe. They can also be seen as limits of the interior region of BTZ black holes as the AdS_3 radius goes to infinity. The lightest flat space cosmology has $p_0 = 0$ and is separated from Minkowski space-time $p_{\text{vac}} = -c_2/24$ by a classical mass gap; the latter is filled by conical deficits. This is very similar to the mass gap separating BTZ black holes from AdS_3.

Note that, in contrast to AdS_3, no cosmic censorship is needed to ensure the absence of closed time-like curves at infinity (although closed time-like curves

[3]Indeed, changing the normalization of p would also change the value of the central charge that ensures that the bracket $\{\mathcal{J}, \mathcal{P}\}$ takes the canonical form in Eq. (9.37).

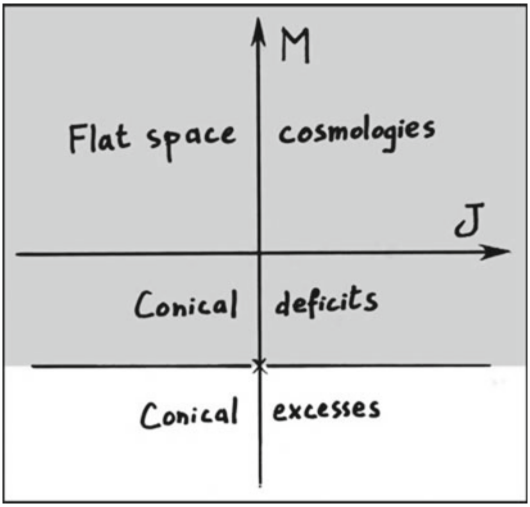

Fig. 9.1 The zero-mode solutions of asymptotically flat gravity with BMS$_3$ fall-offs. The origin of the coordinate system (J, M) is the null orbifold (9.30); the Minkowski metric is located below, on the M axis, right between conical deficits and conical excesses. Flat space cosmologies are located in the region $M > 0$. Conical deficits are such that $-c_2/24 < M < 0$ while excesses have $M < -c_2/24$. Anticipating Sect. 9.3.3, we have shaded the solutions whose orbit has energy bounded from below under BMS$_3$ transformations; those are all flat space cosmologies and all conical excesses, plus Minkowski space. Note that this figure is a flat limit of Fig. 8.5, as the slope of the curve $\ell M = J$ in the plane (J, M) goes to zero when $\ell \to +\infty$

generally do exist in the bulk). In fact, the whole classification of flat zero-mode metrics may be seen as a limit $\ell \to +\infty$ of that of zero-mode metrics in AdS$_3$. The family of flat zero-mode solutions is plotted in Fig. 9.1.

9.2 The BMS$_3$ Group

This section is devoted to a detailed description of the BMS$_3$ group and its central extension. This will rely on a level of abstraction that may seem offputting at first sight, but one should keep in mind that BMS$_3$ is an extension of Poincaré symmetry so that almost all statements on BMS have an analogue in special relativity. We urge the reader to adopt this point of view whenever there is a risk of getting lost in mathematical formulas. In particular, our notation will be consistent with the analogies between Poincaré and BMS$_3$ (Table 9.1).

The plan of this section is as follows. Motivated by the structure of the Poincaré group (4.93), we start by defining a notion of "exceptional semi-direct products" (generally centrally extended) and work out their adjoint and coadjoint representations. We then use asymptotic symmetries to motivate the definition of the BMS$_3$ group and its central extension, which turn out to be exceptional semi-direct products

Table 9.1 Symmetry-based objects for the Poincaré group, and their BMS$_3$ analogues

Notation	Poincaré	BMS$_3$
f	Finite Lorentz tsf.	Finite superrotation
X	Infinitesimal Lorentz tsf.	Infinitesimal superrotation
α	Translation	Supertranslation
$j_m,\ \mathcal{J}_m,\ J_m$	Lorentz generator	Superrotation generator
$p_m,\ \mathcal{P}_m,\ P_m$	Translation generator	Supertranslation generator
j	Relativistic angular momentum	Angular supermomentum
p	Energy-momentum	Supermomentum
$\mathcal{Z}_1,\ c_1$	/	Superrotational central charge
$\mathcal{Z}_2,\ c_2$	/	Supertranslational central charge

based on the Virasoro group. Finally, we write down the adjoint representation, the Lie algebra and the coadjoint representation of the (centrally extended) BMS$_3$ group. Throughout the section, these structures are compared to their Poincaré counterparts and to three-dimensional asymptotically flat gravity. Note that the material presented here relies heavily on Chaps. 2, 4 and 6.

9.2.1 *Exceptional Semi-direct Products*

Here we study a general family of semi-direct products, whose structure turns out to be common to the Poincaré group (in three dimensions) and the BMS$_3$ group. We start by describing this structure and its central extension, then display the corresponding adjoint and coadjoint representations.

Defining Exceptional Semi-direct Products

Definition Let G be a Lie group with Lie algebra $\mathfrak{g}$. The associated *exceptional semi-direct product* is the group

$$G \ltimes_{\text{Ad}} \mathfrak{g}_{\text{Ab}} \equiv G \ltimes \mathfrak{g} \tag{9.38}$$

where $\mathfrak{g}_{\text{Ab}}$ denotes the Lie algebra of G seen as an Abelian vector group acted upon by G according to the adjoint representation. Its group operation is given by (4.6) with the action σ replaced by the adjoint.

As usual we denote elements of (9.38) as pairs (f, α) where $f \in G$ is a "rotation" while $\alpha \in A$ is a "translation". For instance the (double cover of the) Poincaré group (4.93) takes the exceptional form with $G = \text{SL}(2, \mathbb{R})$. It is straightforward to obtain central extensions of this structure: if $\widehat{G}$ is a central extension of G with group operation (2.11) in terms of some two-cocycle C and if $\widehat{\mathfrak{g}}$ is its Lie algebra, one can consider the exceptional semi-direct product

$$\widehat{G} \ltimes_{\widehat{\mathrm{Ad}}} \widehat{\mathfrak{g}}_{\mathrm{Ab}} \tag{9.39}$$

where $\widehat{\mathrm{Ad}}$ denotes the adjoint representation (6.98) of $\widehat{G}$. Its elements are quadruples

$$(f, \lambda; \alpha, \mu) \tag{9.40}$$

where λ, μ are real numbers, being understood that the pair (f, λ) belongs to $\widehat{G}$ while (α, μ) belongs to $\widehat{\mathfrak{g}}_{\mathrm{Ab}}$. The notation emphasizes the fact that "centrally extended rotations" (f, λ) play a role radically different from "centrally extended translations" (α, μ). In fact the notation $\big((f, \lambda), (\alpha, \mu)\big)$ would be more accurate, but to reduce clutter we stick to (9.40).

The group operation in (9.39) is that of an exceptional semi-direct product based on the centrally extended group $\widehat{G}$. Explicitly, using the centrally extended adjoint representation (6.98), we have

$$\begin{aligned}&(f, \lambda; \alpha, \mu) \cdot (g, \rho; \beta, \nu) = \\ &\overset{(6.98)}{=} \Big(f \cdot g, \lambda + \rho + \mathsf{C}(f, g)\,;\ \alpha + \mathrm{Ad}_f\beta, \mu + \nu - \frac{1}{12}\langle \mathsf{S}[f], \beta\rangle\Big)\end{aligned} \tag{9.41}$$

where C is the two-cocycle that defines $\widehat{G}$, S is the associated Souriau one-cocycle (6.79), and the pairing $\langle\cdot,\cdot\rangle$ is that of $\mathfrak{g}^*$ with $\mathfrak{g}$. This is exactly the structure that we will find in the centrally extended BMS$_3$ group below, but for now we first investigate the adjoint and coadjoint representations of (9.39) in general terms.

Adjoint Representation and Lie Algebra

Consider a centrally extended exceptional semi-direct product $\widehat{G} \ltimes \widehat{\mathfrak{g}}$. Owing to the general form (5.103), its Lie algebra is a semi-direct sum

$$\widehat{\mathfrak{g}} \inplus_{\widehat{\mathrm{ad}}} \widehat{\mathfrak{g}}_{\mathrm{Ab}} \tag{9.42}$$

where $\widehat{\mathrm{ad}}$ is the adjoint representation of $\widehat{\mathfrak{g}}$, i.e. the Lie bracket (6.102). The elements of this algebra are quadruples $(X, \lambda; \alpha, \mu)$ where (X, λ) belongs to $\widehat{\mathfrak{g}}$ while (α, μ) belongs to $\widehat{\mathfrak{g}}_{\mathrm{Ab}}$.

The adjoint representation of the group (9.39) follows from formula (5.105). Starting for simplicity with the centreless group (9.38), it is given by

$$\mathrm{Ad}_{(f,\alpha)}(X, \beta) = \big(\mathrm{Ad}_f X, \mathrm{Ad}_f\beta - \mathrm{ad}_{\mathrm{Ad}_f X}\alpha\big) = \Big(\mathrm{Ad}_f X, \mathrm{Ad}_f\beta - [\mathrm{Ad}_f X, \alpha]\Big) \tag{9.43}$$

where the "Ad" on the right denotes the adjoint representation of G alone.[4] In the second equality we abuse notation by writing a bracket between $\mathrm{Ad}_f X \in \mathfrak{g}$ and

[4]In case of identical notations, the subscript indicates which group we are referring to.

$\alpha \in \mathfrak{g}_{\text{Ab}}$, being understood that we use the Lie bracket of $\mathfrak{g}$ and interpret the result as an element of $\mathfrak{g}_{\text{Ab}}$. The Lie bracket of the centreless Lie algebra $\mathfrak{g} \Subset_{\text{ad}} \mathfrak{g}_{\text{Ab}}$ follows:

$$\big[(X, \alpha), (Y, \beta)\big] = \big([X, Y], \text{ad}_X\beta - \text{ad}_Y\alpha\big) = \big([X, Y], [X, \beta] - [Y, \alpha]\big), \quad (9.44)$$

in accordance with the general formula (5.106). Note that this is precisely the form of the Lie bracket (9.23) of BMS$_3$ asymptotic Killing vectors.

The centrally extended adjoint representation corresponding to (9.43) can be obtained in a similar fashion. Using (6.98) and (6.102) we find explicitly

$$\begin{aligned}&\widehat{\text{Ad}}_{(f,\alpha)}(X, \lambda; \beta, \mu) =\\ &= \Big(\text{Ad}_f X, \lambda - \frac{1}{12}\langle \mathsf{S}[f], X\rangle\,; \text{Ad}_f\beta - [\text{Ad}_f X, \alpha], \mu - \frac{1}{12}\langle \mathsf{S}[f], \beta\rangle + \frac{1}{12}\big\langle \mathsf{s}[\text{Ad}_f X], \alpha\big\rangle\Big)\end{aligned} \quad (9.45)$$

where Ad on the right-hand side denotes the adjoint representation of G and $[\cdot, \cdot]$ is the Lie bracket of $\mathfrak{g}$. On the left-hand side we have neglected central terms in the subscript of the adjoint representation, since they act trivially.

From the adjoint representation one can read off, by differentiation, the Lie bracket of the centrally extended algebra (9.42). One expects the contribution of central terms to include a cocycle c given by (6.101), and indeed one finds

$$\big[(X, \lambda; \alpha, \mu), (Y, \rho; \beta, \nu)\big] = \Big([X, Y], \mathsf{c}(X, Y); [X, \beta]-[Y, \alpha], \mathsf{c}(X, \beta)-\mathsf{c}(Y, \alpha)\Big) \quad (9.46)$$

where we abuse notation as in (9.44). Already note that the last entry precisely takes the form of the central extension in the Poisson bracket (9.36) of flat surface charges.

The appearance of the same cocycle c in both central entries of (9.46) is due to the exceptional semi-direct product structure of (9.39). It implies that, when written in terms of generators, the brackets of rotations with translations take the same form as the brackets of rotations with themselves, including central terms. Explicitly, suppose we are given a basis of $\widehat{\mathfrak{g}} \Subset \widehat{\mathfrak{g}}_{\text{Ab}}$ consisting of non-central generators

$$\mathcal{J}_a \equiv (j_a, 0; 0, 0)\,, \qquad \mathcal{P}_a \equiv (0, 0; p_a, 0)$$

where the j_a's and p_a's respectively generate $\mathfrak{g}$ and $\mathfrak{g}_{\text{Ab}}$, together with two central elements

$$\mathcal{Z}_1 \equiv (0, 1; 0, 0)\,, \qquad \mathcal{Z}_2 \equiv (0, 0; 0, 1)\,. \quad (9.47)$$

Suppose also that the Lie brackets of $\mathcal{J}_a$'s take the form (2.27) with some structure constants $f_{ab}{}^c$ and some central coefficients c_{ab}, and let us choose the basis elements $\mathcal{P}_a$ such that their bracket with $\mathcal{J}_a$'s takes the same form as the bracket of $\mathcal{J}_a$'s with themselves. This is allowed by the exceptional semi-direct product structure. Then the bracket (9.46) implies that the commutation relations of $\widehat{\mathfrak{g}} \Subset \widehat{\mathfrak{g}}_{\text{Ab}}$ are

$$\begin{aligned}[\mathcal{J}_a, \mathcal{J}_b] &= f_{ab}{}^c\, \mathcal{J}_c + c_{ab}\, \mathcal{Z}_1\,, \\ [\mathcal{J}_a, \mathcal{P}_b] &= f_{ab}{}^c\, \mathcal{P}_c + c_{ab}\, \mathcal{Z}_2\,, \\ [\mathcal{P}_a, \mathcal{P}_b] &= 0\,. \end{aligned} \tag{9.48}$$

The fact that $\mathcal{J}$'s act on $\mathcal{P}$'s according to the adjoint representation is now manifest since the structure constants of the two first lines are identical. Note in particular that the central generator $\mathcal{Z}_1$ pairs rotations with themselves, while $\mathcal{Z}_2$ pairs rotations with translations. The centrally extended BMS$_3$ algebra (9.37) illustrates this phenomenon, as does the Poincaré algebra (9.10), albeit without central extension.

Note that the definition of (9.39) rules out all central extensions in the bracket $[\mathcal{P}, \mathcal{P}]$ of (9.48), and indeed we will show in Sect. 9.2.5 that such central extensions never take place in the centrally extended BMS$_3$ algebra. However, for other semi-direct product groups, such extensions may occur; an example is the symmetry group of warped conformal field theories [45], $\mathrm{Diff}(S^1) \ltimes C^\infty(S^1)$.

Coadjoint Representation

The space of coadjoint vectors dual to the algebra (9.42) is a direct sum $\widehat{\mathfrak{g}}^* \oplus \widehat{\mathfrak{g}}^*$, or more accurately $\widehat{\mathfrak{g}}^* \oplus \widehat{\mathfrak{g}}^*_{\mathrm{Ab}}$. Following the notation of Sect. 5.4, its elements are quadruples

$$(j, c_1; p, c_2) \tag{9.49}$$

where (j, c_1) is a centrally extended angular momentum dual to $\widehat{\mathfrak{g}}$, while (p, c_2) is a centrally extended momentum dual to $\widehat{\mathfrak{g}}_{\mathrm{Ab}}$. The real numbers c_1, c_2 are central charges; the first pairs rotation generators with themselves, while the second pairs rotations with translations. The pairing of (9.49) with $\widehat{\mathfrak{g}} \in \widehat{\mathfrak{g}}_{\mathrm{Ab}}$ is

$$\big\langle (j, c_1; p, c_2), (X, \lambda; \alpha, \mu)\big\rangle = \langle j, X\rangle + \langle p, \alpha\rangle + c_1\lambda + c_2\mu\,, \tag{9.50}$$

where the two pairings $\langle\cdot,\cdot\rangle$ on the right-hand side are those of $\mathfrak{g}^*$ with $\mathfrak{g}$ and $\mathfrak{g}^*_{\mathrm{Ab}}$ with $\mathfrak{g}_{\mathrm{Ab}}$, respectively. This is a centrally extended generalization of (5.109).

Recall that the coadjoint representation of a semi-direct product involves a cross product (5.110). For the centreless exceptional semi-direct product (9.38), we have

$$\langle \alpha \times p, X\rangle \overset{(5.110)}{=} \langle p, \mathrm{ad}_X \alpha\rangle = -\langle p, \mathrm{ad}_\alpha X\rangle \overset{(5.11)}{=} \big\langle \mathrm{ad}^*_\alpha p, X\big\rangle$$

where ad and ad* denote the adjoint and coadjoint representations of $\mathfrak{g}$, respectively. In other words,

$$\alpha \times p = \mathrm{ad}^*_\alpha\, p \tag{9.51}$$

where we abuse notation slightly by acting with an element of $\mathfrak{g}_{\mathrm{Ab}}$ on an element of $\mathfrak{g}^*_{\mathrm{Ab}}$. Using (5.113), it readily follows that the coadjoint representation of the centreless semi-direct product (9.38) is given by

$$\mathrm{Ad}^*_{(f,\alpha)}(j, p) = \big(\mathrm{Ad}^*_f j, +\mathrm{ad}^*_\alpha \mathrm{Ad}^*_f p, \mathrm{Ad}^*_f p\big) \tag{9.52}$$

where the Ad* on the right-hand side is the coadjoint representation of G. For example, when $G = \mathrm{SL}(2, \mathbb{R})$, this formula is the transformation law of relativistic angular momentum j and energy-momentum p under Poincaré transformations in three dimensions. From (9.52) we also find that the coadjoint representation of the Lie algebra $\mathfrak{g} \Subset \mathfrak{g}_{\mathrm{Ab}}$ is

$$\mathrm{ad}^*_{(X,\alpha)}(j, p) = \left(\mathrm{ad}^*_X j + \mathrm{ad}^*_\alpha p, \mathrm{ad}^*_X p\right) \tag{9.53}$$

in accordance with Eq. (5.114).

The centrally extended generalization of these considerations is straightforward, if mildly technical. Using Eq. (6.104) for the coadjoint action of $\widehat{G}$, formula (9.52) yields the coadjoint representation of $\widehat{G} \ltimes \widehat{\mathfrak{g}}_{\mathrm{Ab}}$:

$$\begin{aligned}&\widehat{\mathrm{Ad}}^*_{(f,\alpha)}(j, c_1; p, c_2) = \\ &= \left(\mathrm{Ad}^*_f j - \frac{c_1}{12}\mathsf{S}[f^{-1}] + \mathrm{ad}^*_\alpha \left[\mathrm{Ad}^*_f p - \frac{c_2}{12}\mathsf{S}[f^{-1}]\right] + \frac{c_2}{12}\mathsf{s}[\alpha], c_1; \mathrm{Ad}^*_f p - \frac{c_2}{12}\mathsf{S}[f^{-1}], c_2\right).\end{aligned} \tag{9.54}$$

Here it is understood that all Ad*'s and ad*'s on the right-hand side are centre*less* — they are the coadjoint representations of G and $\mathfrak{g}$, respectively.

Formula (9.54) looks a bit scary but it is crucial for our purposes, so let us briefly point out two of its important features. First, the central charges c_1, c_2 are left invariant by the action of the group, as expected. Second, note that the transformation law of momentum is

$$f \cdot p = \mathrm{Ad}^*_f p - \frac{c_2}{12}\mathsf{S}[f^{-1}], \tag{9.55}$$

where the Ad* on the right-hand side is that of G (not $\widehat{G}$). This formula says that p is invariant under translations (since it is unaffected by α) and that its transformation law is blind to the central charge c_1, but *not* to c_2. In fact, Eq. (9.55) is the coadjoint representation (6.104) of the centrally extended group $\widehat{G}$ at central charge c_2. As a corollary we can already conclude that the orbits of momenta labelling unitary representations of (9.39) are coadjoint orbits of the group $\widehat{G}$ at fixed central charge c_2; there is no need to master the much more complicated transformation law of angular momentum in (9.54) in order to classify such representations. This will have key consequences for the BMS$_3$ group below.

Remark Property (9.51) explains why we refer to the map (5.110) as a *cross product*. Indeed, the (double cover of the) Euclidean group in three dimensions is an exceptional semi-direct product $\mathrm{SU}(2) \ltimes_{\mathrm{Ad}} \mathfrak{su}(2)_{\mathrm{Ab}}$. Since the coadjoint representation of SU(2) is equivalent to the adjoint, one may identify vectors with covectors and the cross product (9.51) for the Euclidean group can be rewritten as $\alpha \times p = \mathrm{ad}_\alpha p = [\alpha, p]$. Here the Lie bracket is that of $\mathfrak{su}(2)$, so in components one has $(\alpha \times p)_i = \epsilon_{ijk}\alpha^j p^k$, which is the standard definition of the cross product in mechanics.

9.2.2 *Defining* BMS$_3$

Now that we are acquainted with exceptional semi-direct products, let us show how this structure occurs in three-dimensional BMS symmetry.

Centerless BMS$_3$ Group

Our first task is to move backwards from the centreless BMS$_3$ algebra (9.23) to the corresponding group. The algebra consists of pairs $\big(X(\varphi), \alpha(\varphi)\big)$, where $X(\varphi)\partial_\varphi$ is a vector field on the circle while $\alpha(\varphi)$ is a priori just a function on the celestial circle. These two quantities were referred to above as infinitesimal superrotations and supertranslations, respectively. Together, they generate finite transformations (9.12) of the cylinder at null infinity, where $f(\varphi)$ is a diffeomorphism of the circle. Thus we already know that the BMS$_3$ group consists of pairs (f, α), where f is a diffeomorphism of the circle while α is a function. It only remains to work out the group operation; the latter is given by the composition of two transformations (9.12):

$$(u, \varphi) \xmapsto{(g,\beta)} \big(g'(\varphi)u + \beta(g(\varphi)), g(\varphi)\big)$$
$$\xmapsto{(f,\alpha)} \Big(f'(g(\varphi))\big[g'(\varphi)u + \beta(g(\varphi))\big] + \alpha\big(f(g(\varphi))\big), f(g(\varphi))\Big).$$

Here the last result on the right-hand side can be rewritten as

$$\Big((f \circ g)'(\varphi)u + [\alpha + \mathrm{Ad}_f\beta]\Big|_{(f\circ g)(\varphi)}, (f \circ g)(\varphi)\Big) \tag{9.56}$$

where $\mathrm{Ad}_f\beta$ denotes the adjoint representation (6.17) of Diff(S^1) acting on β, that is, the transformation law of a vector field $\beta(\varphi)\partial_\varphi$ under $f(\varphi)$:

$$(\mathrm{Ad}_f\beta)\Big|_{f(\varphi)} = f'(\varphi)\beta(\varphi). \tag{9.57}$$

Expression (9.56) indicates three things:

1. The group operation of superrotations is given by composition (6.8); hence finite (as opposed to infinitesimal) superrotations span a group Diff(S^1).[5]
2. If it wasn't for superrotations, the group operation of supertranslations would just be addition, $\alpha \cdot \beta \equiv \alpha + \beta$. Thus supertranslations span an Abelian additive group whose elements are certain functions on the circle.
3. The action of superrotations on supertranslations is that of diffeomorphisms on vector fields, i.e. it is the adjoint representation (9.57) of Diff(S^1). In particular, supertranslations, which so far we thought of as functions $\alpha(\varphi)$ on the circle, should better be seen as vector fields $\alpha(\varphi)\partial_\varphi$. The only subtlety is that these vector fields do *not* generate diffeomorphisms of celestial circles, but rather

[5]As in Chap. 6 we describe diffeomorphisms of the circle by their lifts belonging to the universal cover $\widetilde{\mathrm{Diff}}^+(S^1)$, which we abusively denote simply as Diff(S^1).

angle-dependent translations (9.11) of retarded time u. Equivalently, each supertranslation is a density $\alpha = \alpha(\varphi)(d\varphi)^{-1}$ on the circle.

These observations motivate the following definition:

Definition The centreless *BMS group* in three dimensions is the exceptional semi-direct product

$$\boxed{\mathrm{BMS}_3 \equiv \mathrm{Diff}(S^1) \ltimes_{\mathrm{Ad}} \mathrm{Vect}(S^1)_{\mathrm{Ab}}} \tag{9.58}$$

where $\mathrm{Diff}(S^1)$ is the group of diffeomorphisms of the circle while $\mathrm{Vect}(S^1)_{\mathrm{Ab}}$ is its Lie algebra, seen as an Abelian vector group acted upon by $\mathrm{Diff}(S^1)$ according to the adjoint representation. Its elements are pairs (f, α); its group operation follows from the general definition (4.6) and is given by

$$(f, \alpha) \cdot (g, \beta) = \big(f \circ g, \alpha + \mathrm{Ad}_f \beta\big) \tag{9.59}$$

where Ad is the action (9.57) of $\mathrm{Diff}(S^1)$ on vector fields. With this definition the action (9.12) of BMS_3 on null infinity reproduces the group operation (9.59).

The BMS_3 group is infinite-dimensional and has the announced form (9.38), with $G = \mathrm{Diff}(S^1)$. Since we saw in Sect. 6.1 that $\mathrm{PSL}(2, \mathbb{R})$ is a subgroup of $\mathrm{Diff}(S^1)$, the Poincaré group is obviously a subgroup of BMS_3. We therefore introduce officially the following terminology:

Definition In the BMS_3 group (9.58), elements of $\mathrm{Diff}(S^1)$ are known as *superrotations* while elements of $\mathrm{Vect}(S^1)_{\mathrm{Ab}}$ are called *supertranslations*.

Remark The name "superrotation" has come to be standard, but the geometric interpretation of $\mathrm{Diff}(S^1)$ makes the terminology "superboosts" somewhat more appropriate. Indeed, recall from Sect. 6.1 that the group $\mathrm{Diff}(S^1)$ is homotopic to a circle, so that the only superrotations spanning a compact group are those conjugate to rigid rotations $f(\varphi) = \varphi + \theta$. The other one-parameter subgroups of $\mathrm{Diff}(S^1)$ are all non-compact and should be seen as boost groups.

Universal Cover of BMS_3

As in Sect. 6.1 we should be careful about what we mean by $\mathrm{Diff}(S^1)$. Strictly speaking, $\mathrm{Diff}(S^1)$ consists of all diffeomorphisms of the circle with the composition law (6.2); its connected subgroup $\mathrm{Diff}^+(S^1)$ consists of orientation-preserving diffeomorphisms. Since the group of supertranslations is a vector space, it is also connected and we define the *connected* BMS_3 group as

$$\mathrm{BMS}_3^+ \equiv \mathrm{Diff}^+(S^1) \ltimes_{\mathrm{Ad}} \mathrm{Vect}(S^1)_{\mathrm{Ab}}\,. \tag{9.60}$$

If we think of the group of superrotations as an extension of the Lorentz group in three dimensions, then $\mathrm{Diff}(S^1)$ corresponds to the disconnected orthochronous Lorentz group $\mathrm{O}(2, 1)^{\uparrow}$ while $\mathrm{Diff}^+(S^1)$ corresponds to the connected (orthochronous *and* proper) Lorentz group $\mathrm{SO}(2,1)^{\uparrow}$. It appears that no $\mathrm{Diff}(S^1)$ transformation corresponds to time reversal (which sounds reasonable since BMS symmetry is defined separately at future and past null infinity).

The group $\mathrm{Diff}^+(S^1)$ of orientation-preserving superrotations is homotopic to a circle, so it admits topological projective transformations that can be dealt with by trading it for its universal cover, $\widetilde{\mathrm{Diff}}^+(S^1)$. Since the vector group of supertranslations is homotopic to a point, the BMS_3 group has the homotopy type of a circle.

Definition The *universal cover of the BMS group* in three dimensions is the exceptional semi-direct product

$$\widetilde{\mathrm{BMS}}_3^+ \equiv \widetilde{\mathrm{Diff}}^+(S^1) \ltimes_{\mathrm{Ad}} \mathrm{Vect}(S^1)_{\mathrm{Ab}} \tag{9.61}$$

where $\widetilde{\mathrm{Diff}}^+(S^1)$ is the universal cover of the connected group $\mathrm{Diff}^+(S^1)$ and consists of $2\pi\mathbb{Z}$-equivariant superrotations (6.7).

In particular, exact representations of $\widetilde{\mathrm{BMS}}_3^+$ generally correspond to projective representations of BMS_3^+. The groups BMS_3, BMS_3^+ and $\widetilde{\mathrm{BMS}}_3^+$ are well-defined infinite-dimensional Lie-Fréchet groups. In what follows, motivated by quantum-mechanical applications, we always focus (implicitly) on the universal cover. Accordingly we abuse notation and denote the universal cover simply by BMS_3, neglecting the superscript "+" and the tilde.

Centrally Extended BMS_3 Group

In order to define the central extension of BMS_3, we apply the prescription (9.39) for centrally extended exceptional semi-direct products to the case $G = \mathrm{Diff}(S^1)$:

Definition The *centrally extended BMS group* in three dimensions is the exceptional semi-direct product

$$\boxed{\widehat{\mathrm{BMS}}_3 \equiv \widehat{\mathrm{Diff}}(S^1) \ltimes_{\widehat{\mathrm{Ad}}} \widehat{\mathrm{Vect}}(S^1)} \tag{9.62}$$

where $\widehat{\mathrm{Diff}}(S^1)$ is the (universal cover of the) Virasoro group.

Since this thesis is concerned with the group $\widehat{\mathrm{BMS}}_3$, let us make its definition a bit more explicit before going further. The elements of $\widehat{\mathrm{BMS}}_3$ are quadruples $\big(f,\lambda;\alpha,\mu\big)$ where f is a superrotation, α a supertranslation, while λ,μ are real numbers, extending Poincaré transformations as before. In $\widehat{\mathrm{BMS}}_3$, centrally extended superrotations (f,λ) span a Virasoro group while extended supertranslations (α,μ) span an infinite-dimensional Abelian group acted upon by superrotations according to the Virasoro adjoint representation. Explicitly, the group operation in $\widehat{\mathrm{BMS}}_3$ takes the form (9.41) where $f\cdot g = f\circ g$, while C is the Bott-Thurston cocycle (6.69) and S is the Schwarzian derivative (6.76). The pairing $\langle\cdot,\cdot\rangle$ is that of $\mathrm{Vect}(S^1)$ with its dual, given by (6.34).

The centreless BMS_3 group (9.58) is perfect, in the same way as $\mathrm{Diff}(S^1)$; this implies that it admits a universal central extension. As it turns out, this is precisely achieved by $\widehat{\mathrm{BMS}}_3$ (see Sect. 9.2.5 for the proof):

Theorem The centrally extended BMS$_3$ group (9.62) is the universal central extension of the centreless BMS$_3$ group (9.58).

9.2.3 Adjoint Representation and $\mathfrak{bms}_3$ Algebra

Lie Algebra

Since BMS$_3$ is an exceptional semi-direct product, its centreless Lie algebra takes the form $\mathfrak{g} \in \mathfrak{g}_{\text{Ab}}$, where $\mathfrak{g}$ is the Lie algebra of Diff(S^1):

$$\mathfrak{bms}_3 = \text{Vect}(S^1) \in_{\text{ad}} \text{Vect}(S^1)_{\text{Ab}} . \tag{9.63}$$

Its elements are pairs (X, α) where $X = X(\varphi)\partial_\varphi$ is an infinitesimal superrotation and $\alpha = \alpha(\varphi)(d\varphi)^{-1}$ an infinitesimal supertranslation. These functions determine the components of vector fields (9.17) generating asymptotic symmetries, so that elements of $\mathfrak{bms}_3$ can be seen as infinitesimal BMS$_3$ transformations. In particular the Poincaré subalgebra of $\mathfrak{bms}_3$ consists of pairs (X, α) whose only non-vanishing Fourier modes are the three lowest ones, as in (9.7)–(9.9). The centrally extended generalization (9.42) of this definition is immediate:

Definition The Lie algebra of $\widehat{\text{BMS}}_3$ is an exceptional semi-direct sum

$$\widehat{\mathfrak{bms}}_3 \equiv \widehat{\text{Vect}}(S^1) \in_{\widehat{\text{ad}}} \widehat{\text{Vect}}(S^1)_{\text{Ab}} . \tag{9.64}$$

Its elements are quadruples $(X, \lambda; \alpha, \mu)$ where $X = X(\varphi)\partial_\varphi$ is an infinitesimal superrotation, $\alpha = \alpha(\varphi)(d\varphi)^{-1}$ an infinitesimal supertranslation, while λ, μ are real numbers.

Adjoint Representation

The adjoint representation of the centreless BMS$_3$ group is given by formula (9.43), where the adjoint action of Diff(S^1) is the transformation law of vector fields (6.18). An important subtlety is that the Lie bracket appearing on the right-hand side is that of the Lie algebra of Diff(S^1) and is therefore the *opposite* (6.21) of the standard bracket of vector fields. Accordingly, in terms of the usual Lie bracket of vector fields on the circle one would write the adjoint representation of BMS$_3$ as

$$\text{Ad}_{(f,\alpha)}(X, \beta) = \big(\text{Ad}_f X, \text{Ad}_f \beta + [\text{Ad}_f X, \alpha]\big) , \tag{9.65}$$

with a plus sign instead of a minus sign in the second entry of (9.43). The centrally extended generalization of that expression is provided by Eq. (9.45), where S is the Schwarzian derivative (6.76), s is its infinitesimal version (6.74), and $\langle \cdot, \cdot \rangle$ is the standard pairing (6.34). Again, when writing the adjoint representation in terms of the standard Lie bracket of vector fields, the sign in front of the bracket of the third entry of (9.45) is a plus instead of a minus. Since we will not explicitly need the adjoint representation of the $\widehat{\text{BMS}}_3$ group, we do not display it here.

Lie Brackets

From the adjoint representation one can read off the Lie bracket of the $\widehat{\mathfrak{bms}}_3$ algebra. In order to absorb the minus sign of (6.21) we *define* the bracket to be

$$\big[(X,\lambda;\alpha,\mu),(Y,\rho;\beta,\nu)\big]\equiv-\frac{d}{dt}\,\widehat{\text{Ad}}_{(e^{tX},t\alpha)}(Y,\rho;\beta,\nu)\big|_{t=0}\,.$$

With this definition the Lie bracket in $\widehat{\mathfrak{bms}}_3$ takes the form (9.46) where the brackets on the right-hand side are standard Lie brackets of vector fields while c is the Gelfand–Fuks cocycle (6.43). This is consistent with the algebra of surface charges (9.36).

The Lie algebra structure can be made more apparent by writing the bracket (9.46) in a suitable basis. As in (9.24) we define the complex superrotation and supertranslation generators of the centreless $\mathfrak{bms}_3$ algebra,

$$j_m\equiv\left(e^{im\varphi}\partial_\varphi,0\right),\qquad p_m\equiv\left(0,e^{im\varphi}(d\varphi)^{-1}\right),\tag{9.66}$$

where the index m runs over all integers. Their brackets take the form (9.10). The corresponding basis of the centrally extended $\widehat{\mathfrak{bms}}_3$ algebra is

$$\begin{aligned}\mathcal{J}_m&\equiv\left(j_m,0;0,0\right)\overset{(9.66)}{=}\left(e^{im\varphi}\partial_\varphi,0;0,0\right),\\ \mathcal{P}_m&\equiv\left(0,0;p_m,0\right)\overset{(9.66)}{=}\left(0,0;e^{im\varphi}(d\varphi)^{-1},0\right),\end{aligned}\tag{9.67}$$

together with two central elements (9.47), i.e. $\mathcal{Z}_1=(0,1;0,0)$ and $\mathcal{Z}_2=(0,0;0,1)$. In these terms the centrally extended bracket (9.46) yields

$$\begin{aligned}i[\mathcal{J}_m,\mathcal{J}_n]&=(m-n)\mathcal{J}_{m+n}+\frac{\mathcal{Z}_1}{12}m^3\delta_{m+n,0}\,,\\ i[\mathcal{J}_m,\mathcal{P}_n]&=(m-n)\mathcal{P}_{m+n}+\frac{\mathcal{Z}_2}{12}m^3\delta_{m+n,0}\,,\\ i[\mathcal{P}_m,\mathcal{P}_n]&=0\,.\end{aligned}\tag{9.68}$$

Up to central terms this is of the same form as the asymptotic symmetry algebra (9.10), and it is consistent with the general form (9.48) for centrally extended exceptional semi-direct products. In the first line we see that superrotations close according to a Virasoro algebra (6.110) with central generator $\mathcal{Z}_1$, while the second line shows that brackets of superrotations with supertranslations take the Virasoro form with a different central element $\mathcal{Z}_2$. The algebra of surface charges (9.37) takes that form, with definite values $c_1=0$, $c_2=3/G$ for the central generators $\mathcal{Z}_1$, $\mathcal{Z}_2$.

Remark The canonical Poincaré subgroup of BMS$_3$ is the one spanned by superrotations (6.88) and supertranslations (9.7), or equivalently the one generated by basis elements j_m, p_m with $m = -1, 0, 1$. But in fact, BMS_3 admits infinitely many other Poincaré subgroups: each of them has a Lie algebra spanned by j_n, j_0, j_{-n} and p_n, p_0, p_{-n}, consisting of superrotations of the form (6.95) and supertranslations

$$\alpha(\varphi) = \alpha^0 - \alpha^1 \cos(n\varphi) - \alpha^2 \sin(n\varphi)$$

whose only non-vanishing Fourier modes are the zero-mode and the n^{th} modes.

9.2.4 Coadjoint Representation

Angular and Linear Supermomentum

The coadjoint vectors of $\widehat{BMS}_3$ are quadruples $\big(j, c_1; p, c_2\big)$ where $j = j(\varphi)d\varphi^2$ and $p = p(\varphi)d\varphi^2$ are quadratic densities on the circle, respectively dual to infinitesimal superrotations and supertranslations. The coefficients c_1 and c_2 are central charges. The pairing of $(j, c_1; p, c_2)$ with the Lie algebra $\widehat{\mathfrak{bms}}_3$ is given by formula (9.50), or explicitly

$$\big\langle (j, c_1; p, c_2), (X, \lambda, \alpha, \mu)\big\rangle = \frac{1}{2\pi}\int_0^{2\pi} d\varphi\big[j(\varphi)X(\varphi) + p(\varphi)\alpha(\varphi)\big] + c_1\lambda + c_2\mu\,.$$

The right-hand side of this expression coincides (up to central terms) with the surface charge (9.31). Inspired by the terminology of superrotations and supertranslations, we introduce the following nomenclature:

Definition Let (j, p) be a coadjoint vector for the BMS_3 group. Then $p = p(\varphi)d\varphi^2$ is called a *supermomentum* while $j = j(\varphi)d\varphi^2$ is an *angular supermomentum*.

The embedding of the Poincaré algebra in $\mathfrak{bms}_3$ suggests an interpretation for the lowest Fourier modes of

$$j(\varphi) = \sum_{m\in\mathbb{Z}} j_m e^{-im\varphi} \qquad \text{and} \qquad p(\varphi) = \sum_{m\in\mathbb{Z}} p_m e^{-im\varphi}. \tag{9.69}$$

Indeed, p_0 is dual to time translations and should be interpreted as the energy associated with $p(\varphi)$; similarly, the components p_1 and $p_{-1} = p_1^*$ are complex linear combinations of the spatial components of momentum. As for j_0, it is the angular momentum associated with $j(\varphi)$, while j_1 and j_{-1} are centre of mass charges. More generally, the function $p(\varphi)$ should be seen as an energy density on the circle — essentially a stress tensor — while $j(\varphi)$ is an angular momentum density on the circle. In particular, it is natural to give dimensions of energy to the function $p(\varphi)$ and the central charge c_2, while the function $j(\varphi)$ and the central charge c_1 are dimensionless. This interpretation is confirmed by the surface charges (9.31), since $p(\varphi)$ is a Bondi mass aspect while $j(\varphi)$ is an angular momentum aspect; furthermore the central charge (9.29) is indeed a mass scale.

Remark To our knowledge the terminology of "supermomentum" for duals of supertranslations dates back to [46], and has subsequently been used throughout the BMS literature (see e.g. [47–51]). The terminology of "angular supermomentum", on the other hand, seems to have first appeared in [52, 53] in relation to the problem of angular momentum, but apparently without direct relation to BMS symmetry. It was later independently introduced in [8] in the BMS context. In [51], angular supermomentum is referred to as "super centre of mass".

Coadjoint Representation

As in the Virasoro case, one should think of the pair (j, p) as the stress tensor of a BMS$_3$-invariant theory; its transformations under BMS$_3$ then coincide with the coadjoint representation, given for centrally extended exceptional semi-direct products by formula (9.54). In that expression, the central charges are invariant (as they should) while the transformation law of supermomentum coincides with the coadjoint representation (6.114) of the Virasoro group at central charge c_2:

$$\big(f \cdot p\big)\big(f(\varphi)\big) = \frac{1}{\big(f'(\varphi)\big)^2}\left[p(\varphi) + \frac{c_2}{12}\mathsf{S}[f](\varphi)\right], \tag{9.70}$$

where S denotes the Schwarzian derivative (6.76). We stress once more that supermomentum is left invariant by supertranslations, as it should.

The transformation law of angular supermomentum is a bit more involved and translates the fact that j is sensitive both to superrotations and to supertranslations, as it should since angular momentum and centre of mass charges are always defined with respect to an arbitrarily chosen origin. We refrain from describing this transformation law any further at this point, as we shall return to it in Sect. 9.3.1 when showing that the phase space of metrics (9.25) is a hyperplane at central charges $c_1 = 0$, $c_2 = 3/G$ embedded in the coadjoint representation of $\widehat{\text{BMS}}_3$.

Kirillov–Kostant Bracket

A prerequisite for showing that the asymptotically flat phase space is a coadjoint representation is to understand the Kirillov–Kostant Poisson bracket of the asymptotic symmetry group. Let us do this here for $\widehat{\text{BMS}}_3$; we proceed as in Sect. 6.4. Thus let $\big\{\mathcal{J}_m^*, \mathcal{P}_m^*, \mathcal{Z}_1^*, \mathcal{Z}_2^*\big\}$ be the dual basis corresponding to (9.67) and (9.47). Writing any coadjoint vector as

$$\big(j(\varphi)d\varphi^2, c_1; p(\varphi)d\varphi^2, c_2\big) = \sum_{m\in\mathbb{Z}} \big(j_m \mathcal{J}_m^* + p_m \mathcal{P}_m^*\big) + c_1 \mathcal{Z}_1^* + c_2 \mathcal{Z}_2^*\,,$$

the components $\{j_m, p_m, c_1, c_2\}$ are global coordinates on the dual space $\widehat{\mathfrak{bms}}_3{}^*$. Their Poisson brackets (5.28) take the form

$$
\begin{aligned}
i\{j_m, j_n\} &= (m-n)j_{m+n} + \frac{c_1}{12}m^3\delta_{m+n,0}\,,\\
i\{j_m, p_n\} &= (m-n)p_{m+n} + \frac{c_2}{12}m^3\delta_{m+n,0}\,,\\
i\{p_m, p_n\} &= 0\,.
\end{aligned}
\tag{9.71}
$$

As is obvious here, c_1 is a genuine Virasoro central charge for superrotations, while c_2 is the (generally dimensionful) central charge pairing superrotations with supertranslations. The surface charges of asymptotically flat gravity satisfy the exact same algebra (9.37), with the values of central charges $c_1 = 0$, $c_2 = 3/G$. We will return to this in Sect. 9.3.

*9.2.5 Some Cohomology**

To conclude our abstract description of BMS$_3$ symmetry, we now show that the centrally extended group (9.62) is in fact the universal central extension of the BMS$_3$ group (9.58). (In both cases $\text{Diff}(S^1)$ is understood to denote the universal cover of the group of orientation-preserving diffeomorphisms of the circle.) We use the notation of Sect. 2.2. Since the proof is very similar to the construction of the Gelfand–Fuks cocycle (6.43), this section may be skipped in a first reading.

The $\mathfrak{bms}_3$ algebra is perfect: it is equal to its Lie bracket with itself. This can be seen, for instance, by noting that the right-hand sides of the brackets (9.10) span all possible $\mathfrak{bms}_3$ generators. Accordingly it follows from (2.19) that the first real cohomology of $\mathfrak{bms}_3$ vanishes: $\mathcal{H}^1(\mathfrak{bms}_3) = 0$. Since the same is true of the centreless BMS$_3$ group, its central extension is universal, and it only remains to establish the second cohomology of $\mathfrak{bms}_3$.

Theorem The second real cohomology space of $\mathfrak{bms}_3$ is two-dimensional. It is generated by the classes of the two-cocycles

$$
\mathsf{c}_1\big((X,\alpha),(Y,\beta)\big) = \mathsf{c}(X,Y) \quad\text{and}\quad \mathsf{c}_2\big((X,\alpha),(Y,\beta)\big) = \mathsf{c}(X,\beta) - \mathsf{c}(Y,\alpha)
\tag{9.72}
$$

where c is the Gelfand–Fuks cocycle (6.43). Their expression in the basis (9.66) is

$$
\mathsf{c}_1(j_m, j_n) = \mathsf{c}_2(j_m, p_n) = -i\frac{m^3}{12}\delta_{m+n,0}\,,
\tag{9.73}
$$

while their other components vanish. As a consequence, the Lie algebra (9.64) is the universal central extension of (9.63), and the group (9.62) is the universal central extension of (9.58).

Proof The fact that the cocycle c_1 is the only non-trivial cocycle pairing superrotation generators with themselves follows from the fact that infinitesimal superrotations span a Witt subalgebra of $\mathfrak{bms}_3$. The considerations of Sect. 6.2 then carry over directly to $\mathfrak{bms}_3$. Now let us ask whether there exists a two-cocycle c such that $\mathsf{c}(p_m, p_n) \neq 0$. The cocycle identity (2.21) with trivial $\mathscr{T}$ implies

$$\mathsf{c}(j_0, [p_m, p_n]) + \tilde{\mathsf{c}}(p_m, [p_n, j_0]) + \tilde{\mathsf{c}}(p_n, [j_0, p_m]) = (m+n)\tilde{\mathsf{c}}(p_m, p_n) \stackrel{!}{=} 0\,,$$

where we used the Lie brackets (9.10). This yields $\tilde{\mathsf{c}}(p_m, p_n) = \tilde{c}_m \delta_{m+n,0}$ where the coefficients $\tilde{c}_m = -\tilde{c}_{-m}$ are to be determined. We now attempt to find a recursion relation for these coefficients; using the cocycle identity

$$\mathsf{c}(p_{-1}, [j_{-m+1}, p_m]) + \mathsf{c}(j_{-m+1}, [p_m, p_{-1}]) + \mathsf{c}(p_m, [p_{-1}, j_{-m+1}]) = 0\,,$$

the $\mathfrak{bms}_3$ algebra (9.10) implies $(2m-1)\tilde{c}_1 + (m-2)\tilde{c}_m = 0$. Since this must be true for all integer values of m we conclude that $\tilde{c}_1 = 0$, which in turn implies $\tilde{c}_m = 0$ for all $m \in \mathbb{Z}$. Thus, any two-cocycle on the $\mathfrak{bms}_3$ algebra has vanishing components $\tilde{\mathsf{c}}(p_m, p_n) = 0$. Finally, suppose that c is a two-cocycle on the $\mathfrak{bms}_3$ algebra and let us ask whether one can have $\mathsf{c}(j_m, p_n) \neq 0$. As in (6.46) we start by ensuring that the cocycle c is rotation-invariant by adding to it a suitable coboundary. Consider therefore the cocycle relation

$$\mathsf{c}(j_0, [j_m, p_n]) = \mathsf{c}([j_0, j_m], p_n) + \mathsf{c}(j_m, [j_0, p_n])\,.$$

The left-hand side can be interpreted as the differential of the one-cochain $\mathsf{k} = \mathsf{c}(j_0, \cdot)$ evaluated at (j_m, p_n), while the right-hand side is the Lie derivative of c with respect to j_0. Since the left-hand side is exact we know that the cohomology class of c is left invariant by rotations; in particular we can add to c the differential db of the one-cochain

$$\mathsf{b}(j_m) \equiv 0\,, \quad \mathsf{b}(p_m) \equiv \frac{i}{m}\mathsf{c}(j_0, p_m)\,,$$

which is such that

$$\mathcal{L}_{j_0}(\mathsf{c} + \mathsf{db})(j_m, p_n) = 0\,. \tag{9.74}$$

From now on we simply write c to denote $\mathsf{c} + \mathsf{db}$. Then, analogously to (6.50), Eq. (9.74) implies $(m+n)\mathsf{c}(j_m, p_n) = 0$ by virtue of the brackets (9.10). In particular we can now write $\mathsf{c}(j_m, p_n) = c_m \delta_{m+n,0}$ and it only remains to find the coefficients c_m. For this we derive a recursion relation using the cocycle identity

$$\mathsf{c}(p_1, [j_{-m-1}, j_m]) + \mathsf{c}(j_{-m-1}, [j_m, p_1]) + \mathsf{c}(j_m, [p_1, j_{-m-1}]) = 0\,,$$

which implies

$$(2m+1)c_{-1} + (m-1)c_{-m-1} + (m+2)c_m = 0 \tag{9.75}$$

by virtue of the $\mathfrak{bms}_3$ algebra (9.10). In particular we have $c_0 = 0$ and $c_1 = -c_{-1}$, which then gives $c_m = -c_{-m}$ and the recursion relation (9.75) can be rewritten as

$$c_{m+1} = \frac{(m+2)c_m - (2m+1)c_1}{m-1}\,.$$

This is the same relation as in the Virasoro case, Eq. (6.52). In particular it is solved by $c_m = m$ and $c_m = m^3$, the former being a coboundary. The result (9.73) follows. ■

9.3 The BMS$_3$ Phase Space

As explained in Sect. 8.3, the space of solutions of a Hamiltonian system coincides with its phase space. Accordingly the on-shell metrics (9.25) span the phase space of asymptotically flat gravity in three dimensions. Here we show that this space is a hyperplane at fixed central charges $c_1 = 0$, $c_2 = 3/G$ embedded in the coadjoint representation of the $\widehat{\text{BMS}}_3$ group. We also discuss this result from a holographic perspective and derive a positive energy theorem.

9.3.1 *Phase Space as a Coadjoint Representation*

The space of on-shell metrics (9.25) is spanned by pairs (j, p) transforming under BMS$_3$ according to (9.27)–(9.28). We now show that these formulas coincide with the coadjoint representation of the $\widehat{\mathfrak{bms}}_3$ algebra at central charges $c_1 = 0, c_2 = 3/G$. This is trivially true for the transformation law of $p(\varphi)$ since (9.28) coincides with the coadjoint representation (6.115) of the Virasoro algebra, which in turn is the infinitesimal version of the transformation law (9.70). As pointed out in Sect. 9.2.4, the case of the angular momentum aspect is more intricate since its coadjoint transformation law is the first entry on the right-hand side of (9.54). When applied to BMS$_3$, the latter formula must be modified slightly to match our conventions for Diff(S^1). Namely, due to the minus sign of the Lie bracket (6.21), the ad* of Eq. (9.54) should be replaced by $-$ad*. Taking this subtlety into account, the transformation law of angular supermomentum is

$$(f,\alpha)\cdot j = \text{Ad}^*_f j - \frac{c_1}{12}\mathsf{S}[f^{-1}] - \text{ad}^*_\alpha\left[\text{Ad}^*_f p - \frac{c_2}{12}\mathsf{S}[f^{-1}]\right] + \frac{c_2}{12}\mathsf{s}[\alpha]\,. \quad (9.76)$$

Here Ad* denotes the coadjoint representation (6.36) of Diff(S^1), S is the Schwarzian derivative (6.76), s is its infinitesimal cousin (6.74), and ad* is the infinitesimal coadjoint representation (6.37) so that $\text{ad}^*_\alpha\, p = \alpha p' + 2\alpha' p$. In order to relate formula (9.76) to the transformation law of the angular momentum aspect, we take an infinitesimal superrotation $f(\varphi) = \varphi + \epsilon X(\varphi)$, an infinitesimal supertranslation $\epsilon\,\alpha(\varphi)$, and define the variation of j by

$$\delta_{(X,\alpha)} j \equiv -\frac{(f, \epsilon\,\alpha)\cdot j - j}{\epsilon}\,.$$

As a result we obtain

$$\delta_{(X,\alpha)} j = Xj' + 2X'j - \frac{c_1}{12}X''' + \alpha p' + 2\alpha' p - \frac{c_2}{12}\alpha''',$$

which exactly coincides with (9.27) when $c_1 = 0$, as expected.

Using the fact that the Poisson algebra of surface charges (9.37) coincides with the Kirillov–Kostant bracket (9.71), we conclude that the (covariant) phase space of three-dimensional asymptotically flat gravity with BMS boundary conditions is a hyperplane $c_1 = 0$, $c_2 = 3/G$ embedded in the space of the coadjoint representation of the $\widehat{\text{BMS}}_3$ group and endowed with its Kirillov–Kostant Poisson bracket. This observation is the flat space analogue of the statement that the subleading components of an AdS space-time metric contain one-point functions of the dual CFT stress tensor. As in AdS, this observation should not come as a surprise. Indeed, the coadjoint representation of BMS$_3$ was bound to appear in the transformation law of the momentum map of the system, and it just so happens that this map is determined by the entries of the metric (9.25). The truly surprising aspect of this observation is the fact that it is the entries of the metric, and not some non-linear combinations thereof, that determine the momentum map. In particular, as in AdS$_3$, the set of solutions (9.25) is a vector space.

In view of these results, one may ask whether the subleading components of asymptotically flat metrics can be interpreted as the components of the stress tensor of some dual theory, similarly to AdS/CFT. The notion of "dual theory" appears to be elusive in the asymptotically flat case, essentially because the metric becomes degenerate at null infinity, but the question can be answered regardless of this complication. Indeed, whatever the dual theory is, it *must* be such that its stress tensor transforms under the coadjoint representation of the BMS$_3$ group (generally with some non-zero central charges), by virtue of the very nature of momentum maps. Accordingly, the stress tensor T of any BMS$_3$-invariant theory is necessarily such that $T_{uu} = p(\varphi)$ is a supermomentum generating supertranslations, while $T_{u\varphi} = j(\varphi)$ is an angular supermomentum generating superrotations.

This being said, it would be reassuring to have explicit field-theoretic illustrations of the fact that (j, p) actually *is* the stress tensor of some two-dimensional field theory. Such an illustration is provided by [34] (see also [54]), where a two-dimensional field theory invariant under BMS$_3$ was obtained thanks to the "dimensional reduction" of three-dimensional gravity through the Chern–Simons formalism. As expected, the stress tensor of that theory is a pair (j, p) that coincides with the functions specifying the metric (9.25), and whose BMS$_3$ transformations exactly take the form of the coadjoint representation (9.54) with central charges $c_1 = 0$, $c_2 = 3/G$ [55]. The higher-spin [56] and supersymmetric [57, 58] generalizations of these considerations confirm this statement, so known examples of BMS$_3$-invariant field theories do support our claim that the functions (j, p) coincide with the components of a "dual" stress tensor.

9.3.2 Boundary Gravitons and BMS_3 *Orbits*

If one picks a metric (9.25) at random, the pair $\big(j(\varphi), p(\varphi)\big)$ is most likely to consist of functions that are not constant on the circle. This is actually implied by BMS_3 symmetry: if we let (j, p) be any seed solution (with j, p constant or not), the set of metrics obtained from it by asymptotic symmetry transformations spans an infinite-dimensional coadjoint orbit of the $\widehat{\mathrm{BMS}}_3$ group at central charges $c_1 = 0$, $c_2 = 3/G$,

$$\mathcal{W}_{(j,c_1;p,c_2)}\,. \tag{9.77}$$

The metrics belonging to this orbit are infinite-dimensional analogues of Poincaré transforms of the state of a particle with momentum p and angular momentum j. As in Sect. 8.3 one may refer to the orbit (9.77) as a space of classical "boundary gravitons" around the background (j, p).

The fact that the phase space of flat gravity coincides with (a hyperplane in) the coadjoint representation of $\widehat{\mathrm{BMS}}_3$ allows us to use the orbits (9.77) as an organizing principle. As in Fig. 8.6, the space of solutions is foliated into disjoint $\widehat{\mathrm{BMS}}_3$ orbits, each of which is a symplectic manifold. Since the classification of coadjoint orbits of $\widehat{\mathrm{BMS}}_3$ follows from the results of Sect. 5.4, we may claim to control the full covariant phase space of asymptotically flat gravity. In particular the classification of zero-mode solutions in Fig. 9.1 is a first step towards the full classification: each point in the plane (J, M) determines an orbit (9.77), and different points define disjoint orbits. Since not all orbits have constant representatives, Fig. 9.1 is an incomplete representation of the full phase space of the system. The complete picture would involve the BMS_3 analogue of Fig. 7.3. Note that the relation between metrics and $\widehat{\mathrm{BMS}}_3$ orbits hints that the quantization of asymptotically flat gravity produces unitary representations of BMS_3. We will investigate this proposal in Chaps. 10 and 11.

9.3.3 Positive Energy Theorem

Positive energy theorems in general relativity are commonly formulated in asymptotically flat space-times, so we are now in position to address the three-dimensional version of that problem. The question that we wish to ask is the following: which asymptotically flat metrics (9.25) have energy bounded from below under BMS_3 transformations?

The answer follows from the fact that asymptotically flat metrics transform under BMS_3 according to the coadjoint representation (9.54). For our purposes the key property of that formula is the fact that the transformation law of p is blind to supertranslations. In this sense the positive energy theorem in three-dimensional flat space is even simpler than in AdS_3:

Positive energy theorem The asymptotically flat metric (j, p) has energy bounded from below under BMS_3 transformations if and only if p belongs to a Virasoro coadjoint orbit (at central charge $c_2 = 3/G$) with energy bounded from below. This is to say that either p is superrotation-equivalent to a constant $p_0 \geq -c_2/24$, or p belongs to the unique massless Virasoro orbit with bounded energy.

As a corollary, we now know that all conical deficits and all flat space cosmologies have energy bounded from below under BMS_3 transformations. By contrast, all conical excesses have energy unbounded from below.

9.4 Flat Limits

There are many similarities between the asymptotic symmetries of three-dimensional Anti-de Sitter and flat space-times. Intuitively, this is because the limit $\ell \to +\infty$ (i.e. $\Lambda \to 0$) of AdS_3 is just Minkowski space. It is tempting to ask if the phenomenon can be formulated in a mathematically precise way such that the conclusions of Sect. 9.1 follow from those of Sect. 8.2 by a suitably defined flat limit. This question was addressed in [59], and the answer is yes. In short, upon reformulating Brown–Henneaux boundary conditions in Bondi-like coordinates at null (rather than spatial) infinity, the Minkowskian asymptotic Killing vectors (9.17), the on-shell metrics (9.25) and the surface charges (9.31) are flat limits of their AdS_3 counterparts displayed in Eqs. (8.30), (8.38) and (8.42) respectively. In particular, BMS_3 symmetry may be seen as a limit of two-dimensional conformal symmetry.

In this section we explore this flat limit from the point of view of group theory, starting from the definition of the AdS_3 asymptotic symmetry group as a set of conformal transformations of a time-like cylinder. We describe the limit at the level of groups, then at the level of Lie algebras, and finally at the level of the coadjoint representation. We end by pointing out a different contraction that produces the Galilean conformal algebra in two dimensions. Considerations related to flat limits of unitary representations are relegated to Sect. 10.2.

9.4.1 *From* Diff(S^1) *to* BMS_3

In Anti-de Sitter space, spatial infinity coincides with null infinity. This observation allows one to reformulate Brown–Henneaux boundary conditions (originally defined at spatial infinity) in terms of Bondi-like coordinates (r, φ, u) at null infinity [59]. The conclusion of this reformulation is that AdS_3 results take the same form as in the standard Fefferman–Graham gauge, up to the replacement of the time coordinate t by a retarded time coordinate u. In particular one can introduce light-cone coordinates

$$x^{\pm} \equiv \frac{u}{\ell} \pm \varphi \tag{9.78}$$

in terms of which the asymptotic symmetry group acts on the cylinder at (null) infinity according to conformal transformations (8.37). Our goal here is to start from these transformations and rediscover the BMS_3 transformations (9.12).

The way to go is to expand everything in powers of a "small" parameter $\epsilon = 1/\ell$. In practice ℓ is dimensionful so it makes no sense to think of it as being "large"; a more precise statement would be that the dimensionless Brown–Henneaux central charge, proportional to ℓ/G, must go to infinity. Despite this subtlety we will keep referring to ℓ as a "large" parameter, keeping in mind that there exists a more precise formulation of the procedure.

In order to distinguish Virasoro elements from those of BMS_3, we denote elements of the group $\text{Diff}(S^1) \times \text{Diff}(S^1)$ as pairs $(\mathcal{F}, \bar{\mathcal{F}})$ where $\mathcal{F}$ and $\bar{\mathcal{F}}$ are lifts of orientation-preserving diffeomorphisms of the circle satisfying the conditions (6.7). Let then $(\mathcal{F}, \bar{\mathcal{F}})$ be a conformal transformation of the cylinder with coordinates (9.78). In the large ℓ limit the transformation of the angular coordinate φ becomes

$$\varphi \mapsto \frac{1}{2}\big(\mathcal{F}(x^+) - \bar{\mathcal{F}}(x^-)\big) \overset{\ell\to+\infty}{\to} \frac{1}{2}\big(\mathcal{F}(\varphi) - \bar{\mathcal{F}}(-\varphi)\big), \tag{9.79}$$

where the combination of $\mathcal{F}$'s on the far right-hand side was obtained by Taylor-expanding functions around $\pm\varphi$ in terms of the small parameter u/ℓ, and neglecting all terms of order $\mathcal{O}(1/\ell)$. The combination of diffeomorphisms in (9.79) is itself a (lift of a) diffeomorphism of the circle. Indeed one readily verifies that

$$f(\varphi) \equiv \frac{1}{2}(\mathcal{F}(\varphi) - \bar{\mathcal{F}}(-\varphi)) \tag{9.80}$$

satisfies the conditions (6.7) when $\mathcal{F}$ and $\bar{\mathcal{F}}$ do. Let us investigate what happens with the time coordinate u in the same limit. Using (9.80) we find

$$u \mapsto \frac{\ell}{2}\big(\mathcal{F}(x^+) + \bar{\mathcal{F}}(x^-)\big) \overset{\ell\to+\infty}{\to} \frac{\ell}{2}\big(\mathcal{F}(\varphi) + \bar{\mathcal{F}}(-\varphi)\big) + f'(\varphi)u,$$

where all terms $\mathcal{O}(1/\ell)$ were neglected once more. The first term on the far right-hand side is potentially divergent: typical diffeomorphisms are independent of ℓ, so the first term goes to infinity in the large ℓ limit. Note, however, that the combination $\mathcal{F}(\varphi) + \bar{\mathcal{F}}(-\varphi)$ is 2π-periodic. Thus, in order for the limit $\ell \to +\infty$ to work we require that there be a finite, ℓ-independent function α on the circle such that

$$\mathcal{F}(\varphi) + \bar{\mathcal{F}}(-\varphi) \equiv \frac{2}{\ell}\,\alpha(f(\varphi)) + \mathcal{O}(1/\ell^2) \tag{9.81}$$

where the argument of α is taken to be $f(\varphi)$ for convenience. This is to say that the diffeomorphisms $\mathcal{F}$ and $\bar{\mathcal{F}}$ are required to depend on ℓ in such a way that

$$\bar{\mathcal{F}}(-\varphi) = -\mathcal{F}(\varphi) + \mathcal{O}(1/\ell). \tag{9.82}$$

For instance, if $\mathcal{F}(\varphi) = \varphi + \theta$ and $\bar{\mathcal{F}}(\varphi) = \varphi + \bar{\theta}$ are rotations, this condition says that $\theta - \bar{\theta}$ goes to zero at least as fast as $1/\ell$ in the large ℓ limit. With this choice the transformation law of u reduces to $u \mapsto f'(\varphi)u + \alpha(f(\varphi))$. Including (9.79), we have thus reproduced the BMS$_3$ transformations (9.12) from a flat limit of $\text{Diff}(S^1) \times \text{Diff}(S^1)$. In this sense the centreless BMS$_3$ group (9.58) is a flat limit of the asymptotic symmetry group of AdS$_3$ with Brown–Henneaux boundary conditions. In particular superrotations arise in the form (9.80) while supertranslations (9.81) measure how fast $\bar{\mathcal{F}}(\varphi)$ goes to $-\mathcal{F}(-\varphi)$ as ℓ goes to infinity. Similar considerations would reproduce the centrally extended BMS$_3$ group (9.62) as a contraction of the direct product of two Virasoro groups.

Note that the condition (9.82) does not imply that there are less elements in the BMS$_3$ group than in the group $\text{Diff}(S^1) \times \text{Diff}(S^1)$. Indeed, both groups are infinite-dimensional Lie groups consisting of two spaces of functions on the circle and have the same cardinality in this sense.

9.4.2 *From Witt to* $\mathfrak{bms}_3$

The limit from $\text{Diff}(S^1) \times \text{Diff}(S^1)$ to BMS$_3$ can be reformulated in terms of Lie algebras. Again, our notation will be slightly different from that of the previous chapters so as to distinguish AdS$_3$ quantities from Minkowskian quantities. Thus we consider a vector field $\mathcal{X}(x^+)\partial_+ + \bar{\mathcal{X}}(x^-)\partial_-$ on a two-dimensional cylinder and use $\partial_\pm = \frac{1}{2}(\ell\partial_u \pm \partial_\varphi)$ to rewrite it as

$$\frac{\ell}{2}\big(\mathcal{X}(x^+) + \bar{\mathcal{X}}(x^-)\big)\partial_u + \frac{1}{2}\big(\mathcal{X}(x^+) - \mathcal{X}(x^-)\big)\partial_\varphi\,. \tag{9.83}$$

In the flat limit $\ell \to +\infty$ the angular component becomes

$$\frac{1}{2}\big(\mathcal{X}(x^+) - \bar{\mathcal{X}}(x^-)\big) \overset{\ell\to+\infty}{\to} \frac{1}{2}\big(\mathcal{X}(\varphi) - \bar{\mathcal{X}}(-\varphi)\big) \equiv X(\varphi) \tag{9.84}$$

where $X(\varphi)$ is some function on the circle, later to be interpreted as (the component of) a superrotation generator. For the time component one finds

$$\frac{\ell}{2}\big(\mathcal{X}(x^+)+\bar{\mathcal{X}}(x^-)\big) \overset{\ell\to+\infty}{\to} \frac{\ell}{2}\big(\mathcal{X}(\varphi)+\bar{\mathcal{X}}(-\varphi)\big)+uX'(\varphi) \equiv \alpha(\varphi)+uX'(\varphi) \tag{9.85}$$

where we have once more introduced a function α on the circle, later to be interpreted as (a component of) a supertranslation generator. This time the requirement is

$$\mathcal{X}(\varphi) + \bar{\mathcal{X}}(-\varphi) = \frac{2}{\ell}\,\alpha(\varphi) \tag{9.86}$$

with a finite, ℓ-independent α, and is directly analogous to the condition (9.81). All in all we find that, in the flat limit, the vector field (9.83) turns into

$$\xi_{(X,\alpha)} \equiv X(\varphi)\partial_\varphi + \big(\alpha(\varphi) + uX'(\varphi)\big)\partial_u$$

and thus coincides with the leading non-radial components of the asymptotic Killing vector field (9.17). The Lie brackets of such vector fields satisfy the centreless $\mathfrak{bms}_3$ algebra; the latter is thus a flat limit of the direct sum of two Witt algebras. Note that from this perspective the fact that supertranslations have dimensions of length follows from the fact (9.86) that $\alpha(\varphi)$ is proportional to ℓ.

The limit from Witt to $\mathfrak{bms}_3$ can also be formulated in terms of commutation relations. Indeed, let $\ell_m = e^{imx^+}\partial_+$ and $\bar{\ell}_m = e^{imx^-}\partial_-$ denote the generators of two commuting Witt algebras (6.24). Then the correspondence (9.84)–(9.86) instructs us to define would-be superrotation and supertranslation generators

$$j_m \equiv \ell_m - \bar{\ell}_{-m}\,, \qquad p_m \equiv \frac{1}{\ell}(\ell_m + \bar{\ell}_{-m})\,. \tag{9.87}$$

The terminology here is consistent with the fact that, on the cylinder, $\ell_0 - \bar{\ell}_0$ generates rotations while $\ell_0 + \bar{\ell}_0$ generates time translations. In the basis (9.87), the commutation relations of the direct sum of two Witt algebras take the form

$$i[j_m, j_n] = (m-n)\,j_{m+n}\,, \qquad i[j_m, p_n] = (m-n)\,p_{m+n}\,, \qquad i[p_m, p_n] = \frac{1}{\ell^2}(m-n)\,j_{m+n}\,. \tag{9.88}$$

In the limit $\ell \to +\infty$ the last bracket vanishes and the algebra reduces to (9.10), reproducing $\mathfrak{bms}_3$ as expected. The same argument can be applied to the direct sum of two Virasoro algebras and gives rise to the centrally extended $\widehat{\mathfrak{bms}_3}$ algebra (see Eq. (9.93) below).

This observation can be used to define "flat limits" of Lie algebras in general terms. Consider indeed the Lie algebra $\mathfrak{g} \oplus \mathfrak{g}$, whose generators we denote t_a and $\bar{t}_a$, with identical commutation relations (5.2) in both sectors:

$$[t_a, t_b] = f_{ab}{}^c\, t_c\,, \qquad [\bar{t}_a, \bar{t}_b] = f_{ab}{}^c\, \bar{t}_c\,. \tag{9.89}$$

Then consider the redefinitions

$$j_a \equiv t_a + \bar{t}_a\,, \qquad p_a \equiv \frac{1}{\ell}(t_a - \bar{t}_a) \tag{9.90}$$

where ℓ is some length scale that we will eventually let go to infinity. In terms of j's and p's the commutation relations (9.89) become

$$[j_a, j_b] = f_{ab}{}^c\, j_c\,, \qquad [j_a, p_b] = f_{ab}{}^c\, p_c, \qquad [p_a, p_b] = \frac{1}{\ell^2} f_{ab}{}^c\, j_c \tag{9.91}$$

and the limit $\ell \to +\infty$ reproduces the commutation relations (9.48) of exceptional semi-direct sums (without central terms). Thus, the flat limit of any group $G \times G$ is an exceptional semi-direct product $G \ltimes_{\text{Ad}} \mathfrak{g}_{\text{Ab}}$. The flat limit (9.87) giving rise to $\mathfrak{bms}_3$ from two copies of the Witt algebra is a special case of that construction. Indeed, the map

$$\ell_m \mapsto -\ell_{-m} \tag{9.92}$$

is a Lie algebra isomorphism when the ℓ_m's generate a Witt algebra (6.24), so the redefinitions (9.87) precisely take the form (9.90) with the correspondence $t_a \leftrightarrow \ell_m$ and $\bar{t}_a \leftrightarrow -\bar{\ell}_{-m}$. As it turns out, all symmetry algebras found so far in the realm of asymptotically flat field theories in three dimensions can be seen as flat limits of the type just described when compared to their AdS$_3$ counterparts.

The limiting procedure that turns the sum of two Witt algebras into $\mathfrak{bms}_3$ is an example of *Inönü-Wigner contraction* [60], similar to the relation between the Poincaré group and the Galilei group. Conversely, the direct sum of two Witt algebras is a deformation of $\mathfrak{bms}_3$. The same construction can be used to show that the Poincaré algebra is a flat limit of the AdS$_3$ isometry algebra, $\mathfrak{so}(2,2) \cong \mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{sl}(2,\mathbb{R})$. We will return to this in Sect. 10.2.

9.4.3 Stress Tensors and Central Charges

We now apply the flat limit to the coadjoint representation of two Virasoro groups, generally with non-zero central charges. Let therefore $T(x^+)$ and $\bar{T}(x^-)$ be CFT stress tensors transforming under left and right conformal transformations as Virasoro coadjoint vectors with central charges c and $\bar{c}$, respectively. (In (8.38) we denoted these stress tensors as p, $\bar{p}$, but here we keep the letter p for supermomenta.) They are paired with vector fields $\mathcal{X}(x^+)\partial_+ + \bar{\mathcal{X}}(x^-)\partial_-$ on the cylinder according to

$$\big\langle (T,\bar{T}),(\mathcal{X},\bar{\mathcal{X}})\big\rangle = \frac{1}{2\pi}\int_0^{2\pi} d\varphi\big[T(x^+)\mathcal{X}(x^+) + \bar{T}(x^-)\bar{\mathcal{X}}(x^-)\big]$$

which (up to notation) is just the AdS$_3$ surface charge (8.42). As above we expand the functions T and $\bar{T}$ in powers of $1/\ell$ and we define

$$p(\varphi) \equiv \lim_{\ell\to+\infty}\frac{1}{\ell}\big(T(x^+) + \bar{T}(x^-)\big), \qquad j(\varphi) + up'(\varphi) \equiv \lim_{\ell\to+\infty}\big(T(x^+) - \bar{T}(x^-)\big).$$

Using (9.85) and (9.86) one then verifies that, in the limit $\ell \to +\infty$,

$$T(x^+)\mathcal{X}(x^+) + \bar{T}(x^-)\bar{\mathcal{X}}(x^-) = j(\varphi)X(\varphi) + p(\varphi)\alpha(\varphi) + u(pX)'(\varphi),$$

which coincides up to a total derivative with the integrand of the flat surface charge (9.31). In other words the surface charges of flat space gravity are large ℓ limits of those of AdS_3. In mathematical terms this is to say that the flat limit of the coadjoint representation of the direct product of two Virasoro groups is the coadjoint representation of the (centrally extended) BMS_3 group.

This phenomenon also allows us to relate the central charges of the Virasoro algebra to those of $\widehat{\mathfrak{bms}}_3$. (We could have done this in terms of abstract Lie algebra generators, but for comparison with three-dimensional gravity we do it here in terms of coadjoint vectors.) Let us consider two Virasoro algebras with central charges c and $\bar{c}$ that depend on ℓ as

$$c = A\ell + B + \mathcal{O}(1/\ell), \qquad \bar{c} = A\ell + \bar{B} + \mathcal{O}(1/\ell)$$

where A, B and $\bar{B}$ are ℓ-independent. Then the definitions

$$c_1 \equiv \lim_{\ell\to+\infty} (c - \bar{c}), \qquad c_2 \equiv \lim_{\ell\to+\infty} \frac{c + \bar{c}}{\ell} \tag{9.93}$$

allows us to write the flat limit of the algebra in the $\widehat{\mathfrak{bms}}_3$ form (9.71) in terms of generators (j_m, p_m) related to Virasoro generators $(\ell_m, \bar{\ell}_m)$ by (9.87). This is the centrally extended analogue of the flat limit described in (9.91). Note that for the Brown–Henneaux central charges (8.40) the prescription (9.93) yields $c_1 = 0$ and $c_2 = 3/G$, which are indeed the standard values for asymptotically flat space-times.

Remark The fact that flat space holography can be studied as a flat limit of the AdS/CFT correspondence is an old idea [61, 62]; see also [63–65]. Here we have described its group-theoretic formulation. It should be noted, however, that there is no known limiting construction that yields BMS symmetry in *four* dimensions from some corresponding asymptotic symmetry in AdS_4.

9.4.4 The Galilean Conformal Algebra

The $\mathfrak{bms}_3$ algebra turns out to be isomorphic to the Galilean conformal algebra in two dimensions. We now explain how the latter can be obtained as a non-relativistic contraction of two Witt algebras and discuss the extent to which Galilean conformal symmetry applies to asymptotically flat gravity in three dimensions. As in the earlier sections of this chapter we work only at the classical level. The quantum version of these considerations will be exposed in Sect. 10.2.

The redefinitions (9.90) suggest a contraction of Witt algebras that differs from the flat limit (9.87). Namely, instead of performing the involution (9.92) before taking the limit $\ell \to +\infty$, one can define

$$\tilde{j}_m \equiv \bar{\ell}_m + \ell_m\,, \qquad \tilde{p}_m \equiv \frac{1}{\ell}(\bar{\ell}_m - \ell_m)\,. \tag{9.94}$$

In contrast to (9.87), this redefinition has nothing to do with the flat limit of AdS$_3$, but the limit $\ell \to +\infty$ still gives rise to an algebra with commutation relations (9.10) upon renaming $j_m \to \tilde{j}_m$ and $p_m \to \tilde{p}_m$. The key difference is that now the generator of time translations is $\tilde{j}_0$ (since it coincides with $\ell_0 + \bar{\ell}_0$) while $\tilde{p}_0$ generates rotations (since it is proportional to $\ell_0 - \bar{\ell}_0$). More generally, with the redefinition (9.94), the generators of would-be supertranslations do not commute while those of would-be superrotations do commute. This is the opposite of the behaviour of superrotations and supertranslations in three-dimensional Einstein gravity.

The redefinitions (9.94) can be interpreted as a non-relativistic contraction of the direct sum of two Witt algebras in two dimensions, analogous to the usual Inönü-Wigner contraction of the Poincaré algebra to the Galilei algebra. For this reason the algebra spanned by $\tilde{j}_m$'s and $\tilde{p}_m$'s is known as the *Galilean conformal algebra* in two dimensions [66, 67]. It is the non-relativistic limit of the conformal algebra in two dimensions and, by a geometric coincidence, it is isomorphic to $\mathfrak{bms}_3$. The Galilean conformal algebra has been extensively studied in its own right; see e.g. [68] for its supersymmetric extension and [66, 69] for its highest-weight representations. It is a fundamental tool in the non-relativistic limit of the AdS/CFT correspondence [70]. In what follows we denote it by $\mathfrak{gca}_2$.

At some point the isomorphism $\mathfrak{gca}_2 \cong \mathfrak{bms}_3$ led to the proposal that flat space holography (in three space-time dimensions) is described by a Galilean conformal field theory [67, 71]. In view of the geometric interpretation of superrotations and supertranslations described above, this sounds suspicious: the Galilean conformal algebra is a version of the $\mathfrak{bms}_3$ algebra "rotated by 90 degrees" where the roles of the Hamiltonian and angular momentum are exchanged. In particular the flat limit of AdS$_3$/CFT$_2$, if it exists, should not give rise to a Galilean conformal field theory since the gravitational flat limit (9.87) of two Witt algebras gives rise to standard $\mathfrak{bms}_3$, in which p_0 generates time translations. Nevertheless, at the level of *classical* symmetries, there is essentially no distinction between $\mathfrak{bms}_3$ and $\mathfrak{gca}_2$; the two are interchangeable. This coincidence led to many publications concerned with flat space holography and attempting to describe its dual theory as a Galilean conformal field theory; see e.g. [72–74] and references therein. One of the goals of this thesis is to explain why the dual theory of asymptotically flat gravity, if it exists at all, cannot be a Galilean conformal field theory. The reason for this is rooted in the elementary observation that the correspondence $\mathfrak{bms}_3 \leftrightarrow \mathfrak{gca}_2$ exchanges the Hamiltonian and the angular momentum, but we will go much beyond that. In fact we shall see that the difference between $\mathfrak{bms}_3$ and $\mathfrak{gca}_2$, while classically invisible, becomes apparent at the *quantum* level. This will rely on the induced representations developed in the next chapter and will be studied in much greater detail in Sect. 10.2. As it turns out, the most striking illustration of this distinction will arise in Sect. 11.2 in the realm of quantum higher-spin theories.

This being said, we stress that discarding Galilean conformal field theories as putative duals for asymptotically flat gravity does *not* rule out all the conclusions of the substantial literature on flat space holography approached from the Galilean side. Rather, the point we wish to make is that those computations that did work in flat space while relying on $\mathfrak{gca}_2$ symmetry would have worked equally well in the language of BMS_3 symmetry. More precisely, any computation that holds for $\mathfrak{gca}_2$ but does not rely on its realization as a quantum symmetry algebra also holds for $\mathfrak{bms}_3$, and therefore for asymptotically flat gravity.

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Chapter 10
Quantum BMS_3 Symmetry

This chapter is devoted to irreducible unitary representations of the BMS_3 group, i.e. BMS_3 particles, which we classify and interpret. As we shall see, the classification is provided by supermomentum orbits that coincide with coadjoint orbits of the Virasoro group. Upon identifying supermomentum with the Bondi mass aspect of asymptotically flat metrics, we will be led to interpret BMS_3 particles as relativistic particles dressed with gravitational degrees of freedom.

The plan is as follows. In Sect. 10.1 we classify BMS_3 particles according to orbits of supermomenta under superrotations. We also describe and interpret the resulting Hilbert spaces of wavefunctions, which we relate to the quantization of (coadjoint) orbits of asymptotically flat metrics under BMS_3. Section 10.2 is devoted to the description of BMS_3 particles as representations of the (centrally extended) $\mathfrak{bms}_3$ algebra and their relation to highest-weight representations of the Virasoro algebra; we also briefly touch upon Galilean representations. Finally, in Sect. 10.3 we evaluate characters of BMS_3 particles. To lighten the notation, from now on the words "BMS_3 group" or "$\mathfrak{bms}_3$ algebra" implicitly refer to their centrally extended versions (except if stated otherwise). We also abuse notation by writing $\text{Diff}(S^1)$ to refer either to $\text{Diff}^+(S^1)$ or to $\widetilde{\text{Diff}}^+(S^1)$, depending on the context.

Most of the results exposed in this chapter have been reported in the papers [1–4]. The relation between BMS_3 particles and gravitational one-loop partition functions [5, 6] will be described in the next chapter. Note that the considerations that follow rely heavily on the material of Chap. 4.

10.1 BMS_3 Particles

In high-energy physics a *particle* is usually defined as an irreducible unitary representation of the Poincaré group. If one takes BMS symmetry seriously, it is tempting to apply the same terminology to representations of BMS. Accordingly, in this section our goal is to answer the following question:

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Replace the word "Poincaré" by "BMS" in the definition of a particle.
What new notion of particle does one then obtain?
What new quantum numbers describe its degrees of freedom?

In principle this problem should be addressed in the realistic four-dimensional world. However, as mentioned in the introduction of this thesis, BMS symmetry in four dimensions is very poorly understood at present, so we will content ourselves with the more modest task of understanding irreducible unitary representations of BMS in three dimensions — that is, *BMS_3 particles*. Remarkably, we will discover that BMS_3 particles are labelled by mass and spin, exactly as standard relativistic particles. As in Sect. 8.4, we will interpret their extra degrees of freedom as boundary gravitons, or equivalently soft gravitons.

The plan of this relatively long section is the following. We first describe the supermomentum orbits and little groups that classify BMS_3 particles. We shall see that these orbits are in fact coadjoint orbits of the Virasoro group, which will allow us to define massive, massless and tachyonic BMS_3 particles. We also discuss the existence of integration measures on supermomentum orbits, since such measures are required to define scalar products of wavefunctions. We then describe the states represented by such wavefunctions and interpret them as particles dressed with quantized gravitational degrees of freedom, in accordance with the relation between asymptotically flat metrics and the coadjoint representation of BMS_3. We also apply this interpretation to the vacuum representation and to spinning BMS_3 particles, and we conclude by discussing the extension of our considerations to four space-time dimensions.

10.1.1 Orbits and Little Groups

Our goal is to understand the quantum-mechanical implementation of BMS_3 symmetry, at least as far as irreducible representations are concerned. According to Sect. 2.1 we should leave room for projective representations; to do this we consider exact representations of the universal cover of the universal central extension of the connected BMS_3 group, that is, $\widehat{\mathrm{BMS}}_3$. The latter was defined in (9.62). Since $\widehat{\mathrm{BMS}}_3$ is a semi-direct product, one expects all its irreducible unitary representations to be induced à la Wigner. These representations are classified by the orbits and little groups described in general terms in Sect. 4.1. Here we perform that classification.

Supermomentum Orbits

The key ingredient in the description of BMS_3 particles is the dual of the space of supertranslations, $\widehat{\mathrm{Vect}}(S^1)^*_{\mathrm{Ab}}$. Following the terminology of Sect. 9.2, its elements are *centrally extended supermomenta*

$$\big(p(\varphi)d\varphi^2, c_2\big) \tag{10.1}$$

paired with centrally extended supertranslations (α, λ) according to (6.111) with the replacements $X \to \alpha$ and $c \to c_2$. As mentioned below (6.69), $p(\varphi)$ has dimensions of energy; its three lowest Fourier modes form a Poincaré energy-momentum vector (in particular the zero-mode is the energy of p). More generally $p(\varphi)$ is an energy density on the circle while the central charge c_2 is an energy scale. Supermomentum transforms as a Virasoro coadjoint vector (9.70) under superrotations, so the allowed supermomenta of a BMS$_3$ particle span a coadjoint orbit of the Virasoro group at central charge c_2. This is the first key conclusion of this section:

Theorem The orbit $\mathcal{O}_p$ of a supermomentum (p, c_2) under superrotations is a coadjoint orbit of the Virasoro group at central charge c_2.

When interpreting $p(\varphi)$ as the Bondi mass aspect of an asymptotically flat metric (9.25), the orbit $\mathcal{O}_p$ is a subset of the orbit (9.77) of the metric under BMS$_3$ transformations. In that context the central charge c_2 coincides with the Planck mass (9.29). Accordingly, from now on we restrict our attention to centrally extended supermomenta whose central charge c_2 is strictly positive.

Massive and Massless BMS$_3$ Particles

The statement that supermomentum orbits are Virasoro coadjoint orbits is analogous to the fact that coadjoint orbits of $\mathrm{SL}(2, \mathbb{R})$ classify the momenta of relativistic particles in three dimensions. In particular the map of Poincaré momenta in Fig. 4.3b is embedded in the larger picture of Fig. 7.3, which is now interpreted as a map of BMS$_3$ supermomenta. Thus, supermomentum orbits that contain a constant representative (the vertical line in Fig. 7.3) are the supermomenta of BMS$_3$ particles that admit a rest frame.

Definition A *massive BMS$_3$ particle* is a BMS$_3$ particle whose supermomenta span a Virasoro coadjoint orbit that admits a generic constant representative p_0.

In this definition the word "generic" refers to the fact that p_0 should not take one of the discrete exceptional values $-n^2c_2/24$. Indeed the orbits containing such exceptional constants are better thought of as BMS$_3$ generalizations of the trivial representation of Poincaré; we will return to this interpretation below.

By contrast, supermomentum orbits that do *not* admit a constant representative describe BMS$_3$ particles that have no rest frame. For instance, the discrete dots that do not belong to the vertical line in Fig. 7.3 are BMS$_3$ generalizations of massless Poincaré particles, while the horizontal lines of Fig. 7.3 generalize tachyons.

Definition A *massless BMS$_3$ particle* is a BMS$_3$ particle whose supermomenta span a Virasoro coadjoint orbit with non-degenerate parabolic monodromy and non-zero winding number. A *BMS$_3$ tachyon* is a BMS$_3$ particle whose supermomenta span a Virasoro coadjoint orbit with hyperbolic monodromy and non-zero winding number.

In these definitions the terms "monodromy" and "winding number" refer to the Virasoro invariants defined in Sect. 7.1. They are the BMS$_3$ generalization of the mass squared in the Poincaré group.

Little Groups

The little groups of BMS$_3$ particles coincide with the stabilizers of the corresponding Virasoro coadjoint orbits. Here, for comparison with the Poincaré little groups of Sect. 4.3, we list the little groups obtained by using the central extension of the multiply connected BMS$_3$ group (9.60). The list of orbits is that of Sect. 7.2 and their little groups are summarized in Table 7.1:

- For a massive BMS$_3$ particle, the stabilizer is the group U(1) of spatial rotations.
- For a vacuum-like BMS$_3$ particle whose supermomentum at rest takes the value $-n^2c_2/24$, the little group is an n-fold cover of the Lorentz group in three dimensions, $\text{PSL}^{(n)}(2, \mathbb{R})$ (with $n \geq 1$).
- For a massless particle with winding number $n \geq 1$, the little group is $\mathbb{R} \times \mathbb{Z}_n$.
- For a BMS$_3$ tachyon with winding number $n \geq 1$, the little group is $\mathbb{R} \times \mathbb{Z}_n$.

This list should be compared with Table 4.1. The representations of these little groups will lead to a notion of BMS$_3$ spin. Note that when dealing with the universal cover (9.61) of BMS$_3$, all compact directions of the above little groups get decompactified so that U(1) is replaced by $\mathbb{R}$, $\text{PSL}^{(n)}(2, \mathbb{R})$ is replaced by its universal cover, and $\mathbb{Z}_n$ is replaced by the group $T_{2\pi/n} \cong \mathbb{Z}$ of translations of $\mathbb{R}$ by integer multiples of $2\pi/n$.

BMS$_3$ Particles with Positive Energy

It is natural to declare that physically admissible BMS$_3$ particles have supermomentum orbits such that the energy functional (7.79) is bounded from below under superrotations. Finding these particles is the BMS$_3$ analogue of the question (7.82) encountered in the Virasoro context. The solution is provided by the earlier results (7.103)–(7.104):

Theorem A BMS$_3$ particle has energy bounded from below if and only if its supermomenta span one of the Virasoro orbits coloured in red in Fig. 7.7.

Recall that Poincaré particles with positive energy fall in exactly three classes, two of which contain only one momentum orbit: massive particles, massless particles, and the trivial orbit. The theorem tells us that essentially the same conclusion holds for BMS$_3$ particles, since all supermomentum orbits with energy bounded from below belong to one of the three following classes:

- the unique vacuum orbit containing the supermomentum $p_0 = -c_2/24$,
- one of the massive orbits located above the vacuum and containing a constant supermomentum $p_0 > -c_2/24$,
- the unique massless orbit with energy bounded from below.

From now on, when referring to BMS$_3$ particles we always implicitly refer only to particles with energy bounded from below (except if explicitly stated otherwise). Note that, in contrast with Virasoro representations, BMS$_3$ particles with unbounded energy may provide unitary representations of BMS$_3$. Furthermore the energy spectrum of any BMS$_3$ particle is continuous.

10.1.2 Mass, Supermomentum, Central Charge

In the list of physical BMS_3 particles, the only family with infinitely many members is the class of massive particles. Let us therefore describe these particles in some more detail and interpret the labels (p, c_2) that classify them.

Defining Mass

The starting point is the observation that the vacuum supermomentum is $p_{\text{vac}} = -c_2/24$, while the supermomentum at rest of any massive BMS_3 particle is located above that vacuum value.

Definition Consider a massive BMS_3 particle with supermomentum at rest $p_0 > -c_2/24$. Then the *mass* of the particle is

$$M \equiv p_0 + c_2/24. \tag{10.2}$$

Massive BMS_3 particles with energy bounded from below have positive mass.

The definition of mass in Eq. (10.2) can be rewritten in a manifestly superrotation-invariant way, without invoking any rest frame. Indeed, recall from (7.22) that the value of p_0 determines the trace of the monodromy matrix M. This relation can be inverted and combined with the definition (10.2), which yields

$$M = \frac{c_2}{24}\left[1 + \left(\frac{1}{\pi}\text{arccosh}\big[\text{Tr}(\mathsf{M}/2)\big]\right)^2\right] \tag{10.3}$$

where we assume for definiteness that $p_0 \geq 0$, which is to say that M is hyperbolic and $M \geq c_2/24$. The same relation holds for elliptic M, hence $M < c_2/24$, upon replacing arccosh by $i\,$arccos, with the convention $\arccos(1) = 0$ and $\arccos(-1) = \pi$.

Formula (10.3) is a superrotation-invariant definition of the mass of a BMS_3 particle, since the trace of the monodromy matrix associated with Hill's equation is Virasoro-invariant. It is a BMS_3 analogue of the relation

$$M^2 = E^2 - \mathbf{p}^2 = -p_\mu p^\mu$$

that determines the mass of a Poincaré particle from its energy-momentum p_μ. As a bonus, (10.3) allows us to distinguish massive particles with elliptic and hyperbolic monodromy. This distinction is consistent with three-dimensional gravity, where metrics with a Bondi mass aspect $p_0 < 0$ are conical deficits — i.e. classical particles — while metrics with $p_0 > 0$ are flat space cosmologies — Minkowskian analogues of BTZ black holes. Furthermore, Eq. (10.3) confirms that massless BMS_3 particles (with positive energy) are actually massless. Indeed, the corresponding monodromy matrix is (7.78) with winding number $n = 1$. This implies that $\text{Tr}(\mathsf{M}) = -2$ for physical massless BMS_3 particles, which can be plugged into (10.3) and yields $M = 0$ upon using $\text{arccosh}(-1) = i\arccos(-1) = i\pi$. It would have been impossible to obtain this result with the weaker definition of mass of Eq. (10.2), since massless particles have no rest frame.

Interpreting Supermomentum

In order to develop our intuition about the supermomentum vector $p(\varphi)$, it is useful to rewrite standard Poincaré momenta in terms of functions on the circle. The supermomentum of a Poincaré particle with mass M typically takes the form

$$p(\varphi) = \sqrt{M^2 + p_x^2 + p_y^2} + p_x \cos\varphi + p_y \sin\varphi - \frac{c_2}{24} \tag{10.4}$$

and represents a particle moving in space with a spatial momentum (p_x, p_y). Note the extra factor $-c_2/24$ at the end, which ensures that M actually coincides with the mass of the particle.

One can then use (9.70) to act with superrotations on (10.4) and obtain various boosted momenta. In particular, Lorentz transformations take the form (6.88) and act on the components $(\sqrt{M^2 + p_x^2 + p_y^2}, p_x, p_y)$ according to the vector representation. This point can be verified by taking a supermomentum at rest, $p_0 = M - c_2/24$, and acting on it with a boost (7.100) in the direction $\varphi = 0$,

$$e^{if(\varphi)} = \frac{\cosh(\gamma/2)e^{i\varphi} + \sinh(\gamma/2)}{\sinh(\gamma/2)e^{i\varphi} + \cosh(\gamma/2)} \tag{10.5}$$

where γ is the rapidity (in terms of standard velocity, $\gamma = \operatorname{arctanh}(v)$). Since this superrotation is of the projective form (6.88), its Schwarzian derivative satisfies (6.94) and the corresponding transformation (9.70) of the supermomentum p can be rewritten as

$$\big(f \cdot p\big)(f(\varphi)) + \frac{c_2}{24} = \frac{1}{(f'(\varphi))^2}\left[p(\varphi) + \frac{c_2}{24}\right]. \tag{10.6}$$

This says that the combination $p + c_2/24$ transforms under Lorentz transformations as a centreless coadjoint vector of Diff(S^1). In particular, the supermomentum of a massive particle at rest transforms according to (10.6) with $p(\varphi) = M - c_2/24$. As a result, since $f'(\varphi)$ is given by (7.101), the energy $E[\gamma]$ of the boosted particle is

$$E[\gamma] = \frac{M}{2\pi}\int_0^{2\pi}\frac{d\varphi}{f'(\varphi)} = M\cosh\gamma$$

while the boosted spatial momentum along the x direction is

$$p_x[\gamma] = \frac{M}{2\pi}\int_0^{2\pi}\frac{d\varphi}{f'(\varphi)}\cos(f(\varphi)) = M\sinh\gamma.$$

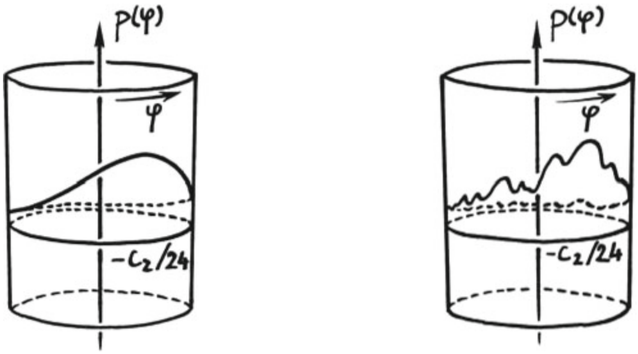

Fig. 10.1 Two possible supermomenta of a massive BMS$_3$ particle. In **a** the supermomentum is that of a boosted Poincaré particle, given by Eq. (10.4); it is a function on the circle located above the line $-c_2/24$ and its only non-zero Fourier modes are the three lowest ones. In **b** the function is dressed with extra non-vanishing Fourier modes, which results in more wiggles. These extra Fourier modes account for gravitational degrees of freedom that do not appear in the pure Poincaré case

The boosted spatial momentum along the y direction vanishes, as it should. This confirms that Lorentz transformations act on a supermomentum at rest exactly as in standard special relativity (albeit in three space-time dimensions).

From these considerations we can now draw a general conclusion on the supermomentum of a (massive) BMS$_3$ particle. Typically, the function $p(\varphi)$ will have some non-trivial profile on the circle; for instance the momentum of a particle moving fast in the x direction is represented by a function $p(\varphi)$ which is larger than $-c_2/24$, has a bump around the point $\varphi = 0$, and almost vanishes in the neighbourhood of the opposite point $\varphi = \pi$. If the particle is obtained by a pure Poincaré boost from a particle at rest, the only non-vanishing components of its supermomentum are its three lowest Fourier modes, p_0, p_1, p_{-1} (as in Eq. (10.4)). Upon switching on superrotations, the supermomentum of the particle acquires extra Fourier modes (p_2, p_3, etc.) that dress the original Poincaré momentum with additional fluctuations. In the upcoming pages we will interpret these extra degrees of freedom as being of gravitational origin (Fig. 10.1).

Interpreting the Central Charge

The central charge c_2 is an energy scale; this is manifest in formula (10.3), where c_2 converts the dimensionless trace of a monodromy matrix into a mass M. The actual value of c_2 is arbitrary in principle, but in Einstein gravity it is proportional to the Planck mass: $c_2 = 3/G$. In particular, note that c_2 is *not* a Virasoro central charge, even though it appears in the transformation law (9.70) of supermomentum as if p was a CFT stress tensor.

It is worth stressing the crucial importance of c_2 for the conclusions of the previous pages. For one thing, the whole classification of supermomentum orbits and the ensuing definition of massive/massless BMS$_3$ particles only makes sense because c_2 is non-zero. If c_2 happened to vanish, none of these results would hold since the corresponding supermomentum orbits would be Virasoro coadjoint orbits at *vanishing* central charge, and we saw in Sect. 7.1 that these orbits are radically different from (and arguably much uglier than) their centrally extended peers. This is not to say that c_2 must be non-zero in order for the supermomentum (10.1) to yield a representation of the BMS$_3$ group; in principle, representations associated with orbits having $c_2 = 0$ are just as acceptable as representations in which $c_2 \neq 0$. However the application to gravity, and the ensuing interpretation of representations as particles, relies crucially on the fact that $c_2 = 3/G$ does *not* vanish. Note that the change of monodromy occurring at $M = c_2/24$ suggests that something radical happens with BMS$_3$ particles whose mass is higher than that bound. This bifurcation reflects the fact that the metric of the gravitational field surrounding the particle changes from that of a conical deficit (when $M < 1/8G$) to that of a flat cosmology (when $M > 1/8G$).

10.1.3 Measures on Superrotation Orbits

Suppose we actually want to describe the space of states of a BMS$_3$ particle with supermomentum orbit $\mathcal{O}_p$. If the particle is scalar, then its Hilbert space consists of complex-valued wavefunctions (4.20) in supermomentum space whose scalar product (3.7) involves an integral over $\mathcal{O}_p$ with some measure μ. The latter needs to be quasi-invariant under superrotations,[1] which motivates the following question:

$$\begin{gathered} Let\ \mathcal{O}_p\ be\ a\ Virasoro\ coadjoint\ orbit\ at\ non-zero\ central\ charge;\\ is\ there\ a\ quasi-invariant\ Borel\ measure\ on\ it? \end{gathered} \tag{10.7}$$

If the answer is affirmative, then the measure is a functional one since $\mathcal{O}_p$ consists of functions on the circle.

A Conjecture

Our viewpoint regarding the problem (10.7) will be pragmatical: path integral measures are used on a daily basis in quantum mechanics, and their efficiency in correctly predicting the values of physical observables is firmly established. Thus, if one is willing to define Hilbert spaces of square-integrable functions thanks to functional measures, their application to BMS$_3$ particles is as acceptable as in quantum physics. In particular one may hope that Virasoro coadjoint orbits do admit quasi-invariant measures:

Conjecture Let $\mathcal{O}_p$ be a Virasoro coadjoint orbit with non-zero central charge (and energy bounded from below). Then there exists a Borel measure $d\mu(q)$ on $\mathcal{O}_p$ which is quasi-invariant under the action of the Virasoro group (where $q \in \mathcal{O}_p$).

[1] We recall that the definition of quasi-invariant measures was given in Sect. 3.2.

In the remainder of this thesis we will rely on this conjecture in order to define the Hilbert space of a BMS$_3$ particle (at least one with bounded energy). The conjecture does not say how the measure $d\mu(q)$ is actually defined, but this is not a problem since the results of Sect. 3.2 imply that induced representations based on different quasi-invariant measures are unitarily equivalent. Thus, assuming that the conjecture is true, we do not really need to know anything specific about the measure.[2] In particular the character computation of Sect. 10.3, although relying on an unknown measure, will produce an unambiguous result.

Aside from these basic observations, we will have very few concrete things to say about the measure. Nevertheless the lines that follow are devoted to a brief review of the literature on Virasoro measure theory, with the intent of further motivating the validity of the conjecture. The reader who is not interested in mathematical subtleties is free to go directly to Sect. 10.1.4.

Remark Recall that all irreducible unitary representations of *regular* semi-direct products are induced representations, where regularity refers to the property defined at the end of Sect. 4.1. Accordingly, in order to claim that all irreducible unitary representations of BMS$_3$ are BMS$_3$ particles whose supermomenta span Virasoro orbits, we would have to prove that the BMS$_3$ group is a regular semi-direct product, which in turn relies on the existence of a measure on the space of supermomenta. We will not address this question here and assume instead that the standard results on finite-dimensional semi-direct products carry over to BMS$_3$.

Measures on Virasoro Orbits

The issue of rigorously defining path integral measures was first addressed by Wiener about a century ago, in the context of stochastic processes. We refer e.g. to the biographical memoir in [7] for more references and a more accurate account of the development of the subject. The problem of defining a Wiener-like quasi-invariant measure on Virasoro coadjoint orbits is more recent, but well known. In the physics literature, as in our conjecture above, the question of the measure is mostly treated in a heuristic way motivated by quantum mechanics; see e.g. [8] for such an approach. By contrast, there is a fair amount of mathematical literature that aims at solving the problem in a rigorous way, and to our knowledge no definite, widely accepted solution is known at present. As announced above, we do not claim to provide an answer here; rather, we shall content ourselves with a brief literature review.

The main motivation for defining Virasoro measures comes from representations of the Virasoro algebra and conformal field theory; the hope is that such measures could provide a rigorous prescription for the geometric quantization of Virasoro orbits. This approach to the problem is adopted for instance in [9–12]. In [13] the authors tackle the issue with a similar motivation, though with different methods; in particular it is suggested there that a measure might be provided by an

[2] Note that the existence of a quasi-invariant measure $d\mu(q)$ on $\mathcal{O}_p$ implies the existence of infinitely many other ones, since one can always multiply the measure by a strictly positive smooth function $\rho(q)$ and obtain a new measure $\rho(q)d\mu(q)$.

infinite-dimensional version of the Liouville measure (5.14) obtained by taking the Kirillov–Kostant symplectic form (8.62) to an infinite power.

A somewhat different approach consists in building measures on Virasoro orbits regardless of their relation to conformal field theory and highest-weight representations. This approach does not simplify the problem of geometric quantization of Virasoro orbits, but it does have the virtue of producing the desired measure. One should keep in mind that the measures needed for BMS$_3$ particles generally have nothing to do with those needed for geometric quantization of the Virasoro group, so the fact that a measure is unrelated to Virasoro representations is not a problem for our purposes. As it turns out, certain results due to Shavgulidze [14–17] precisely point in that direction (see also [18, 19]). Indeed it was shown in [14] that the group of diffeomorphisms of any compact manifold can be endowed with a quasi-invariant Borel measure. This measure then plays a role analogous to the Haar measure (recall Sect. 3.2) and ensures that quotients of the group, such as $\text{Diff}(S^1)/S^1$, can also be endowed with a quasi-invariant measure. As a corollary, Virasoro orbits (all of which are quotients of $\text{Diff}(S^1)$) should generically admit quasi-invariant Borel measures. This is precisely what is needed for BMS$_3$ particles, and it is in fact our main justification for the above conjecture.

10.1.4 States of BMS$_3$ Particles

Under the assumption that there exist quasi-invariant measures on Virasoro coadjoint orbits, we have all the ingredients required to build explicit unitary, irreducible, projective representations of the BMS$_3$ group. Here we describe and interpret the wavefunctions that represent the quantum states of a scalar massive BMS$_3$ particle with mass $M > 0$. Spinning particles and the vacuum representation will be described later.

The States of a BMS$_3$ Particle

The supermomenta of a massive particle span an orbit $\mathcal{O}_p = \text{Diff}(S^1)/S^1$ (with implicit central charge $c_2 > 0$), which we assume to admit a quasi-invariant measure μ. For convenience the orbit representative p is taken to be the supermomentum $p(\varphi) = p_0 = M - c_2/24$ at rest. As in (4.20) the particle's Hilbert space $\mathscr{H}$ consists of complex-valued wavefunctions

$$\Psi : \mathcal{O}_p \to \mathbb{C} : q \mapsto \Psi(q) \tag{10.8}$$

which are square-integrable with respect to μ in the usual sense that the integral

$$\int_{\mathcal{O}_p} d\mu(q)\, |\Psi(q)|^2 \tag{10.9}$$

is finite. As usual, Ψ should be thought of as a wavefunction in (super)momentum space representing a wavepacket that propagates with fuzzy velocity. In contrast to the finite-dimensional case, however, the map (10.8) is really a *functional* since it is defined on a space of functions:

$$\Psi : q(\varphi) \mapsto \Psi[q(\varphi)].$$

Similarly the integral (10.9) is actually a functional integral. Despite these complications we will keep using the simpler notation $\Psi(q)$ for the wavefunctions of BMS$_3$ particles. The scalar product on $\mathscr{H} = L^2(\mathcal{O}_p, \mu, \mathbb{C})$ is then defined by (3.7) with $(\Phi(q)|\Psi(q)) = \Phi^*(q)\Psi(q)$.

The space of states of a BMS$_3$ particle carries an action of BMS$_3$ by unitary transformations. Since we are assuming that the particle is *scalar*, the action of BMS$_3$ on $\mathscr{H}$ is given by formula (4.23):

$$\big(\mathcal{T}[(f,\alpha)]\cdot\Psi\big)(q) = \sqrt{\rho_{f^{-1}}(q)}\; e^{i\langle q,\alpha\rangle}\;\Psi(f^{-1}\cdot q)\,. \tag{10.10}$$

Here (f, α) is an element of the BMS$_3$ group (9.58), with $f(\varphi)$ a superrotation and $\alpha(\varphi)$ a supertranslation. The point $q \in \mathcal{O}_p$ is a supermomentum vector and its pairing $\langle q, \alpha\rangle$ with α is given by (6.34). The function $\rho_f(q)$ is the (unknown) Radon–Nikodym derivative (3.19) of the measure μ; if by chance the measure happens to be invariant, then one can set $\rho_f(q) = 1$. Finally, the action $f \cdot q$ appearing in the argument of the wavefunction on the right-hand side is the BMS$_3$ generalization of the action of boosts on momenta; it is given by formula (9.70), which is the coadjoint representation of the Virasoro group at central charge c_2.

Remark Wavefunctionals are common in quantum field theory. Indeed, the quantum state of a typical field theory is a wavefunctional $\Psi[\phi(\mathbf{x})]$, where $\phi(\mathbf{x})$ is a spatial field configuration. The truly striking aspect of (10.8) is not quite the fact that it belongs to a space of wavefunctionals, but rather that it provides an *irreducible* representation of the symmetry group.

BMS$_3$ Particles as Projective Representations

The central charge $c_2 \neq 0$ turns expression (10.10) into a *projective* representation of the centreless BMS$_3$ group. Indeed, as is manifest in (9.71), c_2 is responsible for an extra term in the commutation relations of superrotations with supertranslations. At the group-theoretic level this difference is due to the extra terms of the group operation (9.41), as opposed to the centreless group operation (9.59). In the latter case we have

$$(f, 0)\cdot(e, \alpha) = (f, \mathrm{Ad}_f\alpha) = (e, \mathrm{Ad}_f\alpha)\cdot(f, 0) \tag{10.11}$$

where e is the identity in Diff(S^1) and Ad is the action of diffeomorphisms on vector fields on the circle. By contrast, in the centrally extended case (9.41) there are two extra slots for central terms and the analogue of (10.11) becomes

$$(f, 0; 0, 0) \cdot (e, 0; \alpha, 0) = \Big(f, 0; \mathrm{Ad}_f \alpha, -\frac{1}{12}\langle \mathsf{S}[f], \alpha\rangle\Big),$$
$$(e, 0; \alpha, 0) \cdot (f, 0; 0, 0) = \Big(f, 0; \mathrm{Ad}_f \alpha, 0\Big)$$

by a term proportional to $\langle \mathsf{S}[f], \alpha\rangle$, with S the Schwarzian derivative (6.76). This phenomenon is analogous to the statement that boosts and translations do not commute in the Bargmann group (4.103). In practice it means that the wavefunction obtained by acting first with (e, α), then by $(f, 0)$, differs from the one obtained by acting first with $(f, 0)$, then with $(e, \mathrm{Ad}_f \alpha)$, by a constant complex phase that can be evaluated using (9.55):

$$\mathcal{T}[(f, 0)] \cdot \mathcal{T}[(e, \alpha)] = \exp\Big[-i\frac{c_2}{12}\langle \mathsf{S}[f], \alpha\rangle\Big]\,\mathcal{T}[(e, \mathrm{Ad}_f \alpha)] \cdot \mathcal{T}[(f, 0)]\,. \quad (10.12)$$

This is indeed the statement (2.7) that the representation $\mathcal{T}$ is projective, when seen as a representation of the centreless BMS$_3$ group (9.58). It is the BMS$_3$ analogue of the Galilean result (4.121).

There is an important subtlety in (10.12): formula (10.10) is a projective representation only if one insists on using the *centreless* group operation (9.59). If instead one uses the centrally extended group (9.62), projectivity is absorbed by the definition of the group operation (9.41). This is a restatement of our earlier observation in Sect. 2.1 that projective representations can be seen in two equivalent ways: either as genuine projective representations of a centreless group, or as exact (non-projective) representations of a centrally extended group. This same argument is the reason why massive BMS$_3$ particles can have arbitrary real values of spin, as we shall discuss below.

Plane Waves

As in Sect. 3.3 we can describe the representation (10.10) in terms of a basis of one-particle states with definite (super)momentum on the orbit $\mathcal{O}_p$. Let therefore δ denote the Dirac distribution associated with the measure μ and defined by the requirement (3.39). In the present case μ is a functional measure, so δ is a functional delta distribution. For $k \in \mathcal{O}_p$, we define the plane wave state with supermomentum k as

$$\Psi_k(q) \equiv \delta(k, q) \quad (10.13)$$

which is now a functional analogue of Eq. (3.43). It is a typical asymptotic state in a scattering experiment. The scalar products of plane waves are given by (3.44):

$$\langle \Psi_k | \Psi_{k'} \rangle = \delta(k, k'). \quad (10.14)$$

Strictly speaking, plane waves are not square-integrable, hence do not belong to the space of states of a BMS$_3$ particle. They should therefore be understood in the weaker sense that any wavepacket (3.45) can be written as an infinite sum of plane waves. With this word of caution, one may say that plane waves form a "basis" of the space of states of a BMS$_3$ particle. Their transformation law under BMS$_3$ transformations is given by Eq. (4.32),

$$\mathcal{T}[(f,\alpha)]\cdot\Psi_k=\sqrt{\rho_f(k)}\;e^{i\langle f\cdot k,\alpha\rangle}\;\Psi_{f\cdot k}\,, \tag{10.15}$$

except that we have removed the spin representation $\mathcal{R}$ since the particle considered here has vanishing spin. This formula reflects the fact that a wavefunction with momentum k boosted by a superrotation f becomes a wavefunction with momentum $f\cdot k$. In short, all results of Chaps. 3 and 4 remain valid, up to the fact that manifolds become spaces of functions while functions become functionals.

10.1.5 Dressed Particles and Quantization

Asymptotically flat gravity enjoys BMS$_3$ symmetry, so its quantization is expected to produce unitary representations of the BMS$_3$ group. More precisely, since the phase space of metrics (9.25) is a hyperplane $c_1=0$, $c_2=3/G$ in the space of the coadjoint representation of $\widehat{\text{BMS}}_3$, the geometric quantization of the orbit (9.77) of a metric (j,p) under asymptotic symmetry transformations is expected to produce an irreducible unitary representation of BMS$_3$ with supermomentum orbit $\mathcal{O}_p$ and spin (5.128) determined by the restriction of j to the Lie algebra of the little group of p. Thus,

A BMS$_3$ particle is the quantization
of the orbit of a metric under BMS$_3$ transformations.

In the following pages we use this observation to compare BMS$_3$ particles with standard relativistic particles. For simplicity we focus on a massive scalar particle whose orbit representative is taken to be the supermomentum $p=M-c_2/24$ at rest.

Leaking Wavefunctions

The transformation law (10.10) is an infinite-dimensional generalization of a scalar representation (with mass M) of the Poincaré group in three dimensions. Indeed, by restricting one's attention to the Poincaré subgroup of BMS$_3$, one obtains a (highly reducible) unitary representation of Poincaré. The latter contains the standard scalar irreducible representation with mass M, but it also contains an uncountable infinity of other representations with higher mass. These extra representations arise because the action of Lorentz transformations on the supermomentum orbit $\mathcal{O}_p\cong\text{Diff}(S^1)/S^1$ is not transitive; in fact, the set of Lorentz-inequivalent supermomenta is an infinite-dimensional manifold

$$\text{PSL}(2,\mathbb{R})\backslash\text{Diff}(S^1)/S^1, \tag{10.16}$$

which is a double quotient of Diff(S^1). The quotient on the left is taken with respect to the Lorentz group PSL$(2,\mathbb{R})$.

This observation can be rephrased in a more intuitive way: it says that the supermomentum orbit $\mathcal{O}_p = \text{Diff}(S^1)/S^1$ contains infinitely many finite-dimensional submanifolds $\text{SL}(2, \mathbb{R})/S^1$ obtained by acting on the orbit with Lorentz transformations. Each submanifold is a standard momentum orbit (4.97) for massive Poincaré particles in three dimensions. Any two of those submanifolds are mutually Lorentz-inequivalent; the set of such inequivalent sub-orbits is the space (10.16).

Now, a wavefunction $\Psi(q)$ of a BMS$_3$ particle is never supported on just one Lorentz sub-orbit of $\mathcal{O}_p$: if it was, some components of its supermomentum would be sharply defined and the uncertainty principle would be violated. (By the way, we stress again that the plane waves (10.13) do *not* actually belong to the Hilbert space.) Rather, the wavefunction spreads over many Lorentz-inequivalent momentum orbits. In other words, the wavefunction $\Psi(q)$ "leaks" into the directions of supermomentum obtained by acting with superrotations that are *not* Lorentz transformations (Fig. 10.2).

Remark The double quotient (10.16) is an application of the *induction-reduction theorem* for induced representations. The latter roughly goes as follows: Let G be a group with two (generally different) subgroups H, H' and let $\mathcal{S}$ be an irreducible representation of H. Then the restriction to H' of the induced representation $\text{Ind}_H^G(\mathcal{S})$

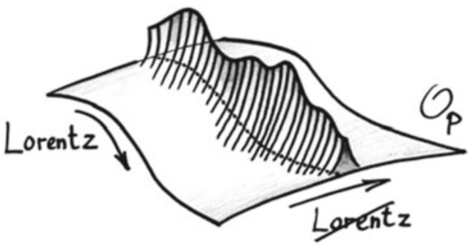

Fig. 10.2 The supermomentum orbit of Fig. 10.3, now with a wavefunction on top. The wavefunction is roughly supported on a Lorentz sub-orbit, but not quite: it leaks into directions that cannot be achieved with Lorentz transformations

Fig. 10.3 A supermomentum orbit $\mathcal{O}_p \cong \text{Diff}(S^1)/S^1$ with embedded Lorentz sub-orbits represented as curvy lines. Lorentz transformations move points along these sub-orbits. Transitions from one sub-orbit to another are only possible with superrotations that do not belong to the Lorentz subgroup of $\text{Diff}(S^1)$. One can define an equivalence relation on the supermomentum orbit by declaring that two points are equivalent if they belong to the same Lorentz sub-orbit. The quotient of $\mathcal{O}_p$ by that equivalence relation is the double quotient (10.16)

is a direct integral of irreducible representations of H' labelled by the points of the double quotient $H'\backslash G/H$. In (10.16) we have applied this theorem to $G = \text{BMS}_3$, $H = \text{U}(1) \ltimes \text{Vect}(S^1)$ and $H' = \text{PSL}(2,\mathbb{R}) \ltimes \mathfrak{sl}(2,\mathbb{R})$. The latter is the Poincaré group in three dimensions while the U(1) of H is the little group for massive particles.

Gravitational Dressing

We have just explained that any supermomentum orbit contains infinitely many Poincaré-inequivalent sub-orbits. It is natural to wonder how the extra directions of the orbit — those that do not lie along Lorentz generators — are to be interpreted. In other words: how should one think of the fact that the wavefunction of a BMS$_3$ particle leaks into directions that are forbidden by Poincaré transformations?

A natural guess is suggested by the very origin of the BMS$_3$ group: as we showed in Sect. 9.1, it is the asymptotic symmetry group of Minkowskian space-times (in three dimensions). Asymptotic symmetries may be thought of as generalizations of isometries that incorporate gravitational fluctuations. Now, the isometry group of Minkowski space-time is the Poincaré group, and its unitary representations are particles in the usual sense. Accordingly,

A BMS particle is a Poincaré particle
dressed with gravitational degrees of freedom. (10.17)

Let us be more precise about what we mean by "gravitational degrees of freedom". Classically, those are the classes of points on the orbit $\mathcal{O}_p$ that cannot be obtained from p by a Lorentz transformation; the set of these classes coincides with the double coset space (10.16). Quantum-mechanically, we would like these gravitational degrees of freedom to correspond to the set of quantum states that *cannot* be obtained from the state of a particle at rest by Lorentz transformations. This can be rephrased in precise terms: on the Hilbert space $\mathscr{H}$ of a BMS$_3$ particle we define an equivalence relation $\sim$ such that $\Psi \sim \Psi'$ if there exists a Poincaré transformation (f, α) for which $\Psi' = \mathcal{T}[(f, \alpha)]\Psi$. Then the space of Poincaré-inequivalent states is the quotient

$$\mathscr{H}/\sim\ . \tag{10.18}$$

The latter can also be seen as the set of quantum states obtained by acting on a state at rest with supertranslations and superrotations that do not belong to the Poincaré subgroup. In this sense it is the three-dimensional analogue of the space of *soft gravitons* in four dimensions, as follows from the recently discovered relation [20, 21] between asymptotic symmetries and soft theorems in gauge theories. Thus,

A BMS particle is a particle dressed with soft gravitons.

Note that the terminology of "soft gravitons" is a bit dangerous here, since three-dimensional Einstein gravity has no local degrees of freedom, hence no genuine (bulk) gravitons. We already pointed out this subtlety in the introduction of the thesis, and our point of view remains the same: owing to the relation between soft gravitons and asymptotic symmetries, any theory with non-trivial asymptotic symmetries can be interpreted as a theory containing soft degrees of freedom, regardless of the presence of bulk degrees of freedom. In this sense three-dimensional gravity is a toy model for soft gravitons.

10.1.6 The BMS$_3$ Vacuum

Having analysed massive scalar BMS$_3$ particles, we now turn to some of their cousins. Here we describe the vacuum BMS$_3$ representation, while spinning particles are studied in Sect. 10.1.7.

The vacuum representation of BMS$_3$ is the scalar induced representation based on the vacuum supermomentum orbit — the one containing the constant $p_{\text{vac}} = -c_2/24$. The corresponding little group is the Lorentz group PSL$(2, \mathbb{R})$, so the orbit is

$$\mathcal{O}_{\text{vac}} \cong \text{Diff}(S^1)/\text{PSL}(2, \mathbb{R}). \tag{10.19}$$

As before we assume that it admits a measure which is quasi-invariant under superrotations. The Hilbert space of the representation is then spanned by square-integrable wavefunctions (10.8) on $\mathcal{O}_{\text{vac}}$ transforming under BMS$_3$ according to (10.10).

The interesting aspect of the vacuum representation is its interpretation. Indeed, while massive particles exist in both the Poincaré group and the BMS$_3$ group, the vacuum representation is non-trivial only in the BMS$_3$ context. This is analogous to the observation of Sect. 8.4 that the vacuum representation of the Virasoro group is non-trivial. In particular, as in Virasoro, there is no fully BMS$_3$-invariant definition of the vacuum at non-zero central charge; the maximal possibility is Poincaré invariance, which is indeed achieved by $p_{\text{vac}} = -c_2/24$ (with $j = 0$). This reduced symmetry is responsible for the non-triviality of the representation and for the fact that wavefunctions of the vacuum representation "leak" into directions that cannot be reached by Poincaré transformations. As in the massive case (10.16), the set of Lorentz-inequivalent supermomenta on the vacuum orbit is a double coset space

$$\text{PSL}(2, \mathbb{R})\backslash\text{Diff}(S^1)/\text{PSL}(2, \mathbb{R}) \tag{10.20}$$

which parameterizes the decomposition of the vacuum BMS$_3$ representation as a direct integral of Poincaré sub-representations.

The presence of a non-trivial vacuum representation can be interpreted in gravitational terms. Indeed, according to our heuristic proposal (10.17), the vacuum BMS$_3$ representation consists *only* of gravitational degrees of freedom (since the corresponding Poincaré particle is trivial). Classically these degrees of freedom span the BMS$_3$ orbit of the Minkowski metric. In the terminology of Sect. 8.3 those would be "boundary gravitons", or equivalently soft gravitons, around Minkowski space.

Dressed Particles Revisited

Let us return to the interpretation of BMS$_3$ particles as dressed particles, now using the vacuum representation as an extra input. We start with the following observation: consider the space $L^2(\mathcal{M} \times \mathcal{N}, \mu)$ of square-integrable functions on the product space $\mathcal{M} \times \mathcal{N}$; suppose the measure μ factorizes as a product $\mu = \mu_{\mathcal{M}} \times \mu_{\mathcal{N}}$, where $\mu_{\mathcal{M}}$ is a measure on $\mathcal{M}$ and $\mu_{\mathcal{N}}$ is a measure on $\mathcal{N}$. Then one has a tensor product decomposition

$$L^2(\mathcal{M} \times \mathcal{N}, \mu_{\mathcal{M}} \times \mu_{\mathcal{N}}) \cong L^2(\mathcal{M}, \mu_{\mathcal{M}}) \otimes L^2(\mathcal{N}, \mu_{\mathcal{N}}). \tag{10.21}$$

Let us use this result to compare massive particles, with supermomentum orbits $\text{Diff}(S^1)/S^1$, to the vacuum whose orbit is $\text{Diff}(S^1)/\text{PSL}(2, \mathbb{R})$. Since both of these infinite-dimensional manifolds are homotopic to a point, we can relate them as

$$\text{Diff}(S^1)/S^1 \cong \big(\text{PSL}(2, \mathbb{R})/S^1\big) \times \big(\text{Diff}(S^1)/\text{PSL}(2, \mathbb{R})\big). \tag{10.22}$$

This is to say that the massive supermomentum orbit is a direct product $\mathcal{M} \times \mathcal{N}$. The first factor of the product is the Poincaré momentum orbit (4.97) of a massive particle in three dimensions, which suggests that a BMS$_3$ particle is equivalent to a relativistic particle "times" the vacuum BMS$_3$ representation. This can be made precise using (10.21): if the measure μ used to define the scalar product of wavefunctions for a massive BMS$_3$ particle factorizes into a product on $\text{Diff}(S^1)/\text{PSL}(2, \mathbb{R})$ and $\text{PSL}(2, \mathbb{R})/S^1$, then the Hilbert space of a massive BMS$_3$ particle factorizes into

$$\mathscr{H}_{\text{BMS}} \cong \mathscr{H}_{\text{Poinc}} \otimes \mathscr{H}_{\text{vac}} \tag{10.23}$$

where $\mathscr{H}_{\text{Poinc}}$ is the space of states of a massive Poincaré particle and $\mathscr{H}_{\text{vac}}$ is that of the BMS$_3$ vacuum representation.

One can reformulate the statement (10.23) in the basis of plane wave states (10.13). Indeed, on a massive supermomentum orbit $\mathcal{O}_p = \text{Diff}(S^1)/S^1$, the diffeomorphism (10.22) allows us to write any supermomentum q as a pair $q = (q_{\text{Poinc}}, q_{\text{vac}})$ where q_{Poinc} is a momentum vector with three components belonging to the Poincaré sub-orbit $\mathcal{O}_{\text{Poinc}} = \text{PSL}(2, \mathbb{R})/S^1$, while q_{vac} is a supermomentum that belongs to the vacuum BMS$_3$ orbit $\mathcal{O}_{\text{vac}} = \text{Diff}(S^1)/\text{PSL}(2, \mathbb{R})$. Then, under the assumption that the measure μ on $\mathcal{O}_p$ disintegrates into a product of measures on $\text{PSL}(2, \mathbb{R})/S^1$ and $\mathcal{O}_{\text{vac}}$, any plane wave (10.13) can be written as a tensor product

$$\Psi_k = \Psi_{k_{\text{Poinc}}} \otimes \Psi_{k_{\text{vac}}}$$

where the left and right factors of the product are plane waves on the orbits $\mathcal{O}_{\text{Poinc}}$ and $\mathcal{O}_{\text{vac}}$, respectively. A generic state of a BMS$_3$ particle is an infinite linear combination of such factorized plane waves. Note that none of our upcoming conclusions rely on this phenomenon, so in the sequel we will not necessarily assume that the measure μ disintegrates into a product.

Remark The decomposition (10.22) is similar to that of the Poincaré D-momentum $p_\mu = (E, \mathbf{p})$. In the latter case the truly covariant quantity is p_μ but our non-relativistic intuition splits it into an energy-momentum pair $(E, \mathbf{p})$. In the same way, for BMS$_3$ particles the truly covariant quantity is the full supermomentum $p(\varphi)$, but our intuition splits it into a pair $(p_{\text{Poinc}}, p_{\text{vac}})$.

10.1.7 Spinning BMS$_3$ Particles

We finally turn to the spinning generalization of the BMS$_3$ representations considered above. We have already addressed almost all the subtleties of the construction, so we display the inclusion of spin merely for completeness. In short, our main conclusion will be that the spin of massive BMS$_3$ particles is not quantized, exactly as in the Poincaré group in three dimensions. The reader who is happy to accept this result, or to deal only with scalar particles, may go to Sect. 10.1.8.

We recall from (4.28) that spin is the label that specifies the representation of the little group chosen for the description of a particle. In the case of BMS$_3$ particles at non-zero c_2, the little groups were described at the end of subsection 10.1.1. They are all either one-dimensional Abelian groups such as U(1) or $\mathbb{R}$ (possibly up to discrete factors), or n-fold covers of the Lorentz group PSL$(2, \mathbb{R})$. All these are finite-dimensional Lie groups and their unitary representations are known, so writing down generic spinning representations of BMS$_3$ is mostly a technical problem.

For definiteness, let us consider a massive BMS$_3$ particle. Its little group U(1) consists of spatial rotations, exactly as for massive Poincaré particles in three dimensions. All irreducible unitary representations of U(1) are of the form (2.13), with s an integer. However, the BMS$_3$ group (9.58) has the same homotopy type as Diff(S^1), which is homotopic to a circle. This implies that it admits topological projective representations of the type described in Sect. 2.1, which can be classified by considering exact (non-projective) representations of the universal cover (9.61) of BMS$_3$. In the latter case, the little group of massive particles gets unwrapped from U(1) to $\mathbb{R}$, whose unitary representations are now labelled by an arbitrary real spin s. Since those are the physically relevant representations, we conclude that the spin of a massive BMS$_3$ particle is generally an arbitrary real number. In particular, most massive BMS$_3$ particles are anyons. This is the same conclusion as in the Poincaré group in three dimensions.

Now suppose we fix a certain value of mass M and spin $s \in \mathbb{R}$ and ask what are the states of the corresponding BMS$_3$ particle. We denote the spin s representation of the little group by $\mathcal{R}$; it is a one-dimensional representation of the form (2.13). Thus the Hilbert space of the BMS$_3$ particle consists again of complex-valued wavefunctions (10.8), but their transformation law under BMS$_3$ contains an extra term with respect to the scalar representation (10.10). That extra term involves the Wigner rotation (4.31) associated with the superrotation f and the supermomentum q,

$$W_q[f] = \mathcal{R}[g_q^{-1} f \, g_{f^{-1} \cdot q}] \,, \tag{10.24}$$

where the superrotations g_q ($q \in \mathcal{O}_p$) are standard boosts such that $g_q \cdot p = q$. Thus we run into the problem of finding standard boosts for a massive supermomentum orbit.

Incidentally we have already defined such standard boosts, though not in the same language. Indeed, Eq. (7.44) is precisely the definition of standard boosts for elliptic Virasoro coadjoint orbits. These boosts are built as follows:

1. Take the supermomentum $q(\varphi)$ with central charge c_2; write down the associated Hill's equation (7.12) with the replacement $(p, c) \to (q, c_2)$.
2. Find two linearly independent solutions ψ_1, ψ_2 of Hill's equation satisfying the Wronskian condition (7.16).
3. Define a vector field $X_q(\varphi)$ by (7.38).
4. Define a diffeomorphism f of S^1 by (7.41), with $p_0 = M - c_2/24$
5. The standard boost associated with q is $g_q = f^{-1}$.

This procedure is somewhat convoluted, but it does provide a family of standard boosts on the orbit of a massive supermomentum, as desired. We refrain here from actually computing these boosts.

Equipped with standard boosts one can write down the transformation law of massive BMS$_3$ particles with non-zero spin, given by Eq. (4.30):

$$\big(\mathcal{T}[(f, \alpha)] \cdot \Psi\big)(q) = \sqrt{\rho_{f^{-1}}(q)}\; e^{i\langle q, \alpha\rangle}\, W_q[f] \cdot \Psi(f^{-1} \cdot q)\,,$$

where the notation is the same as in (10.10) up to the insertion of the Wigner rotation (10.24). This can also be rewritten in terms of plane waves (10.13) as

$$\mathcal{T}[(f, \alpha)] \cdot \Psi_k = \sqrt{\rho_f(k)}\; e^{i\langle f\cdot k, \alpha\rangle}\, \mathcal{R}\left[g_{f\cdot k}^{-1} f\, g_k\right] \cdot \Psi_{f\cdot k}$$

(recall Eq. (4.32)). The interpretation of all these formulas is the same as before, up to the extra Wigner rotation. In particular, a spinning BMS$_3$ particle is a spinning Poincaré particle dressed with soft gravitons.

10.1.8 BMS Particles in Four Dimensions?

Having described BMS particles in three dimensions, we now ask to what extent our observations apply to the realistic four-dimensional case. To answer this we first briefly review some previous literature on BMS in four dimensions and its representations, before exposing our viewpoint on the matter in light of the more recent developments relating BMS symmetry to soft theorems.[3] Our approach will be mostly qualitative and heuristic.

[3]To our knowledge there is, at present, no detailed review on BMS symmetry. Accordingly the literature review provided here cannot fail to be biased by the author's ignorance; we apologize in advance for the references that we may have missed.

A History of BMS Symmetry

BMS symmetry was discovered in the sixties by Bondi, Van der Burg, Metzner [22, 23] and Sachs [24, 25], as a group (1.1) of globally well-defined diffeomorphisms of asymptotically flat space-times. The notion of asymptotic flatness was put on firmer ground shortly thereafter, thanks to the notion of conformal compactifications [26]. Arguably, at the time of the discovery, the presence of an infinite-dimensional group of supertranslations was seen as something of a pathology. Nevertheless it was quickly suggested that full BMS symmetry (with the supertranslations turned on) could be used to discuss aspects of both quantum gravity [27] and S-matrix physics [24]. This led to the study of unitary representations of BMS.

The first suggestion that BMS representations might be relevant to particle physics appeared in [24]. A little later McCarthy and collaborators set out to study the representation theory of the global BMS group in full detail, with scattering amplitudes as a motivation [28]. Owing to the semi-direct product structure of (1.1) the strategy was to build induced representations à la Wigner in terms of orbits and little groups, as described in this thesis in Chap. 4. In particular it was shown in [29, 30] that all little groups are compact, leading to the conclusion that BMS particles in four dimensions cannot have continuous spin, in contrast to their Poincaré counterparts. This was followed by the observation that the restriction of a BMS representation to its Poincaré subgroup is reducible and consists of a tower of Poincaré particles with different spins [31]; in [32] this spin mixing was interpreted as being due to the presence of the gravitational field. It was also shown in [33] that certain BMS representations studied earlier in [34, 35] were in fact reducible induced representations.

Along the way it was realized that the absence of continuous-spin particles exhibited in [29, 30] was due to a delicate choice of topology, and that different topologies lead to radically different conclusions, including particles with continuous spin [36]. These continuous-spin particles were then interpreted as scattering states in [37, 38]. Finally, the whole construction was put on firm mathematical ground in [39], where the theory of induced representations was extended to groups of the type $G \ltimes A$ with an infinite-dimensional Abelian group A. As a corollary, it was shown in [40] that induced representations à la Wigner exhaust all irreducible unitary representations of the global BMS group (1.1). This analysis was later completed by the proof that the global BMS group has no non-trivial central extensions, and therefore admits no projective representations other than those originating from its non-trivial topology [41].

BMS Symmetry and Holography

For about two decades after McCarthy's work on BMS representations, the study of BMS symmetry as such appears to have slowed down, with the exception of the discovery of its supersymmetric version in [42]. Nevertheless, substantial progress was made during that period in closely related areas, particularly in the study of the structure of gravity near null infinity (see e.g. [43–46]). This led for instance to the idea of asymptotic quantization [47, 48], which can roughly be thought of as a quantization of bulk degrees of freedom obtained by quantizing a suitable radiative phase space on the boundary — an idea that sounds prophetic given the development of holography about ten years later.

As mentioned in the introduction of this thesis, holography emerged from general considerations on the nature of quantum gravity [49, 50] guided by the seminal observation by Bekenstein and Hawking that black holes have entropy [51, 52]. Its best known realization occurs in the AdS/CFT duality [53–55], but it was soon suggested that a suitable notion of holography should hold for all families of space-times (see e.g. [56, 57]), and in particular for asymptotically flat gravity. In that context interest in BMS symmetry slowly re-emerged [58–62], leading in particular to the proposal [63–65] that the globally well-defined BMS group (1.1) should be extended to include a Virasoro-like semi-group of local conformal transformations of celestial spheres. This proposal is the origin of the terminology of "superrotations" for asymptotic symmetry transformations that extend Lorentz transformations. The study of asymptotically flat holography then took off, both in four dimensions [66–70] and in three dimensions [2, 3, 71–82], although in the latter case a substantial part of the literature is written in the language of Galilean conformal symmetry [83–97].

Along the way, it was realized by Strominger and collaborators that BMS symmetry does have highly non-trivial implications for the S-matrix, in the form of soft graviton theorems [21, 98–101]. This discovery sparked a flurry of papers discussing the applications of the BMS group (and its gauge-theoretic generalizations [20, 102–108]) to scattering amplitudes [109, 110], memory effects [111–113] and more recently to black holes [114].

BMS$_4$ Particles?

Despite recent progress, we still seem to be quite far from having truly understood BMS symmetry in four dimensions. Representation theory provides an easy way to illustrate the problem. Indeed, one of the cornerstones of the relation between BMS symmetry and soft theorems is the fact that supertranslations generate soft graviton states when acting on the vacuum. Accordingly, if the global BMS group (1.1) is correct, then the representations considered by McCarthy in [29, 30] should account for this effect: they should represent Poincaré particles dressed with soft gravitons.

However, it is easy to see that this is not the case. To illustrate this point, consider the analogue of the global BMS group (1.1) in three dimensions,

$$\text{gBMS}_3 \equiv \text{PSL}(2,\mathbb{R}) \ltimes \text{Vect}(S^1)_{\text{Ab}}\,, \tag{10.25}$$

where $\text{PSL}(2,\mathbb{R})$ is the Lorentz subgroup of $\text{Diff}(S^1)$ consisting of projective transformations (6.88) and acting on superrotations according to (6.17). The representations of this group would be induced exactly in the same way as for the standard BMS$_3$ group (9.58), but there would be two crucial differences:

- There would be no non-trivial central extensions.
- The supermomentum orbits would all be finite-dimensional (since $\text{PSL}(2,\mathbb{R})$ is finite-dimensional).

In particular the Hilbert space of any irreducible unitary representation of the group (10.25) coincides with the space of states of a Poincaré particle; it consists of wavefunctions on a finite-dimensional momentum orbit. In fact, the only difference between the representations of this group and those of Poincaré would be that translations are paired with supermomenta according to the functional formula (6.34) rather than a finite-dimensional product $p_\mu \alpha^\mu$. In particular the vacuum representation would be trivial and there would be no way for quantum supertranslations to create soft graviton states upon acting on the vacuum.

These observations exhibit an important point: the global BMS group (1.1) cannot be the end of the story. It is too small to account for soft graviton degrees of freedom, and it must be extended in some way. Unfortunately the argument does not tell us *how* BMS symmetry should be extended. To our knowledge, two proposals have been formulated so far, both suggesting a superrotational extension of the Lorentz group. The first is the aforementioned idea of turning superrotations into a semi-group of local conformal transformations of celestial spheres [63, 64, 71, 115, 116]; the second suggests that superrotations should instead span a group $\text{Diff}(S^2)$ of diffeomorphisms of the sphere [117, 118]. It appears that there are currently no definitive arguments for selecting one proposal over the other; see however [119], where it is argued that finite singular conformal transformations of celestial spheres in four dimensions are pathological.

The fact that BMS symmetry in four dimensions is ill-defined is a call for further developments. This thesis is one of them: it aims at understanding a three-dimensional toy model and using it as a guide for the realistic problem. Indeed, many properties that we have encountered in our investigation of BMS_3 should remain true in BMS_4. In particular the semi-direct product structure $G \ltimes A$ appears to be a robust feature and implies that BMS particles are classified by supermomentum orbits that coincide with orbits of the Bondi mass aspect under asymptotic symmetry transformations. Furthermore the occurrence of a central extension pairing superrotations with supertranslations has also been observed in BMS_4 [115]. However, a sharp difference between BMS_3 and BMS_4 is that, in the former, supertranslations do *not* create new states when acting on the vacuum. A possibly related difference is that the language suited to the study of BMS_4 appears to be that of *groupoids* rather than groups. We will not have much more to say about this here, and return now to our study of BMS_3.

10.2 BMS Modules and Flat Limits

In the previous pages we have described irreducible, unitary representations of BMS_3. In order to make contact with the representation theory of the Virasoro algebra it is useful to reformulate these representations in Lie-algebraic language, in the form of so-called *induced modules*. This reformulation will also allow us to discuss the flat limit of dressed particles in AdS_3 and to understand the difference between unitarity in BMS_3 and unitarity for Galilean conformal symmetry. The plan is as follows: we first describe induced modules for the Poincaré algebra in three dimensions and interpret them as ultrarelativistic limits of highest-weight representations of $\mathfrak{sl}(2,\mathbb{R})$, before applying the same construction to the $\mathfrak{bms}_3$ algebra. The presentation is adapted from [4].

10.2.1 Poincaré Modules in Three Dimensions

Our goal here is to rewrite the three-dimensional relativistic particles of Sect. 4.3 in Lie-algebraic language. This will allow us to relate Poincaré representations with the ultrarelativistic limit of highest-weight representations of $\mathfrak{sl}(2,\mathbb{R})\oplus\mathfrak{sl}(2,\mathbb{R})$.

Poincaré Algebra

In three dimensions, the Lie algebra of the Poincaré group is spanned by three Lorentz generators j_m and three translation generators p_m ($m=-1,0,1$) with Lie brackets (9.10). As in the case (8.54) of $\mathfrak{sl}(2,\mathbb{R})$, it is more convenient to use a different complexified basis $J_m = ij_m$, $P_m = ip_m$, in terms of which the brackets become

$$[J_m, J_n] = (m-n)\,J_{m+n}\,,\qquad [J_m, P_n] = (m-n)\,P_{m+n}\,,\qquad [P_m, P_n] = 0\,. \quad (10.26)$$

These conventions are such that, in any unitary representation, the operators representing Poincaré generators satisfy Hermiticity conditions of the type (8.56):

$$(P_m)^\dagger = P_{-m}\,,\qquad (J_m)^\dagger = J_{-m}\,. \quad (10.27)$$

As in Sect. 8.4, we abuse notation by denoting with the same letter both the abstract generators J_m, P_n and the operators that represent them.

The Poincaré algebra has two quadratic Casimir operators: the mass squared

$$\mathcal{M}^2 = P_0^2 - P_1 P_{-1} \quad (10.28)$$

and the three-dimensional analogue of the square of the Pauli–Lubanski vector,

$$\mathcal{S} = P_0 J_0 - \frac{1}{4}\left(J_1 P_{-1} + J_{-1} P_1 + P_1 J_{-1} + P_{-1} J_1\right). \quad (10.29)$$

The eigenvalues of these operators classify irreducible representations according to mass and spin, exactly as in Sect. 4.3. See e.g. [120] for the proof of the fact that the operator (10.28) actually takes the value M^2 in the space of states of a relativistic particle with mass M.

Induced Modules

Irreducible unitary representations of the Poincaré group are obtained by considering the Lorentz orbit of a momentum p and building a Hilbert space of wavefunctions on that orbit. A basis of this space is provided by plane waves (10.13), where the Dirac distribution is determined by the choice of measure on the orbit. (For instance one can take the Lorentz-invariant measure (3.4).) Their transformation laws are given by (10.15). Here, in order to make the link with the standard Dirac notation, we denote such plane waves by $\Psi_k \equiv |k, s\rangle$ for any $k \in \mathcal{O}_p$, where s is the spin of the representation.

For future comparison with $\mathfrak{bms}_3$, we focus on a relativistic particle with mass $M > 0$. Its little group U(1) consists of spatial rotations generated by J_0. If we call p the momentum of the representation in the rest frame, then the corresponding plane wave $\Psi_p \equiv |M, s\rangle$ satisfies

$$P_0|M, s\rangle = M|M, s\rangle\,, \quad P_{-1}|M, s\rangle = P_1|M, s\rangle = 0\,, \quad J_0|M, s\rangle = s|M, s\rangle\,. \tag{10.30}$$

From now on we call $|M, s\rangle$ the *rest frame state* of the representation. Any other plane wave $\Psi_k = |k, s\rangle$ with boosted momentum $k \in \mathcal{O}_p$ can be obtained by acting on $|M, s\rangle$ with a Lorentz transformation g_k, where g_k is a standard boost. In this sense the rest frame state determines all the properties of the representation, in the same way that highest-weight representations are determined by their highest-weight state. Note, however, that the conditions (10.30) that define $|M, s\rangle$ are *not* of the same form as the highest-weight conditions (8.57) in that they involve both positive and negative modes.

Let us now understand how the conditions (10.30) induce a representation of the Poincaré algebra. They define a one-dimensional representation of the subalgebra generated by $\{P_m, J_0\}$. This subalgebra consists of infinitesimal translations and spatial rotations, i.e. it is a semi-direct sum $\mathfrak{u}(1) \inplus \mathbb{R}^3$ where $\mathfrak{u}(1)$ is generated by J_0 while $\mathbb{R}^3$ is generated by the P_m's. Thus the prescription (10.30) is a Lie-algebraic version of the spin representation (4.28) for the case of a little group U(1) with $\mathcal{R}[\theta] = e^{is\theta}$. Guided by our experience of induced representations, we can attempt to induce a representation of the full Poincaré algebra out of the one-dimensional representation (10.30); the result is known as an *induced module* (see e.g. Sect. 10.7 of [121]). We thus declare that the carrier space $\mathscr{H}$ of the representation is spanned by all states obtained by acting on the rest frame state with operators that do *not* appear in the conditions (10.30):

$$|k, l\,\rangle = (J_{-1})^k (J_1)^l |M, s\rangle\,, \tag{10.31}$$

where k, l are non-negative integers. Such states are infinitesimally boosted states analogous to the descendant states (8.58) that span Verma modules for the Virasoro algebra. By definition, they form a basis of the space $\mathscr{H}$. The latter provides a Poincaré representation as it should, since acting from the left on the states (10.31) yields linear operators on $\mathscr{H}$ whose commutators coincide with (10.26). Moreover, the Casimir operators (10.28) and (10.29) have the same eigenvalue on each state (10.31), since they commute by construction with all elements of the algebra. This readily implies that the representation is irreducible.

Note that unitarity is not obvious in this picture: if one did not know that the induced module follows from a manifestly unitary representation of the Poincaré group in terms of wavefunctions, there would be no straightforward way to define a scalar product on the space $\mathscr{H}$ spanned by the states (10.31), even after enforcing the Hermiticity conditions (10.27). In fact, the norm squared of any plane wave state is strictly infinite because of the delta function in (10.14). This is strikingly different from the highest-weight representations of Sect. 8.4, where the highest-weight conditions were enough to evaluate the norm squared (8.59) of all descendant states.

Remark The definition of the infinitesimally boosted states (10.31) follows from the general construction of induced modules, as follows. Let $\mathfrak{g}$ be a Lie algebra with some subalgebra $\mathfrak{h}$. Let $\mathscr{S}$ be a one-dimensional representation of $\mathfrak{h}$. If $\mathcal{U}(\mathfrak{g})$ denotes the universal enveloping algebra of $\mathfrak{g}$, then the $\mathfrak{g}$-module $\mathscr{T} = \text{Ind}_{\mathfrak{h}}^{\mathfrak{g}}(\mathscr{S})$ induced by $\mathscr{S}$ is the representation of $\mathfrak{g}$ that acts in the space $\mathcal{U}(\mathfrak{g}) \otimes \mathbb{C}$ quotiented by the relations $Y \otimes \lambda = 1 \otimes \mathscr{S}[Y]\lambda$ for all $Y \in \mathfrak{h}$ and all $\lambda \in \mathbb{R}$. For any Lie algebra element $X \in \mathfrak{g}$, the operator $\mathscr{T}[X]$ acts on the carrier space by hitting on vectors from the left, in such a way that commutators of operators $\mathscr{T}[X]$ reproduce the Lie brackets of the Lie algebra $\mathfrak{g}$. In this language the Poincaré module above is induced by the representation (10.30) of the semi-direct sum $\mathfrak{g}_p \in \mathbb{R}^3$, where $\mathbb{R}^3$ is the Lie algebra of translations generated by the P_m's while $\mathfrak{g}_p$ is the Lie algebra of the little group generated by J_0. The conditions (10.30) define a one-dimensional representation $\mathscr{S}$ of $\mathfrak{g}_p \in \mathbb{R}^3$, analogous to the spin representation (4.28).

Ultrarelativistic Limit of $\mathfrak{sl}(2, \mathbb{R})$ Modules

In addition to being convenient for generalizations to infinite-dimensional extensions of the Poincaré algebra, Poincaré modules can be seen as a limits of unitary representations of the AdS_3 isometry algebra $\mathfrak{so}(2, 2) \cong \mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{sl}(2, \mathbb{R})$. The generators of the latter can be divided in two groups, L_m and $\bar{L}_m$ with $m = -1, 0, 1$, whose Lie brackets are two commuting copies of (8.55). In terms of these basis elements the quadratic Casimir of each copy of $\mathfrak{sl}(2, \mathbb{R})$ is (8.60). As usual our conventions are such that, in any unitary representation, the Hermiticity conditions (8.56) hold in both sectors.

The Poincaré algebra (10.26) can be recovered from $\mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{sl}(2, \mathbb{R})$ as a flat limit of the type described in Sect. 9.4. Thus we introduce a length scale ℓ (to be identified with the AdS radius) and define new generators as in (9.87):

$$J_m \equiv L_m - \bar{L}_{-m}, \qquad P_m \equiv \frac{1}{\ell}(L_m + \bar{L}_{-m}). \tag{10.32}$$

The resulting algebra is (9.88) without i's on the left-hand side, and its limit $\ell \to +\infty$ reproduces the Poincaré algebra (10.26). In addition the quadratic Casimir (8.60) can be combined with its barred counterpart $\bar{\mathcal{C}}$, producing

$$\frac{2}{\ell^2}\left(\mathcal{C}+\bar{\mathcal{C}}\right)=\mathcal{M}^2+\mathcal{O}(\ell^{-2}), \qquad \frac{1}{\ell}\left(\mathcal{C}-\bar{\mathcal{C}}\right)=\mathcal{S}, \tag{10.33}$$

where $\mathcal{M}^2$ and $\mathcal{S}$ are the Poincaré Casimirs (10.28) and (10.29).

The matching of Casimir operators suggests that the contraction also relates Poincaré modules to $\mathfrak{sl}(2,\mathbb{R})\oplus\mathfrak{sl}(2,\mathbb{R})$ representations. Concretely, consider the tensor product of two highest-weight representations (8.57) of $\mathfrak{sl}(2,\mathbb{R})$ with weights $h, \bar{h}$:

$$L_1|h,\bar{h}\rangle=\bar{L}_1|h,\bar{h}\rangle=0\,, \qquad L_0|h,\bar{h}\rangle=h|h,\bar{h}\rangle\,, \qquad \bar{L}_0|h,\bar{h}\rangle=\bar{h}|h,\bar{h}\rangle\,. \tag{10.34}$$

This yields an irreducible representation of $\mathfrak{sl}(2,\mathbb{R})\oplus\mathfrak{sl}(2,\mathbb{R})$ whose carrier space is spanned by descendant states $(L_{-1})^m(\bar{L}_{-1})^n|h,\bar{h}\rangle$. Let us rewrite this representation in terms of the operators (10.32). First we define the numbers

$$M\equiv\frac{h+\bar{h}}{\ell}\,, \qquad s\equiv h-\bar{h}\,, \tag{10.35}$$

which are eigenvalues of energy and angular momentum:

$$P_0|h,\bar{h}\rangle=\frac{h+\bar{h}}{\ell}|h,\bar{h}\rangle\,, \qquad J_0|h,\bar{h}\rangle=(h-\bar{h})|h,\bar{h}\rangle \tag{10.36}$$

in terms of operators (10.32). Similarly, in terms of J's and P's, the condition that L_1 and $\bar{L}_1$ annihilate the highest-weight state becomes

$$\Big(P_{\pm1}\pm\frac{1}{\ell}J_{\pm1}\Big)|h,\bar{h}\rangle=0\,. \tag{10.37}$$

This allows us to reformulate the whole representation of $\mathfrak{sl}(2,\mathbb{R})\oplus\mathfrak{sl}(2,\mathbb{R})$ in terms of operators J_m, P_n; it results in expressions of the form

$$P_n|k,l\rangle=\sum_{k',l'}\mathsf{P}^{(n)}_{k',l';\,k,l}(M,s,\ell)|k',l'\rangle\,, \qquad J_n|k,l\rangle=\sum_{k',l'}\mathsf{J}^{(n)}_{k',l';\,k,l}(M,s)|k',l'\rangle \tag{10.38}$$

where the states $|k,l\rangle$ take the form (10.31) with the identification $|M,s\rangle\equiv|h,\bar{h}\rangle$, while $\mathsf{P}^{(n)}$ and $\mathsf{J}^{(n)}$ are infinite matrices. Owing to the definition (10.32) and property (10.37), only negative powers of ℓ appear in (10.38). It follows that the matrix elements $\mathsf{P}^{(n)}_{k',l';\,k,l}$ and $\mathsf{J}^{(n)}_{k',l';\,k,l}$ have a well-defined limit $\ell\to\infty$. This limit coincides with the result that one would find in a Poincaré module spanned by states (10.31), provided that the conformal weights scale as

$$h=\frac{M\ell+s}{2}+\lambda+\mathcal{O}(1/\ell), \qquad \bar{h}=\frac{M\ell-s}{2}+\lambda+\mathcal{O}(1/\ell), \tag{10.39}$$

where λ is an arbitrary parameter independent of ℓ. Thus, in the flat limit, the $\mathfrak{sl}(2,\mathbb{R})$ highest-weight conditions (10.37) are turned into a rest frame condition (10.30) and the Poincaré Casimirs $\mathcal{M}^2$ and $\mathcal{S}$ take the values M^2 and Ms, respectively. In short, Poincaré modules are flat limits of $\mathfrak{sl}(2,\mathbb{R})\oplus\mathfrak{sl}(2,\mathbb{R})$ modules.

Note again that unitarity is subtle: starting from scalar products of $\mathfrak{sl}(2,\mathbb{R})\oplus\mathfrak{sl}(2,\mathbb{R})$ descendants, the flat limit gives rise to scalar products of states (10.31) that diverge like positive powers of ℓ. Indeed the norms (8.59) diverge when $\ell\to+\infty$, owing to the fact that h is proportional to ℓ in (10.39). Equivalently, the wavefunctions corresponding to states (10.31) become (derivatives of) delta functions in the flat limit; from this point of view ℓ is an infrared regulator. Nevertheless, upon recognizing these divergent scalar products as delta functions (10.14), one concludes that the Poincaré module is a unitary representation in disguise.

Relation (10.39) shows that the contraction defined by (10.32) is an ultrarelativistic/high-energy limit from the viewpoint of AdS_3. Poincaré modules are thus remnants of $\mathfrak{so}(2,2)$ representations whose energy becomes large in the limit $\ell\to\infty$. In Sect. 10.2.3 we shall see that the *non*-relativistic contraction from $\mathfrak{so}(2,2)$ to $\mathfrak{iso}(2,1)$ gives rise to representations of a different type, that have been discussed in [122, 123].

We should mention that highest-weight representations of $\mathfrak{sl}(2,\mathbb{R})$ can also be interpreted as induced modules. Indeed Eq. (8.57) defines a one-dimensional representation of the subalgebra spanned by $\{L_0, L_1\}$, while the vector space of descendant states can be identified with a quotient of $\mathcal{U}(\mathfrak{sl}(2,\mathbb{R}))\otimes\mathbb{C}$ as discussed in the remark of page 3.15. The main difference with respect to Poincaré is the splitting of the algebra as $\mathfrak{n}^-\oplus\mathfrak{h}\oplus\mathfrak{n}^+$, where $\mathfrak{n}^\pm$ are nilpotent subalgebras, which allows one to evaluate scalar products by enforcing the Hermiticity conditions (8.56).

10.2.2 Induced Modules for $\mathfrak{bms}_3$

Let us now apply the considerations of the previous pages to three-dimensional BMS symmetry. The centrally extended $\mathfrak{bms}_3$ algebra[4] is spanned by superrotation generators $\mathcal{J}_m$ and supertranslation generators $\mathcal{P}_m$ ($m\in\mathbb{Z}$) together with central charges $\mathcal{Z}_1$, $\mathcal{Z}_2$, whose Lie brackets take the form (9.68). Following the Virasoro convention (8.63), we change the normalization and define

$$J_m \equiv i\mathcal{J}_m + i\frac{\mathcal{Z}_1}{24}\delta_{m,0}\,,\qquad P_m \equiv i\mathcal{P}_m + i\frac{\mathcal{Z}_2}{24}\delta_{m,0}\,,\tag{10.40}$$

as well as $Z_1\equiv i\mathcal{Z}_1$ and $Z_2\equiv i\mathcal{Z}_2$. The constant shifts in P_0 and J_0 ensure that the vacuum state has zero eigenvalues under these operators. According to this definition the operators representing J_m and P_m in any unitary representation satisfy the Hermiticity conditions (10.27). Furthermore, in any irreducible representation the

[4]Recall that we use the same notation for both the $\mathfrak{bms}_3$ algebra and its central extension.

central charges Z_1 and Z_2 take definite values c_1, c_2, so we can write the commutation relations of the $\mathfrak{bms}_3$ algebra in a form analogous to (8.65):

$$\begin{aligned} [J_m, J_n] &= (m-n)J_{m+n} + \frac{c_1}{12}\, m(m^2-1)\,\delta_{m+n,0}\,, \\ [J_m, P_n] &= (m-n)P_{m+n} + \frac{c_2}{12}\, m(m^2-1)\,\delta_{m+n,0}\,, \\ [P_m, P_n] &= 0\,. \end{aligned} \tag{10.41}$$

In contrast to Poincaré, the quadratic operators (10.28)–(10.29) no longer commute with the algebra (10.41). Nevertheless, the classification of BMS$_3$ representations in Sect. 10.1 provides at least one obvious, yet non-trivial, Casimir operator.[5] Indeed, in the Hilbert space of any BMS$_3$ particle, the "mass operator" (10.3) takes a definite value when M is the monodromy matrix whose trace is given by the Wilson loop (7.21), with c replaced by c_2 and $p(\varphi)$ replaced by the "supermomentum operator"

$$\hat{p}(\varphi) = \sum_{m\in\mathbb{Z}} P_m e^{-im\varphi} - \frac{c_2}{24}$$

where the P_m's are the supertranslation generators appearing in (10.41). Accordingly, upon writing the right-hand side of (10.3) in terms of P_m's, one obtains a highly non-linear combination of operators that commutes, by construction, with the entire $\mathfrak{bms}_3$ algebra. (That it commutes with P_m's is trivial, since all supertranslations commute; that it commutes with J_m's follows from the fact that (10.3) is invariant under superrotations.) The value of that Casimir operator can be used to classify BMS$_3$ particles, as we have done in Sect. 10.1.

Aside from the mass operator (10.3), any function of the BMS$_3$ central charges c_1, c_2 is clearly a Casimir. To our knowledge, whether this list exhausts all possible $\mathfrak{bms}_3$ Casimirs is an open question, though it seems plausible that it does since the only Casimirs of the Virasoro algebra are functions of its central charges [124]. In particular, it is not clear whether there exists a $\mathfrak{bms}_3$ Casimir whose value specifies the spin of a BMS$_3$ particle, analogously to the Poincaré combination (10.29).

We now describe induced modules for the $\mathfrak{bms}_3$ algebra (10.41), built analogously to the Poincaré modules above and classified by their mass and spin (and central charges). We discuss separately generic massive modules and the vacuum module, and end by showing how they can all be obtained as ultrarelativistic limits of highest-weight representations of Virasoro algebras.

Massive Modules

Consider a BMS$_3$ particle with mass $M > 0$ and spin s. Its supermomentum orbit contains a constant $p = M - c_2/24$; the corresponding plane wave state $\Psi_p \equiv |M, s\rangle$ is such that

[5] I am indebted to Axel Kleinschmidt for this observation.

$$P_0|M, s\rangle = M|M, s\rangle, \qquad P_m|M, s\rangle = 0 \text{ for } m \neq 0, \qquad J_0|M, s\rangle = s|M, s\rangle. \tag{10.42}$$

Thus $|M, s\rangle$ is a supermomentum eigenstate with vanishing eigenvalues under P_m, $m \neq 0$. In analogy with (10.30), we call $|M, s\rangle$ the *rest frame state* of the module.

As in the Poincaré case, the conditions (10.42) define a one-dimensional representation of the subalgebra of (10.41) spanned by $\{P_n, J_0, c_1, c_2\}$. This representation can be used to define an induced module $\mathscr{H}$ with basis vectors analogous to (10.31),

$$J_{n_1} J_{n_2} \cdots J_{n_N} |M, s\rangle, \tag{10.43}$$

where the n_i's are non-zero integers such that $n_1 \leq n_2 \leq \cdots \leq n_N$. With this ordering, states (10.43) with different combinations of n_i's are linearly independent within the universal enveloping algebra of $\mathfrak{bms}_3$, and acting on them from the left with the generators of the algebra provides linear operators on $\mathscr{H}$ whose commutators coincide with (10.41). Thus one readily obtains a representation of the $\mathfrak{bms}_3$ algebra.

As in the Poincaré case above, unitarity is hidden in this picture because there is no straightforward way to compute scalar products of states (10.43). In fact, since $|M, s\rangle$ is a delta function, all such states strictly have infinite norm. This is because realistic states of BMS_3 particles are smeared wavefunctions that consist of infinite linear combinations of plane waves. Unitarity can then be recognized in the fact that acting with (finite) superrotations on $|M, s\rangle$ produces a "basis" of plane waves that generate a space of square-integrable wavefunctionals on the supermomentum orbit. In particular the representation is automatically irreducible in the sense that all basis states are obtained by acting with symmetry transformations on the single state $|M, s\rangle$.

Vacuum Module

Recall that the BMS_3 vacuum is the scalar representation whose supermomentum orbit $\text{Diff}(S^1)/\text{PSL}(2, \mathbb{R})$ contains the vacuum configuration $p = p_{\text{vac}} = -c_2/24$. The corresponding induced module can be described similarly to massive ones. Owing to the normalization (10.40), the Hilbert space of the vacuum representation contains a plane wave $\Psi_p \equiv |0\rangle$ such that

$$P_m|0\rangle = 0 \;\text{ for all } m \in \mathbb{Z} \qquad \text{and} \qquad J_n|0\rangle = 0 \;\text{ for } n = -1, 0, 1. \tag{10.44}$$

Here the condition $P_0|0\rangle = 0$ says that the vacuum has zero mass for the normalization (10.40), while the extra conditions $J_{\pm 1}|0\rangle = 0$ enforce Lorentz-invariance. They reflect the fact that the little group of the vacuum is the whole Lorentz group, rather than the group of spatial rotations that occurs for massive particles.

If we were dealing with the Poincaré algebra, the requirements (10.44) would produce a trivial representation. Here, by contrast, there exist non-trivial "boosted vacua" of the form (10.43), where now the n_i's are integers different from $-1, 0, 1$. These vacua are Lie-algebraic analogues of the boundary gravitons described earlier. The fact that the vacuum is not invariant under the full BMS_3 symmetry, but only under its Poincaré subgroup, suggests that the boosted states (10.43) (with all n_i's $\neq -1, 0, 1$) can be interpreted as Goldstone-like states associated with broken symmetry generators; see the discussion surrounding (8.81). Note that, in contrast to the realistic four-dimensional case, BMS$_3$ supertranslations do *not* create new states when acting on the vacuum.

Ultrarelativistic Limit of Virasoro Modules

In analogy with the observations of Sect. 10.2.1, $\mathfrak{bms}_3$ modules may be seen as limits of tensor products of highest-weight representations of Virasoro. Let therefore L_m, $\bar{L}_m$ be generators of two commuting copies of the Virasoro algebra (8.65) with definite central charges c, $\bar{c}$. Highest-weight representations are then obtained starting from a primary state $|h, \bar{h}\rangle$ which satisfies (10.34) together with

$$L_m|h, \bar{h}\rangle = \bar{L}_m|h, \bar{h}\rangle = 0 \quad \text{for } m > 0. \tag{10.45}$$

The carrier space is spanned by descendant states

$$L_{-n_1} \dots L_{-n_k} \bar{L}_{-\bar{n}_1} \dots \bar{L}_{-\bar{n}_l}|h, \bar{h}\rangle \tag{10.46}$$

with $1 \leq n_1 \leq n_2 \leq \cdots \leq n_k$ and $1 \leq \bar{n}_1 \leq \cdots \leq \bar{n}_l$. Since we eventually wish to take the ultrarelativistic limit of this representation, we will be interested in large values of h and $\bar{h}$, where the representation is irreducible and unitary thanks to the standard Hermiticity conditions (8.64).

As in the Poincaré case, one can define new generators (10.32), now including also the central charges c_1, c_2 defined by (9.93). In particular, the space of Virasoro descendants can be rewritten in the basis (10.43) with the identification $|M, s\rangle \equiv |h, \bar{h}\rangle$, where M and s are the eigenvalues of P_0 and J_0 related to h and $\bar{h}$ by (10.35). The change of basis from descendant states (10.46) to infinitesimally boosted states (10.43) is invertible because none of the J_n's annihilate the highest-weight state. The resulting Virasoro representation takes a form analogous to (10.38), where now each state is labelled by the quantum numbers n_i of (10.43) and the matrices $\mathsf{P}^{(n)}$ and $\mathsf{J}^{(n)}$ also depend on the central charges (9.93). As before, only negative powers of ℓ enter $\mathsf{P}^{(n)}$ via the highest-weight conditions (10.45) written in the new basis:

$$\Big(P_{\pm n} \pm \frac{1}{\ell} J_{\pm n}\Big)|h, \bar{h}\rangle = 0. \tag{10.47}$$

A limit $\ell \to \infty$ performed at fixed M, s and c_1, c_2 (rather than fixed h, $\bar{h}$ say) then yields a massive $\mathfrak{bms}_3$ module of the type described above. In particular, the limit maps the highest-weight state (10.45) on the rest frame state (10.42). In this sense $\mathfrak{bms}_3$ modules are high-energy limits of tensor products of Virasoro modules, since h and $\bar{h}$ go to infinity in the flat limit. By the way, this provides an intuitive picture of why the energy spectrum of BMS$_3$ particles is continuous: the typical distance between two consecutive eigenvalues of $P_0 = (L_0 + \bar{L}_0)/\ell$ is $1/\ell$, which shrinks to zero when ℓ goes to infinity.

10.2.3 Representations of the Galilean Conformal Algebra

Here we revisit the Galilean conformal algebra introduced in Sect. 9.4. As explained there, $\mathfrak{gca}_2$ coincides with $\mathfrak{bms}_3$, which effectively makes them classically interchangeable. We now argue that this equivalence does *not* hold at the quantum level; this observation will be the basis of our arguments in Sect. 11.3, when explaining why the quantization of asymptotically flat gravity cannot be a Galilean conformal field theory. The highest-weight representations that we shall describe here were first obtained in [122], but their identification with induced representations of BMS_3 is new.

Highest-Weight Representations of $\mathfrak{gca}_2$

The $\mathfrak{gca}_2$ algebra is isomorphic to $\mathfrak{bms}_3$, but their interpretations differ: in $\mathfrak{bms}_3$ the non-Abelian generators generalize the angular momentum operator and span superrotations, while the Abelian ones generalize the Hamiltonian and span supertranslations. By contrast, in $\mathfrak{gca}_2$, the non-Abelian generators are the ones that generalize the Hamiltonian, and the Abelian ones generalize (angular) momentum. Due to this difference, one is naturally led to look for unitary representations of $\mathfrak{gca}_2$ where the operator J_0 of (10.41) is bounded from below. From the BMS_3 perspective this is an awkward choice (since it explicitly breaks parity by forcing all states to have the same sign of angular momentum), but from the Galilean viewpoint it is perfectly well motivated.

To describe these representations we use the method of induced modules applied to the algebra (10.41), with the Hermiticity conditions (10.27). To stress that we are dealing with *Galilean* rather than *relativistic* representations, we denote all $\mathfrak{gca}_2$ generators with a tilde on top, such as $\tilde{J}_m$, $\tilde{P}_m$, plus central charges $\tilde{c}_1$, $\tilde{c}_2$. In order to obtain a representation where the spectrum of $\tilde{J}_0$ is bounded from below, we start from a state $|\tilde{M},\tilde{s}\rangle$ which has highest weight for the Virasoro subalgebra generated by $\tilde{J}$'s in the sense that

$$\tilde{J}_0|\tilde{M},\tilde{s}\rangle = \tilde{s}|\tilde{M},\tilde{s}\rangle, \qquad \tilde{P}_0|\tilde{M},\tilde{s}\rangle = \tilde{M}|\tilde{M},\tilde{s}\rangle \tag{10.48}$$

and

$$\tilde{J}_m|\tilde{M},\tilde{s}\rangle = \tilde{P}_m|\tilde{M},\tilde{s}\rangle = 0 \qquad \text{for } m > 0\,. \tag{10.49}$$

We stress that $\tilde{s}$ is now interpreted as a (dimensionless) energy while $\tilde{M}$ is a (dimensionful) momentum. In analogy with Virasoro representations, one can then define descendant states of the form

$$\tilde{P}_{-k_1}\ldots\tilde{P}_{-k_n}\tilde{J}_{-l_1}\ldots\tilde{J}_{-l_m}|\tilde{M},\tilde{s}\rangle \tag{10.50}$$

with $1 \leq k_1 \leq \cdots \leq k_n$, $1 \leq l_1 \leq \cdots \leq l_m$, and declare that they form a basis of the carrier space. The conditions (10.49) allow one to evaluate the would-be scalar products of such descendants upon using the Hermiticity conditions (10.27), and one finds that the representation is *non-unitary* whenever $\tilde{M} \neq 0$ or $\tilde{c}_2 \neq 0$ [122]. When $\tilde{M} = c_2 = 0$, unitarity requires in addition that $\tilde{s} > 0$ and that the superrotation central charge $\tilde{c}_1$ be non-negative. Thus, unitary representations of $\mathfrak{gca}_2$ boil down to highest-weight representations of its Virasoro subalgebra generated by the $\tilde{J}$'s.

A similar construction can be applied to a vacuum-like highest-weight representation of $\mathfrak{gca}_2$, whose highest-weight state $|\tilde{0}\rangle$ is annihilated by all Poincaré generators $\tilde{J}_{-1}, \tilde{J}_0, \tilde{J}_1, \tilde{P}_{-1}, \tilde{P}_0, \tilde{P}_1$ and by all positive modes as in (10.48):

$$\tilde{P}_m|\tilde{0}\rangle = 0\,, \qquad \tilde{J}_m|\tilde{0}\rangle = 0 \qquad \text{for } m \geq -1. \tag{10.51}$$

Again, one concludes in that case that the representation is unitary if and only if $\tilde{c}_2 = 0$ and $\tilde{c}_1 \geq 0$.

The highest-weight representations of the type just described which are *unitary* (i.e. have $\tilde{M} = 0$ and $\tilde{c}_2 = 0$) are special cases of induced representations of BMS$_3$ as described in Sect. 10.1. Indeed, consider the vanishing supermomentum $(\tilde{p}, \tilde{c}_2) = (0, 0)$. Its orbit under superrotations is trivial and its little group is the whole Virasoro group, so the corresponding induced representation is entirely determined by its spin $\tilde{s}$. The latter labels a unitary highest-weight representation of Virasoro, with central charge $\tilde{c}_1$ say. At the Lie-algebraic level this spin representation takes the form of a highest-weight representation (10.48)–(10.49) with $\tilde{M} = 0$ and $\tilde{c}_2 = 0$. There is an analogue of this construction in the Poincaré group: the vanishing momentum vector $p = 0$ has a trivial orbit and its little group is the whole Lorentz group, so the corresponding induced representation of Poincaré is just a unitary representation of the Lorentz group; it is a "vacuum with spin" of the type mentioned in Sect. 4.2.

The difference between the BMS$_3$ vacuum (10.44) and the Galilean vacuum (10.51) implies sharp differences for all quantum systems enjoying such symmetries, since it affects the definition of normal ordering. For example, the normal-ordered product $:J_2P_{-3}:$ equals J_2P_{-3} in a BMS$_3$-invariant theory, while in a Galilean conformal field theory one has $:\tilde{J}_2\tilde{P}_{-3}: = \tilde{P}_{-3}\tilde{J}_2$. We shall see explicit illustrations of this phenomenon in Sect. 11.3 below, when dealing with non-linear higher-spin symmetry algebras. It suggests in particular that theories enjoying $\mathfrak{bms}_3$ symmetry or $\mathfrak{gca}_2$ symmetry differ greatly at the quantum level, despite the isomorphism $\mathfrak{bms}_3 \cong \mathfrak{gca}_2$.

Galilean Limit of Virasoro Modules

We now recover the (generally non-unitary) Galilean highest-weight representations defined by (10.48)–(10.49) as a non-relativistic limit of Virasoro modules. As before we let the generators L_m, $\bar{L}_n$ satisfy the algebra (8.65) with central charges $c, \bar{c}$ respectively, and we consider a highest-weight representation of the type (10.34)–(10.45). In order to take the non-relativistic limit (9.94), we introduce a length scale ℓ and define

$$\tilde{J}_n \equiv \bar{L}_n + L_n\,, \qquad \tilde{P}_n \equiv \frac{1}{\ell}\left(\bar{L}_n - L_n\right)\,. \tag{10.52}$$

We stress that the combinations of L_m's appearing here differ from those of the ultrarelativistic limit (10.32). In particular, $\tilde{J}_0$ now generates time translations while

$\tilde{P}_0$ generates spatial translations; the parameter ℓ should no longer be interpreted as the AdS_3 radius, and there is no mixing between positive and negative modes. In these terms, the limit $\ell \to +\infty$ of the direct sum of two Virasoro algebras reduces to a $\mathfrak{gca}_2$ algebra (10.41) with tildes on top of all generators, including the central charges

$$\tilde{c}_1 = \bar{c} + c \,, \qquad \tilde{c}_2 = \frac{\bar{c} - c}{\ell} \,. \tag{10.53}$$

Note that, up to central charges, the same redefinitions applied to $\mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{sl}(2, \mathbb{R})$ reproduce the Poincaré algebra $\mathfrak{iso}(2, 1)$; this is a *non*-relativistic limit to be contrasted with the *ultra*relativistic limit described in Sect. 10.2.1.

In analogy with Sect. 10.2.2, let us rewrite the tensor product of two highest-weight representations of Virasoro in terms of the operators (10.52). Given the weights $h, \bar{h}$ we define

$$\tilde{s} \equiv \bar{h} + h \,, \qquad \tilde{M} \equiv \frac{\bar{h} - h}{\ell}$$

which we stress is radically different from the ultrarelativistic redefinition (10.35). Upon identifying $|\tilde{M}, \tilde{s}\rangle \equiv |h, \bar{h}\rangle$, the highest-weight state satisfies (10.48) and (10.49). These conditions hold for any value of ℓ, including the limit $\ell \to +\infty$. The descendant states (10.50) then provide a representation of the sum of two Virasoro algebras, which in the non-relativistic limit $\ell \to +\infty$ becomes a generically *non-unitary* representation of $\mathfrak{gca}_2$. Unitarity is recovered if $\tilde{M} = \tilde{c}_2 = 0$ and $\tilde{s}, \tilde{c}_1 \geq 0$. Again, this is strikingly different from the ultrarelativistic contraction described above.

Remark The difference between $\mathfrak{gca}_2$ modules and $\mathfrak{bms}_3$ modules has been known, albeit in disguise, ever since the nineties. Namely, the tensionless limit of string theory gives rise to so-called *null strings* [125], whose worldsheet is a null surface and thus provides a stringy generalization of null geodesics. It was observed in [126] that the algebra of constraints arising from worldsheet reparameterization invariance of null strings is the $\mathfrak{bms}_3$ algebra, although the name "BMS" was not used at the time.[6] In the same paper the authors observed that a suitable normal-ordering prescription gives rise to a consistent quantization of the null string in *any* space-time dimension, and systematically results in a continuous mass spectrum. This result is the stringy analogue of the $\mathfrak{bms}_3$ modules described in Sect. 10.2.2. Only later was it realized that a different normal ordering prescription [128, 129] gives rise to the same critical dimension as in standard string theory (26 for the bosonic string and 10 for the superstring), but that the resulting spectrum is massless and discrete; in fact, the spectrum then coincides with the massless part of the spectrum of standard string theory. The latter result is the stringy analogue of the $\mathfrak{gca}_2$ modules described here and the critical dimension is analogous to the requirement $\tilde{c}_2 = 0$ that ensures unitarity for non-relativistic modules; see [130–132] for a recent account of these results.

[6]This occurrence of the $\mathfrak{bms}_3$ algebra predates its gravitational description [127] by a decade!

10.3 Characters of the BMS$_3$ Group

Now that we are acquainted with BMS$_3$ particles, we can start using them as a computational tool. In this section we use the Frobenius formula (4.33),[7]

$$\chi[(f,\alpha)] = \mathrm{Tr}\big(\mathcal{T}[(f,\alpha)]\big) = \int_{\mathcal{O}_p} d\mu(k)\,\delta(k, f\cdot k)\,e^{i\langle k,\alpha\rangle}\chi_{\mathcal{R}}[g_k^{-1} f g_k]\,, \quad (10.54)$$

to evaluate characters of rotations f and supertranslations α in induced representations of the (centrally extended) BMS$_3$ group. Remarkably, the localization effect due to the delta function will allow us to compute characters despite the fact that we do not know explicit measures on supermomentum orbits. We focus on massive BMS$_3$ particles and on the BMS$_3$ vacuum, and compare the results to the ultrarelativistic limit of Virasoro characters. The results reviewed here were first reported in [3]; their application to partition functions [5, 6] will be exposed in the next chapter.

10.3.1 Massive Characters

We consider a BMS$_3$ particle with mass $M > 0$ and spin $s \in \mathbb{R}$, so that the little group representation is $\mathcal{R}[\theta] = e^{is\theta}$. Our goal is to evaluate the character (10.54) for arbitrary BMS$_3$ transformations (f, α), along the lines described in Sect. 4.2 for the Poincaré group. The only subtlety is that now $\mathcal{O}_p \cong \mathrm{Diff}(S^1)/S^1$ is an infinite-dimensional supermomentum orbit; the g_k's of Eq. (10.54) are standard boosts and the pairing $\langle k, \alpha\rangle$ is given by (6.34). We assume as before that there exists a quasi-invariant measure μ on the supermomentum orbit $\mathcal{O}_p$, but we stress again that different measures yield equivalent representations so that the end result will be independent of μ. We shall verify this point explicitly below.

The character (10.54) vanishes if f is not conjugate to an element of the little group U(1); furthermore it is a class function, so we may take $f(\varphi) = \varphi + \theta$ to be a pure rotation by some angle θ, without loss of generality. We assume for simplicity that θ is non-zero; the case $\theta = 0$ is radically different, and we shall briefly comment about it below. When $\theta \neq 0$, the delta function $\delta(k, f \cdot k)$ of (10.54) localizes the supermomentum integral to the unique point of $\mathcal{O}_p$ that is left invariant by rotations, namely the supermomentum at rest $p = M - c_2/24$. This allows us to pull the little

[7] Here δ denotes the delta function associated by (3.39) with the measure μ.

group character $\chi_{\mathcal{R}}[f] = e^{is\theta}$ out of the integral and to reduce the pairing $\langle k, \alpha \rangle$ to a product $p\alpha^0 = M\alpha^0 - c_2\alpha^0/24$, so the whole character (10.54) boils down to a BMS$_3$ analogue of eq. (4.55):

$$\chi[(f, \alpha)] = e^{is\theta} e^{i\alpha^0(M-c_2/24)} \int_{\mathcal{O}_p} d\mu(k)\, \delta(k, f \cdot k)\,. \tag{10.55}$$

To evaluate the character it only remains to integrate the delta function. This requires local coordinates on the orbit in a neighbourhood of p, which can be obtained by Fourier-expanding each supermomentum $k(\varphi)$ as in Eq. (6.116). The Fourier modes $k_n = k^*_{-n}$ then transform under rotations $f(\varphi) = \varphi + \theta$ according to

$$k_n \mapsto \big[f \cdot k\big]_n = k_n\, e^{in\theta}.$$

As we shall see, the character that follows from this transformation is divergent due to the fact that the group is infinite-dimensional. To cure this divergence we consider complex rotations rather than real ones and introduce a complex parameter

$$\tau \equiv \frac{1}{2\pi}(\theta + i\epsilon) \tag{10.56}$$

where $\epsilon > 0$. We then define the transformation of supermomentum Fourier modes under complex rotations to be

$$k_n \mapsto \big[f \cdot k\big]_n = \begin{cases} k_n e^{2\pi i n\tau} & \text{if } n > 0, \\ k_0 & \text{if } n = 0, \\ k_n e^{2\pi i n\bar{\tau}} & \text{if } n < 0. \end{cases} \tag{10.57}$$

We will see below that this modification can be justified by thinking of BMS$_3$ representations as high-energy limits of Verma modules. Note that this prescription leaves room for "Euclidean" rotations (i.e. rotations by an imaginary angle) while preserving the reality condition $(k_n)^* = k_{-n}$.

The problem now is to express the measure μ and the corresponding delta function δ in terms of Fourier modes. On the massive supermomentum orbit $\mathcal{O}_p \cong \text{Diff}(S^1)/S^1$, the non-zero Fourier modes of $k(\varphi)$ determine its energy k_0. This is analogous to the statement that the energy of a relativistic particle is determined by its momentum according to $E = \sqrt{M^2 + \mathbf{k}^2}$. Let us prove this in a neighbourhood of the supermomentum at rest, $p = M - c_2/24$, by acting on it with an infinitesimal superrotation X that we Fourier-expand as

$$X(\varphi) = i \sum_{n\in\mathbb{Z}} X_n e^{-in\varphi}.$$

Since the action of superrotations on supermomenta is the coadjoint representation (6.115) of the Virasoro algebra, we find a variation

$$(\delta_X p)(\varphi) = \sum_{n\in\mathbb{Z}} 2n\left(M + \frac{c_2}{24}(n^2-1)\right) X_n e^{-in\varphi} \equiv \sum_{n\in\mathbb{Z}} \delta p_n\, e^{-in\varphi}. \tag{10.58}$$

Here the variation of the zero-mode, δp_0, vanishes for any choice of X. By contrast, all other Fourier modes are acted upon in a non-trivial way and can therefore take arbitrary values by a suitable choice of X. This implies that (at least in a neighbourhood of p) the non-zero modes of supermomenta provide local coordinates on $\mathcal{O}_p$. In terms of the Fourier decomposition (6.116), this is to say that when $k(\varphi) = p + \varepsilon(\delta_X p)(\varphi)$, the non-zero modes k_n coincide with $\varepsilon\delta p_n$ (while $k_0 = p$ to first order in ε).

It follows that in terms of k_n's, the supermomentum measure μ of (10.55) reads

$$d\mu(k) = (\text{Some } k - \text{dependent prefactor}) \times \prod_{n\in\mathbb{Z}^*} dk_n \tag{10.59}$$

where the prefactor is unknown. In quantum mechanics one would write the infinite product $\prod_{n\in\mathbb{Z}^*} dk_n$ as a path integral measure $\mathcal{D}k$, with the extra rule that the zero-mode of k is not to be integrated over. The definition of the delta function (3.39) associated with μ ensures that

$$\delta(q,k) = (\text{Some } k - \text{dependent prefactor})^{-1} \times \prod_{n\in\mathbb{Z}^*} \delta(q_n - k_n),$$

where the δ on the right-hand side is the usual Dirac distribution in one dimension. Crucially, the prefactor appearing in front of the delta function is the inverse of the prefactor of the measure (10.59). As in Eq. (4.56) this implies that the combination $d\mu(k)\delta(k,\cdot)$ is invariant under changes of measures, and it allows us to rewrite the character (10.55) as

$$\begin{aligned}\chi[(f,\alpha)] &= e^{is\theta} e^{i\alpha^0(M-c_2/24)} \int_{\mathbb{R}^{2\infty}} \prod_{n\in\mathbb{Z}^*} dk_n \prod_{n\in\mathbb{Z}^*} \delta\left(k_n - [f\cdot k]_n\right) \\ &\overset{(10.57)}{=} e^{is\theta} e^{i\alpha^0(M-c_2/24)} \left| \int_{\mathbb{R}^\infty} \prod_{n=1}^{+\infty} dk_n \prod_{n=1}^{+\infty} \delta\left(k_n(1-e^{2\pi i n\tau})\right)\right|^2, \end{aligned} \tag{10.60}$$

where we have replaced the real angle θ by its complex counterpart $2\pi\tau$ given by (10.56). Denoting $q \equiv \exp[2\pi i\tau]$ and evaluating the integral, the character of a massive BMS$_3$ particle finally reduces to

$$\boxed{\chi[(f,\alpha)] = e^{is\theta} e^{i\alpha^0(M-c_2/24)} \frac{1}{\prod_{n=1}^{+\infty} |1-q^n|^2}\,.} \tag{10.61}$$

We stress that this holds only provided f is conjugate to a rotation by θ. We recognize here the ubiquitous factor $(1-q)^{-1}$ arising from the Atiyah–Bott fixed point theorem (4.61). The result can also be rewritten in terms of the Dedekind eta function (8.75),

$$\chi[(f,\alpha)] = \frac{|q|^{1/12}}{|\eta(\tau)|^2} e^{is\theta} e^{i\alpha^0(M-c_2/24)},$$

with $|q| = 1$ in the (pathological) limit $\epsilon \to 0$.

Remark At this stage, and in contrast to conformal field theory, the coefficient τ should not be seen as a modular parameter. The small parameter ϵ in (10.56) was merely introduced to ensure convergence of the determinant arising from the integration of the delta function in (10.60). This being said, the occurrence of the Dedekind eta function is compatible with the modular transformations used in [74, 87] to derive a Cardy-like formula reproducing the entropy of flat space cosmologies.

10.3.2 Comparison to Poincaré and Virasoro

Formula (10.61) extends the Poincaré character (4.98) in three dimensions. Indeed, taking $\epsilon = 0$ in (10.61) and forgetting about all convergence issues, one finds

$$\chi[(f,\alpha)] = e^{is\theta} e^{i\alpha^0(M-c_2/24)} \prod_{n=1}^{+\infty} \frac{1}{4\sin^2(n\theta/2)}\,.$$

Here the term $n = 1$ coincides (4.98), while the contribution of higher Fourier modes is due, loosely speaking, to the infinitely many Poincaré subgroups of BMS$_3$. This is analogous to the fact that the Virasoro character (8.74) may be seen as a product of infinitely many SL$(2,\mathbb{R})$ characters (5.102) labelled by an integer n.

The divergence of the BMS$_3$ character (10.61) as $\epsilon \to 0$ is identical to that of the Virasoro character (8.74) as τ becomes real. In this sense, the divergence is not a pathology of BMS$_3$, but rather a general phenomenon to be expected from infinite-dimensional groups; here we have cured this divergence by adding an imaginary part $i\epsilon$ to the angle. The origin of this imaginary part can be traced back to the fact that BMS$_3$ representations are ultrarelativistic limits of Virasoro representations, as discussed at length in Sect. 10.2.2. Indeed, suppose we are given a tensor product of two Virasoro representations with highest weights h, $\bar{h}$ and central charges c, $\bar{c}$. The corresponding character generalizes the partition function (8.80) as

$$\text{Tr}\left(q^{L_0-c/24}\bar{q}^{\bar{L}_0-\bar{c}/24}\right) \overset{(8.74)}{=} \frac{q^{h-c/24}\bar{q}^{\bar{h}-\bar{c}/24}}{\prod_{n=1}^{+\infty}|1-q^n|^2}, \qquad q = e^{2\pi i\tau}. \tag{10.62}$$

Writing the modular parameter in the form (8.78) with an ℓ-independent β and introducing a mass M and a spin s defined by (10.35), the large ℓ limit of the quantities appearing in the right-hand side of (10.62) is

$$\tau \sim \frac{1}{2\pi}(\theta + i\epsilon), \qquad q^{h-c/24}\bar{q}^{\bar{h}-\bar{c}/24} \sim e^{i\theta(s-c_1/24)}e^{-\beta(M-c_2/24)}. \tag{10.63}$$

(Here the imaginary part of τ goes to zero, but we keep writing it as $\epsilon > 0$ to reproduce the regularization used in (10.61).) Thus the flat limit of (10.62) is

$$\lim_{\ell\to+\infty} \operatorname{Tr}\left(q^{L_0-c/24}\bar{q}^{\bar{L}_0-c/24}\right) = e^{i\theta(s-c_1/24)}e^{-\beta(M-c_2/24)}\frac{1}{\prod_{n=1}^{+\infty}|1-q^n|^2},$$

and coincides (up to a redefinition of spin) with the BMS$_3$ character (10.61) for a supertranslation whose zero-mode is a Euclidean time translation, $\alpha = i\beta$. The left-hand side of this expression can be interpreted as a trace

$$\lim_{\ell\to+\infty} \operatorname{Tr}\left(q^{L_0-c/24}\bar{q}^{\bar{L}_0-c/24}\right) \overset{(10.63)}{=} \operatorname{Tr}\left(e^{i\theta(J_0-c_1/24)}e^{-\beta(P_0-c_2/24)}\right) = \chi[(f,\alpha)], \tag{10.64}$$

where f is a rotation by θ and the operators J_m, P_n are normalized so as to satisfy the commutation relations (10.41). In this form, the matching between the flat limit of the Virasoro character (10.62) and the BMS$_3$ character (10.61) is manifest.

Universality of BMS$_3$ Characters

Even though BMS$_3$ and Virasoro characters are related by the limit just described, they are strikingly different in that the result (10.61) holds for any value of the central charge c_2, any mass M, and any spin s. By contrast, the characters of irreducible, unitary highest weight representations of the Virasoro algebra depend heavily on the values of the central charge c and the highest weight h: when $c \leq 1$, only certain discrete values of c and h lead to unitary representations, and the resulting character is not given by (10.62) [133–135]. In that sense, induced representations of the BMS$_3$ group are less intricate than highest weight representations of the Virasoro algebra. Since the former are high-energy, high central charge limits of the latter, this could have been expected: all complications occurring at small c vanish when ℓ goes to infinity, since c scales linearly with ℓ by assumption.

We could also have guessed that such a simplification would occur thanks to dimensional arguments. Indeed, both M and c_2 are dimensionful parameters labelling BMS$_3$ representations, so their values can be tuned at will by a suitable choice of units. Accordingly, in contrast to Virasoro highest weight representations, one should not expect to find sharp bifurcations in the structure of BMS$_3$ particles as M and c_2 vary. In this sense formula (10.61) is a universal character.

10.3.3 Vacuum Character

We now turn to the character of the BMS_3 vacuum, that is, the scalar representation whose supermomentum orbit is that of $p_{\text{vac}} = -c_2/24$. The computation is identical to that of Sect. 10.3.1, save for the fact that the little group is the Lorentz group $PSL(2, \mathbb{R})$ rather than U(1), so the orbit $\mathcal{O}_{\text{vac}} \cong \text{Diff}(S^1)/PSL(2, \mathbb{R})$ has codimension three rather than one in $\text{Diff}(S^1)$.

As before, the quantity we wish to compute is $\chi[(f, \alpha)]$, where α is any supertranslation. Since the little group is now larger than U(1), one can obtain non-trivial characters even when f is not conjugate to a rotation. We will not consider such cases here and stick instead to our earlier convention that $f(\varphi) = \varphi + \theta$ is a rotation by $\theta \neq 0$. (Equivalently we may take f to be merely conjugate to a rotation since the character is a class function.) Then the integral of the Frobenius formula (10.54) localizes to the unique rotation-invariant point p_{vac} on the orbit and the character can be written as

$$\chi_{\text{vac}}[(f, \alpha)] = e^{-i\alpha^0 c_2/24} \int_{\mathcal{O}_{\text{vac}}} d\mu(k)\, \delta(k, f \cdot k). \tag{10.65}$$

Here μ is some quasi-invariant measure on the vacuum orbit. Using Fourier expansions (6.116), we can think of Fourier modes as redundant coordinates on the orbit; the subtlety is to understand which of these modes should be modded out so as to provide genuine, non-redundant local coordinates on $\mathcal{O}_{\text{vac}}$.

As in the case of massive characters we work in a neighbourhood of the rest frame supermomentum p_{vac} and rely on the action (10.58) of infinitesimal superrotations. Taking $M = 0$ in that equation, we now find that all three modes δp_1, δp_0 and δp_{-1} vanish for any choice of X. This is an infinitesimal restatement of the fact that the little group is $PSL(2, \mathbb{R})$. Thus, in a neighbourhood of p_{vac}, we can use the higher Fourier modes p_n with $|n| \geq 2$ as local coordinates. In particular the measure μ now takes the form

$$d\mu(k) = (\text{Some } k - \text{dependent prefactor}) \times \prod_{n=2}^{+\infty} dk_n dk_{-n}\,,$$

where the prefactor is again unknown, but eventually irrelevant since it is cancelled by the prefactor of the corresponding delta function. The vacuum character (10.65) thus boils down to

$$\begin{aligned}\chi_{\text{vac}}[(f, \alpha)] &= e^{-i\alpha^0 c_2/24} \int_{\mathbb{R}^{2\infty-2}} \prod_{n=2}^{+\infty} dk_n dk_{-n} \prod_{n=2}^{+\infty} \delta(k_n - [f \cdot k]_n)\delta(k_{-n} - [f \cdot k]_{-n}) \\ &\overset{(10.57)}{=} e^{-i\alpha^0 c_2/24} \left| \int_{\mathbb{R}^{\infty-1}} \prod_{n=2}^{+\infty} dk_n \prod_{n=2}^{+\infty} \delta\left(k_n(1-q^n)\right) \right|^2,\end{aligned}$$

where $q \equiv \exp[2\pi i\tau]$ and $\tau = (\theta + i\epsilon)/2\pi$ contains an imaginary part ϵ that regularizes the divergence of the infinite product. Integrating the delta functions and taking into account the determinant, we finally obtain

$$\boxed{\chi_{\text{vac}}[(f,\alpha)] = e^{-i\alpha^0 c_2/24} \frac{1}{\prod_{n=2}^{+\infty} |1-q^n|^2}\,.} \tag{10.66}$$

Note the truncated product starting at $n = 2$, which reflects Lorentz-invariance. As in the massive case above, this expression can be interpreted as a trace (10.64), now taken in the Hilbert space of the vacuum representation. It can also be recovered as a flat limit of the product of two Virasoro vacuum characters (8.77).

Remark In this section we have systematically assumed that f is a rotation by some non-zero angle θ. In doing so we have left aside the interesting problem of computing characters of pure supertranslations. This includes for instance Euclidean time translations, whose characters coincide with canonical partition functions of BMS$_3$ particles. Analogously to the Poincaré results (4.68) or (4.99), all such characters are infrared-divergent and rely on an integral taken over the *whole* supermomentum orbit, due to the lack of a localizing delta function. We will not attempt to evaluate these characters here.

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Chapter 11
Partition Functions and Characters

The asymptotic symmetries described in Chap. 9 suggest that the quantization of asymptotically flat gravitational fields in three dimensions provides unitary representations of BMS_3. In particular, it should be possible to identify BMS_3 particles with quantized gravitational fluctuations around suitable background metrics. The purpose of this chapter is to confirm this identification by matching one-loop partition functions of gravity with BMS_3 characters. As a by-product, the method of heat kernels that we shall use for this computation also allows us to evaluate partition functions for fields with arbitrary spin, which will lead us to higher-spin extensions of BMS_3 symmetry. As we will show, the resulting irreducible unitary representations can be classified analogously to the BMS_3 particles of Chap. 10, and their characters match one-loop partition functions of combinations of higher-spin fields in three-dimensional Minkowski space.

The plan is as follows. We start in Sect. 11.1 by evaluating one-loop partition functions of free fields with arbitrary mass and spin in D-dimensional Minkowski space at finite temperature and angular potentials. We show that the result is an exponential of Poincaré characters which, for spin two in $D = 3$, coincides with the vacuum BMS_3 character (10.66). In Sect. 11.2 we extend this matching to higher-spin theories in three dimensions by describing a method for obtaining induced irreducible unitary representations of the corresponding asymptotic symmetry groups. Section 11.3 is devoted to the Lie-algebraic counterpart of that method, which we compare to earlier proposals in the literature [1]. We show in particular that ultrarelativistic and non-relativistic limits of quantum $\mathcal{W}$ algebras differ, which singles out induced representations as the correct approach to flat space holography. Finally, in Sect. 11.4 we define supersymmetric extensions of the BMS_3 group, describe their irreducible unitary representations and show that their characters coincide with one-loop partition functions of asymptotically flat hypergravity. Sections "From Mixed Traces to Bosonic Characters" and "From Mixed Traces to Fermionic Characters" are technical appendices that summarize computations related to $SO(n)$ characters which are useful for Sects. 11.1 and 11.4, respectively.

B. Oblak, *BMS Particles in Three Dimensions*, Springer Theses,
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The results described in this chapter first appeared in [2–4]. They are Minkowskian analogues of earlier observations on partition functions in AdS_3 [5–7] that we already referred to in Sect. 8.4. Note that our language in this chapter will be somewhat different than in the previous ones, as we will rely much more heavily on quantum field theory. On the other hand the group-theoretic tools that we will be using are essentially the same as in Chap. 10.

11.1 Rotating Canonical Partition Functions

We wish to study one-loop partition functions of higher-spin fields in D-dimensional Minkowski space at finite temperature $1/\beta$, and with non-zero angular potentials. As in Sect. 4.2, we denote these potentials by $\vec{\theta} = (\theta_1, \ldots, \theta_r)$, where $r = \lfloor (D-1)/2 \rfloor$ is the rank of $SO(D-1)$, that is, the maximal number of independent rotations in $(D-1)$ space dimensions; we assume $D \geq 3$. The computation involves a functional integral over fields living on a quotient of $\mathbb{R}^D$, where the easiest way to incorporate one-loop effects is the heat kernel method. Accordingly we now briefly review this approach, before applying it to bosonic fields and rewriting the resulting partition function as an exponential of Poincaré characters; for spin two and $D = 3$, the result coincides with the vacuum BMS_3 character. Fermions will be treated separately in Sect. 11.4.1.

11.1.1 Heat Kernels and Method of Images

Our goal is to evaluate partition functions of the form

$$Z(\beta, \vec{\theta}\,) = \int \mathcal{D}\phi \, e^{-S[\phi]} \tag{11.1}$$

where ϕ is some collection of fields (bosonic or fermionic) defined on a thermal quotient $\mathbb{R}^D/\mathbb{Z}$ of flat Euclidean space, satisfying suitable (anti)periodicity conditions. (The explicit action of $\mathbb{Z}$ on $\mathbb{R}^D$, with its dependence on β and $\vec{\theta}$, will be displayed below — see Eq. (11.7).) The functional $S[\phi]$ is a Euclidean action for these fields. Expression (11.1) can be evaluated perturbatively around a saddle point ϕ_c of S, leading to the semi-classical (one-loop) result

$$Z(\beta, \vec{\theta}\,) \sim e^{-S[\phi_c]} \left[\det \left(\frac{\delta^2 S}{\delta\phi\delta\phi} \right) \bigg|_{\phi_c} \right]^{\#} \tag{11.2}$$

where the exponent # depends on the nature of the fields that were integrated out. The quantity $\delta^2 S/\delta\phi(x)\delta\phi(y)$ appearing in this expression is a differential operator acting on sections of a suitable vector bundle over $\mathbb{R}^D/\mathbb{Z}$. The evaluation of the one-loop contribution to the partition function thus boils down to that of a functional determinant.

After gauge-fixing, such determinants reduce to expressions of the form $\det(-\Delta + M^2)$, where Δ is a Laplacian operator on $\mathbb{R}^D/\mathbb{Z}$. These, in turn, can be evaluated thanks to the method of heat kernels. In short (see e.g. [5, 8] for details), one can express $\det(-\Delta + M^2)$ on $\mathbb{R}^D$ as an integral

$$-\log\det(-\Delta + M^2) = \int_0^{+\infty} \frac{dt}{t} \int_{\mathbb{R}^D} d^D x \, \mathrm{Tr}\left[K(t, x, x)\right], \tag{11.3}$$

up to an ultraviolet divergence that can be regularized with standard methods. Here $K(t, x, x')$ is a matrix-valued bitensor known as the *heat kernel* associated with $(-\Delta + M^2)$. It satisfies the heat equation

$$\frac{\partial}{\partial t} K(t, x, x') - (\Delta_x - M^2)\, K(t, x, x') = 0\,, \tag{11.4}$$

with the initial condition

$$K(t = 0, x, x') = \delta^{(D)}(x - x')\, \mathbb{I} \tag{11.5}$$

where $\mathbb{I}$ is an identity matrix having the same tensor structure as K (here omitted for brevity) while $\delta^{(D)}$ is the Dirac delta function associated with the translation-invariant Lebesgue measure on $\mathbb{R}^D$.

Heat kernels are well suited for the computation of functional determinants on quotient spaces. Indeed, suppose Γ is a discrete subgroup of the isometry group of $\mathbb{R}^D$, acting freely on $\mathbb{R}^D$. Introducing the equivalence relation $x \sim y$ if there exists a $\gamma \in \Gamma$ such that $\gamma(x) = y$, we define the quotient manifold $\mathbb{R}^D/\Gamma$ as the corresponding set of equivalence classes. Given a differential operator Δ on $\mathbb{R}^D$, it naturally induces a differential operator on $\mathbb{R}^D/\Gamma$, acting on fields that satisfy suitable (anti)periodicity conditions. Because the heat Eq. (11.4) is linear, the heat kernel on the quotient space can be obtained from the heat kernel on $\mathbb{R}^D$ by the method of images:

$$K^{\mathbb{R}^D/\Gamma}(t, x, x') = \sum_{\gamma \in \Gamma} K\big(t, x, \gamma(x')\big)\,. \tag{11.6}$$

Here, abusing notation slightly, x and x' denote points both in $\mathbb{R}^D$ and in its quotient. In writing (11.6) we are assuming, for simplicity, that the tensor structure of K is trivial, but as soon as K carries tensor or spinor indices (i.e. whenever the fields under consideration have non-zero spin), the right-hand side involves Jacobians that account for the non-trivial transformation law of K. Once $K^{\mathbb{R}^D/\mathbb{Z}}$ is known, the determinant of the operator $-\Delta + M^2$ is given by (11.3) with K replaced by $K^{\mathbb{R}^D/\mathbb{Z}}$ and $\mathbb{R}^D$ replaced by $\mathbb{R}^D/\mathbb{Z}$.

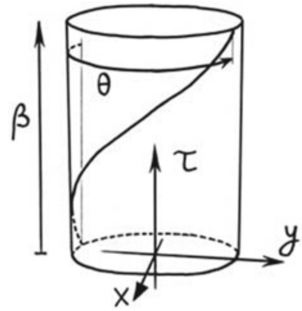

Fig. 11.1 The quotient space $\mathbb{R}^3/\mathbb{Z}$ defined by identifications of $\mathbb{R}^3$ generated by the group action (11.7); β is an inverse temperature while θ is an angular potential

We shall be concerned with thermal quantum field theories on rotating Minkowski space, so we define our fields on a quotient $\mathbb{R}^D/\mathbb{Z}$ of Euclidean space with the action of $\mathbb{Z}$ obtained as follows. For odd D, we endow $\mathbb{R}^D$ with Cartesian coordinates (x_i, y_i) (where $i = 1, \ldots, r$) and a Euclidean time coordinate τ, so that an integer $n \in \mathbb{Z}$ acts on $\mathbb{R}^D$ according to (see Fig. 11.1)

$$\gamma^n \begin{pmatrix} x_i \\ y_i \end{pmatrix} = \begin{pmatrix} \cos(n\theta_i) & -\sin(n\theta_i) \\ \sin(n\theta_i) & \cos(n\theta_i) \end{pmatrix} \cdot \begin{pmatrix} x_i \\ y_i \end{pmatrix} \equiv R(n\theta_i) \cdot \begin{pmatrix} x_i \\ y_i \end{pmatrix}, \qquad \gamma^n(\tau) = \tau + n\beta\,. \tag{11.7}$$

For even D we add one more spatial coordinate z, invariant under $\mathbb{Z}$. In terms of the coordinates $\{x_1, y_1, \ldots, x_r, y_r, \tau\}$ (and also z if D is even), the Euclidean Lorentz transformation implementing the rotation (11.7) is the nth power of the rotation matrix

$$J = \begin{pmatrix} R(\theta_1) & 0 & \cdots & 0 \\ 0 & \ddots & 0 & \vdots \\ \vdots & 0 & R(\theta_r) & 0 \\ 0 & \cdots & 0 & 1 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} R(\theta_1) & 0 & \cdots & 0 & 0 \\ 0 & \ddots & 0 & \vdots & 0 \\ \vdots & 0 & R(\theta_r) & 0 & 0 \\ 0 & \cdots & 0 & 1 & 0 \\ 0 & \cdots & 0 & 0 & 1 \end{pmatrix} \tag{11.8}$$

for D odd or D even, respectively. Being isometries of flat space, these transformations are linear maps in Cartesian coordinates, so their nth power coincides with the Jacobian matrix $\partial\gamma^n(x)^\mu/\partial x^\nu$ that will be needed later for the method of images. Throughout this chapter we take all angles $\theta_1, \ldots, \theta_r$ to be non-vanishing and combine them in a vector $\vec{\theta} = (\theta_1, \ldots, \theta_r)$. We now display the computation of one-loop partition functions on $\mathbb{R}^D/\mathbb{Z}$ for bosonic higher-spin fields.

11.1.2 Bosonic Higher Spins

Here we study the rotating one-loop partition function of a free bosonic field with spin s and mass M (including the massless case). For $M > 0$ its Euclidean action can be presented either (i) using a symmetric traceless field $\phi_{\mu_1\dots\mu_s}$ of rank s together with a tower of auxiliary fields of ranks $s-2, s-3, \dots, 0$ that do not display any gauge symmetry [9]; or (ii) using a set of doubly traceless fields of ranks $s, s-1, \dots, 0$ subject to a gauge symmetry generated by traceless gauge parameters of ranks $s-1, s-2, \dots, 0$ [10]. In the latter case, the action is a sum of Fronsdal actions [11] for each of the involved fields, plus a set of cross-coupling terms with one derivative proportional to M, and a set of terms without derivatives proportional to M^2. In the massless limit, all these couplings vanish and one can consider independently the (Euclidean) Fronsdal action for the field of highest rank:

$$S[\phi_{\mu_1\dots\mu_s}] = -\frac{1}{2}\int d^D x\ \phi^{\mu_1\dots\mu_s}\left(\mathcal{F}_{\mu_1\dots\mu_s} - \frac{1}{2}\,\delta_{(\mu_1\mu_2}\mathcal{F}_{\mu_3\dots\mu_s)\lambda}{}^{\lambda}\right), \tag{11.9}$$

where indices are raised and lowered thanks to the Euclidean metric, while

$$\mathcal{F}_{\mu_1\cdots\mu_s} \equiv \Delta\,\phi_{\mu_1\dots\mu_s} - \partial_{(\mu_1|}\partial^{\lambda}\phi_{|\mu_2\dots\mu_s)\lambda} + \partial_{(\mu_1}\partial_{\mu_2}\phi_{\mu_3\dots\mu_s)\lambda}{}^{\lambda}\,. \tag{11.10}$$

Parentheses denote the symmetrization of the indices they enclose, with the minimum number of terms needed and without any overall factor. The massless action (11.9) has a gauge symmetry $\phi_{\mu_1\dots\mu_s} \mapsto \phi_{\mu_1\dots\mu_s} + \partial_{(\mu_1}\xi_{\mu_2\dots\mu_s)}$, where $\xi_{\mu_2\dots\mu_s}$ is a symmetric tensor field. When $s = 2$ the action reduces to that of a metric perturbation $h_{\mu\nu}$ around a flat background. We refer for instance to [12] for many more details on this topic.

Massive Case

Applying e.g. the techniques of [7] to the presentation of the Euclidean action of a massive field of spin s of [10], one finds that the partition function is given by

$$\log Z = -\frac{1}{2}\log\det(-\Delta^{(s)} + M^2) + \frac{1}{2}\log\det(-\Delta^{(s-1)} + M^2)\,, \tag{11.11}$$

where $\Delta^{(s)}$ is the Laplacian $\partial_\mu\partial^\mu$ acting on periodic,[1] symmetric, traceless tensor fields with s indices on $\mathbb{R}^D/\mathbb{Z}$. We denote the heat kernel associated with $(-\Delta^{(s)} + M^2)$ on $\mathbb{R}^D$ by $K_{\mu_s,\nu_s}(t, x, x')$, where μ_s and ν_s are shorthands that denote sets of s symmetrized indices. The differential equation (11.4) with initial condition (11.5) for $K_{\mu_s,\nu_s}(t, x, x')$ then reads

$$(\Delta^{(s)} - M^2 - \partial_t)K_{\mu_s,\,\nu_s} = 0\,, \quad K_{\mu_s,\,\nu_s}(t=0, x, x') = \mathbb{I}_{\mu_s,\,\nu_s}\delta^{(D)}(x - x')\,, \tag{11.12}$$

where $\mathbb{I}_{\mu_s,\nu_s}$ is an identity matrix with the same tensor structure as K_{μ_s,ν_s}. Sets of repeated covariant or contravariant indices denote sets of indices that are symmetrized with the minimum number of terms required and without multiplicative factors, while contractions involve as usual a covariant and a contravariant index. For instance the tracelessness condition on the heat kernel amounts to

[1] More precisely, the field at time $\tau + \beta$ is rotated by $\vec{\theta}$ with respect to the field at time τ.

$$\delta^{\mu\mu} K_{\mu_s,\,\nu_s} = \delta^{\nu\nu} K_{\mu_s,\,\nu_s} = 0\,. \tag{11.13}$$

The unique solution of (11.12) fulfilling this condition is

$$K_{\mu_s,\,\nu_s}(t,x,x') = \frac{1}{(4\pi t)^{D/2}}\, e^{-M^2 t - \frac{1}{4t}|x-x'|^2}\, \mathbb{I}_{\mu_s,\,\nu_s} \tag{11.14}$$

where $|x-x'|$ is the Euclidean distance between x and x', while the spin-s identity matrix is

$$\mathbb{I}_{\mu_s,\,\nu_s} = \sum_{n=0}^{\lfloor \frac{s}{2}\rfloor} \frac{(-1)^n 2^n n!\,[D+2(s-n-2)]!!}{s!\,[D+2(s-2)]!!}\, \delta^n_{\mu\mu}\delta^{s-2n}_{\mu\nu}\delta^n_{\nu\nu}\,. \tag{11.15}$$

Note that the dependence of this heat kernel on the space-time points x, x' and on Schwinger proper time t is that of a scalar heat kernel, and completely factorizes from its spin/index structure which is entirely accounted for by the matrix $\mathbb{I}$.[2] This simplification is the reason why heat kernel computations are simpler in flat space than in AdS or dS.

To determine the heat kernel associated with the operator $(-\Delta^{(s)}+M^2)$ on $\mathbb{R}^D/\mathbb{Z}$, we use the method of images (11.6), taking care of the non-trivial index structure. Denoting the matrix (11.8) by $J_\alpha{}^\beta$ (it is the Jacobian of the transformation $x \mapsto \gamma(x)$), the spin-s heat kernel on the quotient space $\mathbb{R}^D/\mathbb{Z}$ is

$$K^{\mathbb{R}^D/\mathbb{Z}}_{\mu_s,\,\nu_s}(t,x,x') = \sum_{n\in\mathbb{Z}} (J^n)_\alpha{}^\beta \ldots (J^n)_\alpha{}^\beta K_{\mu_s,\,\beta_s}\big(t,x,\gamma^n(x')\big)\,, \tag{11.16}$$

where we recall again that repeated covariant or contravariant indices are meant to be symmetrized with the minimum number of terms required and without multiplicative factors, while repeating a covariant index in a contravariant position denotes a contraction. Accordingly, Eq. (11.3) gives the determinant of $(-\Delta^{(s)}+M^2)$ on $\mathbb{R}^D/\mathbb{Z}$:

$$\begin{aligned} -\log\det(-\Delta^{(s)}+M^2) &= \int_0^{+\infty}\frac{dt}{t}\int_{\mathbb{R}^D/\mathbb{Z}} d^Dx\,(\delta^{\mu\alpha})^s\, K^{\mathbb{R}^D/\mathbb{Z}}_{\mu_s,\,\alpha_s}(t,x,x) \\ &= \sum_{n\in\mathbb{Z}} (J^n)^{\mu\beta}\cdots(J^n)^{\mu\beta}\,\mathbb{I}_{\mu_s,\beta_s}\int_0^{+\infty}\frac{dt}{t}\int_{\mathbb{R}^D/\mathbb{Z}} d^Dx\,\frac{1}{(4\pi t)^{D/2}}\,e^{-M^2t-\frac{1}{4t}|x-\gamma^n(x)|^2}. \end{aligned} \tag{11.17}$$

In this series the term $n=0$ contains both an ultraviolet divergence (due to the singular behaviour of the integrand as $t\to 0$) and an infrared one (due to the integral of a constant over $\mathbb{R}^D/\mathbb{Z}$), proportional to the product βV where V is the spatial volume of the system. This divergence is a quantum contribution to the vacuum

[2] Note also that the scalar heat kernel coincides with the propagator of a free particle in $\mathbb{R}^D$, whose expression for $D=2$ was written in Eq. (5.158).

energy, which we ignore from now on. The only non-trivial one-loop contribution then comes from the terms $n \neq 0$ in (11.17). Using

$$|x - \gamma^n(x)|^2 = n^2\beta^2 + \sum_{i=1}^{r} 4\sin^2(n\theta_i/2)(x_i^2 + y_i^2)$$

in terms of the coordinates introduced around (11.7), the integrals over t and x give rise to a divergent series

$$-\log\det(-\Delta^{(s)} + M^2) = \sum_{n\in\mathbb{Z}^*} \frac{1}{|n|} \frac{\chi_s[n\vec{\theta}\,]}{\prod_{j=1}^{r} |1 - e^{in\theta_j}|^2} \times \begin{cases} e^{-|n|\beta M} & \text{if } D \text{ odd,} \\ \frac{ML}{\pi} K_1(|n|\beta M) & \text{if } D \text{ even,} \end{cases} \tag{11.18}$$

where K_1 is the first modified Bessel function of the second kind, $L \equiv \int_{-\infty}^{+\infty} dz$ is an infrared divergence (4.64) that arises in even dimensions because the z axis is left fixed by the rotation (11.8), and

$$\chi_s[n\vec{\theta}\,] \equiv (J^n)^{\mu\beta} \ldots (J^n)^{\mu\beta}\, \mathbb{I}_{\mu_s,\,\beta_s} \equiv \left[(J^n)^{\mu\beta}\right]^s \mathbb{I}_{\mu_s,\,\beta_s} \tag{11.19}$$

is the full mixed trace of $\mathbb{I}_{\mu_s,\nu_s}$. As such, expression (11.18) makes no sense because the sum over n diverges. To cure this problem, one needs to choose a regularization procedure. Motivated by the similar situation already encountered in Eq. (10.56), for now we choose to regulate the series by a naive replacement: we let ϵ_j, $j = 1, \ldots, r$ be small positive parameters and replace θ_j by $\theta_j \pm i\epsilon_j$ in all positive powers of $e^{\pm i\theta_j}$. As a result, expression (11.18) is replaced by the convergent series

$$-\log\det(-\Delta^{(s)} + M^2) = \sum_{n\in\mathbb{Z}^*} \frac{1}{|n|} \frac{\chi_s[n\vec{\theta}, \vec{\epsilon}\,]}{\prod_{j=1}^{r} |1 - e^{in(\theta_j + i\epsilon_j)}|^2} \times \begin{cases} e^{-|n|\beta M} & \text{if } D \text{ odd,} \\ \frac{ML}{\pi} K_1(|n|\beta M) & \text{if } D \text{ even,} \end{cases} \tag{11.20}$$

where $\chi_s[n\vec{\theta}, \vec{\epsilon}\,]$ is still given by (11.19), except that now all factors $e^{\pm i\theta_j}$ appearing in the Jacobians are replaced by $e^{\pm i(\theta_j \pm i\epsilon_j)}$.

The regularization described here is motivated by the fact that, for odd D, the resulting expressions look very much like flat limits of AdS one-loop determinants, in which case the parameters $\epsilon_j \propto \beta/\ell$ are remnants of the inverse temperature (with ℓ the AdS radius). The subtlety, however, is that the exact matching of the flat limit of AdS with combinations such as (11.20) requires some of the ϵ_j's to be multiplied by certain positive coefficients; thus Eq. (11.20) is not quite the same as the flat limit of its AdS counterpart — we will illustrate this point for $D = 3$ in Sect. 11.1.3. As for even values of D, the situation is even worse since the flat limit of the AdS result contains an infrared divergence; it is not obvious how this divergence can be regularized so as to reproduce the combination $L \cdot K_1$ of (11.20), though apart from this the other terms of the expression indeed coincide with the flat limit of their

AdS counterparts. From now on we will use the $i\epsilon$ prescription systematically, often omitting to indicate it explicitly. We will keep it only in the final results, and in Sect. 11.1.3 we will introduce a refined regularization such that partition functions in $D = 3$ exactly reproduce characters of suitable asymptotic symmetry algebras, while also matching the flat limit of their AdS peers.

In Eq. (11.20), the divergence as $\epsilon_j \to 0$ is the same as in the BMS_3 character (10.61). The new ingredient is the angle-dependent trace (11.19); in Appendices sections "Mixed Traces and Symmetric Polynomials" and "Symmetric Polynomials and SO (D) Characters" we show that the latter is the character of an irreducible, unitary representation of $\text{SO}(D)$ with highest weight $\lambda_s \equiv (s, 0, \ldots, 0)$. More precisely, let H_i denote the generator of rotations in the plane (x_i, y_i), in the coordinates defined around (11.7). Then the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{so}(D)$ is generated by $H_1, \ldots, H_r$, plus, if D is even, a generator of rotations in the plane (τ, z). Denoting the dual basis of $\mathfrak{h}^*$ by $L_1, \ldots, L_r$ (plus possibly L_{r+1} if D is even), we can consider the weight $\lambda_s = sL_1$ whose only non-zero component (in the basis of L_i's) is the first one. The character of the corresponding highest-weight representation of $\mathfrak{so}(D)$ coincides with expression (11.127):

$$\chi_s[n\vec{\theta}\,] = \chi^{(D)}_{\lambda_s}[n\theta_1, \ldots, n\theta_r] \quad \text{or} \quad \chi^{(D)}_{\lambda_s}[n\theta_1, \ldots, n\theta_r, 0]\,, \tag{11.21}$$

for D odd or even, respectively. From now on, $\chi^{(n)}_\lambda$ denotes a character of $\text{SO}(n)$ with highest weight λ.

We can now display the one-loop partition function (11.11). Using expression (11.20) for the one-loop determinant together with property (11.21), we find

$$Z(\beta, \vec{\theta}\,) = \exp\left[\sum_{n=1}^{+\infty} \frac{n^{-1}}{\prod_{j=1}^{r} |1 - e^{in\theta_j}|^2} \times \begin{cases} \left(\chi^{(D)}_{\lambda_s}[n\vec{\theta}\,] - \chi^{(D)}_{\lambda_{s-1}}[n\vec{\theta}\,]\right) e^{-n\beta M} \\ \left(\chi^{(D)}_{\lambda_s}[n\vec{\theta}, 0] - \chi^{(D)}_{\lambda_{s-1}}[n\vec{\theta}, 0]\right) \frac{ML}{\pi} K_1(n\beta M) \end{cases}\right] \tag{11.22}$$

where the upper (resp. lower) line corresponds to the case where D is odd (resp. even). Remarkably, the differences of $\text{SO}(D)$ characters appearing here can be simplified: according to Eqs. (11.154a) and (11.155), the difference of two $\text{SO}(D)$ characters with weights $(s, 0, \ldots, 0)$ and $(s-1, 0, \ldots, 0)$ is a (sum of) character(s) of $\text{SO}(D-1)$:

$$\left.\begin{array}{ll} \chi^{(D)}_{\lambda_s}[\vec{\theta}\,] - \chi^{(D)}_{\lambda_{s-1}}[\vec{\theta}\,] & (D \text{ odd}) \\ \chi^{(D)}_{\lambda_s}[\vec{\theta}, 0] - \chi^{(D)}_{\lambda_{s-1}}[\vec{\theta}, 0] & (D \text{ even}) \end{array}\right\} = \chi^{(D-1)}_{\lambda_s}[\vec{\theta}\,]\,. \tag{11.23}$$

Since the rank of $\text{SO}(D-1)$ is $r = \lfloor (D-1)/2 \rfloor$, the right-hand side of this equality makes sense regardless of the parity of D, and the partition function (11.22) boils down to

$$Z(\beta,\vec{\theta}\,)=\exp\left[\sum_{n=1}^{+\infty}\frac{1}{n}\frac{\chi^{(D-1)}_{\lambda_s}[n\vec{\theta},\vec{\epsilon}\,]}{\prod_{j=1}^{r}|1-e^{in(\theta_j+i\epsilon_j)}|^2}\times\begin{cases}e^{-n\beta M} & (D\text{ odd})\\ \frac{ML}{\pi}K_1(n\beta M) & (D\text{ even})\end{cases}\right].\tag{11.24}$$

Note that the function of $n\vec{\theta}$ and $n\beta$ appearing here in the sum over n is essentially the character (4.60)–(4.65) of a Poincaré particle with mass M and spin λ_s; we will return to this observation in Sect. 11.1.4. An analogous result holds in Anti-de Sitter space [13–15].

Massless case

We now turn to the one-loop partition function associated with the Euclidean Fronsdal action (11.9), describing a *massless* field with spin s. The extra ingredient with respect to the massive case is the gauge symmetry $\phi_{\mu_1\ldots\mu_s}\mapsto\phi_{\mu_1\ldots\mu_s}+\partial_{(\mu_1}\xi_{\mu_2\ldots\mu_s)}$. This forces one to fix a gauge and introduce ghost fields that absorb the gauge redundancy [7], which adds two more functional determinants to the massive result (11.11) and leads to the following expression for the one-loop term of the partition function:

$$\log Z=-\frac{1}{2}\log\det(-\Delta^{(s)})+\log\det(-\Delta^{(s-1)})-\frac{1}{2}\log\det(-\Delta^{(s-2)})\,.\tag{11.25}$$

As before, $\Delta^{(s)}$ is the Laplacian on $\mathbb{R}^D/\mathbb{Z}$ acting on periodic, traceless, symmetric fields with s indices. The functional determinants can be evaluated exactly as in the massive case, upon setting $M=0$. In particular, using $\lim_{x\to 0}xK_1(x)=1$, the massless version of the functional determinant (11.20) is

$$-\log\det(-\Delta^{(s)})=\sum_{n\in\mathbb{Z}^*}\frac{1}{|n|}\frac{\chi_s[n\vec{\theta},\vec{\epsilon}\,]}{\prod_{j=1}^{r}|1-e^{in(\theta_j+i\epsilon_j)}|^2}\times\begin{cases}1 & \text{if } D \text{ odd,}\\ \frac{L}{\pi|n|\beta} & \text{if } D \text{ even,}\end{cases}\tag{11.26}$$

which has been regularized as in the massive case. The matching (11.21) between χ_s and a character of SO(D) remains valid, but a sharp difference arises upon including all three functional determinants in (11.25). Indeed, the combination of χ_s's now is

$$\chi_s[n\vec{\theta}\,]-2\chi_{s-1}[n\vec{\theta}\,]+\chi_{s-2}[n\vec{\theta}\,]\overset{(11.21)-(11.23)}{=}\chi^{(D-1)}_{\lambda_s}[n\vec{\theta}\,]-\chi^{(D-1)}_{\lambda_{s-1}}[n\vec{\theta}\,]\,.\tag{11.27}$$

It is tempting to use (11.23) once more to rewrite this as a character of SO($D-2$), and indeed this is exactly what happens for even D because in that case the rank of SO($D-1$) equals that of SO($D-2$):

$$Z(\beta,\vec{\theta}\,)=\exp\left[\sum_{n=1}^{+\infty}\frac{1}{n}\frac{\chi^{(D-2)}_{\lambda_s}[n\vec{\theta},\vec{\epsilon}\,]}{\prod_{j=1}^{r}|1-e^{in(\theta_j+i\epsilon_j)}|^2}\frac{L}{\pi n\beta}\right]\qquad(\text{even } D).\tag{11.28}$$

If D is *odd*, however, the rank decreases by one unit in going from SO$(D-1)$ to SO$(D-2)$, so expression (11.27) contains one angle too much to be a character of SO$(D-2)$. In fact, when $D=3$, the right-hand side of (11.27) is the best we can hope to get; for $s \geq 2$ it takes the form

$$\chi^{(1)}_{\lambda_s}[n\theta] \equiv \chi^{(2)}_{\lambda_s}[n\theta] - \chi^{(2)}_{\lambda_{s-1}}[n\theta] = e^{isn\theta} - e^{i(s-1)n\theta} + \text{c.c.} \tag{11.29}$$

where we have used the character $\chi_s[\theta] = e^{is\theta} + e^{-is\theta}$ for parity-invariant unitary representations of SO(2) and "c.c." means "complex conjugate". (For lower spins one has $\chi^{(1)}_{\lambda_0}[\theta] \equiv 1$ and $\chi^{(1)}_{\lambda_1}[\theta] \equiv 2\cos\theta - 1$.) Hence the partition function given by (11.25) becomes

$$Z(\beta,\theta) = e^{-S^{(0)}} \exp\left[\sum_{n=1}^{+\infty} \frac{1}{n} \frac{1}{|1-e^{in(\theta+i\epsilon)}|^2} \left(e^{isn(\theta+i\epsilon)} - e^{i(s-1)n(\theta+i\epsilon)} + \text{c.c.}\right)\right] \qquad (D=3) \tag{11.30}$$

upon using the crude regularization described below Eq. (11.20). For the sake of generality we have included a spin-dependent classical action $S^{(0)}$, whose value is a matter of normalization and is generally taken to vanish, except for spin two (see below). In the more general case where D is odd and larger than three, a simplification does occur on the right-hand side of (11.27): as we show in Appendix section "Differences of SO(D) Characters", the difference (11.27) can be written as a sum of SO$(D-2)$ characters with angle-dependent coefficients (see Eq. (11.154b)). Indeed, let us define

$$\mathcal{A}^r_k(\vec{\theta}\,) \equiv \frac{|\cos((r-i)\theta_j)|_{\theta_k=0}}{|\cos((r-i)\theta_j)|}, \qquad k=1,\dots,r, \tag{11.31}$$

where $|A_{ij}|$ denotes the determinant of an $r \times r$ matrix. Then the rotating one-loop partition function for a massless field with spin s in odd space-time dimension $D \geq 5$ reads

$$Z(\beta,\vec{\theta}\,) = \exp\left[\sum_{n=1}^{+\infty} \frac{1}{n} \frac{\sum_{k=1}^{r} \mathcal{A}^r_k(n\vec{\theta},\vec{\epsilon}\,)\,\chi^{(D-2)}_{\lambda_s}[n\theta_1,\dots,\widehat{n\theta_k},\dots,n\theta_r,\vec{\epsilon}\,]}{\prod_{j=1}^{r} |1-e^{in(\theta_j+i\epsilon_j)}|^2}\right] \qquad (\text{odd } D \geq 5), \tag{11.32}$$

where the hat on top of an argument denotes omission.

Note that the massless partition functions (11.28) and (11.32) are related to the massless limit of (11.24). Indeed, as we show in Appendix section "From SO(D) to SO $(D-1)$", it turns out that

$$\chi^{(D-1)}_{\lambda_s}[\vec{\theta}\,] = \sum_{j=0}^{s} \begin{cases} \sum_{k=1}^{r} \mathcal{A}^r_k(\vec{\theta}\,)\,\chi^{(D-2)}_{\lambda_j}[\theta_1,\dots,\widehat{\theta_k},\dots,\theta_r] & \text{for odd } D, \\ \chi^{(D-2)}_{\lambda_j}[\vec{\theta}\,] & \text{for even } D. \end{cases} \tag{11.33}$$

Accordingly, the massless limit of a massive partition function with spin s is a product of massless partition functions with spins ranging from 0 to s,

$$\lim_{M\to 0} Z_{M,s} = \prod_{j=0}^{s} Z_{\text{massless},\,j}\,, \tag{11.34}$$

consistently with the structure of the massive action [10]. This result stresses again the role of the functions $\mathcal{A}_k^r(\vec{\theta})$ defined in (11.31): when the space-time dimension is odd, one needs angle-dependent coefficients because the rank of the little group of massless particles is smaller than the maximum number of angular velocities, so that a single $SO(D-2)$ character cannot account for all of them. By the way, the results (11.33) and (11.34) also hold in dimension $D = 3$, provided one takes the "characters" $\chi^{(1)}_{\lambda_s}[\theta]$ to be of the form (11.29) with $\chi^{(1)}_{\lambda_0}[\theta] = 1$ and $\chi^{(1)}_{\lambda_1}[\theta] = 2\cos\theta - 1$.

11.1.3 Partition Functions and BMS$_3$ Characters

Let us rewrite the three-dimensional partition function (11.30) in a form more convenient for the group-theoretic discussion of the remainder of this chapter. As mentioned above, the only non-trivial step will be to slightly modify the $i\epsilon$ regularization. Namely, instead of the combination of exponentials appearing in (11.30), consider the expression

$$e^{isn(\theta+i\epsilon)} - e^{i(s-1)n\theta-(s+1)n\epsilon} + \text{c.c.} \tag{11.35}$$

Writing $q \equiv e^{i(\theta+i\epsilon)}$ and plugging (11.35) into the sum over n of Eq. (11.30), one obtains the series

$$\sum_{n=1}^{+\infty} \frac{1}{n} \frac{q^{ns} - q^{ns}\bar{q}^n + \text{c.c.}}{|1-q^n|^2} = \sum_{n=1}^{+\infty} \frac{1}{n}\left(\frac{q^{ns}}{1-q^n} + \text{c.c.}\right) = -\sum_{j=s}^{+\infty} \log(1-q^j) + \text{c.c.} \tag{11.36}$$

where the new regularization (11.35) has ensured that the summand decomposes as the sum of a chiral and an anti-chiral piece in q. (This was *not* the case with the rough regularization of Eq. (11.20)!) In order to write down the full partition function, it only remains to assign a value to the classical action $S^{(0)}$; a convention that has come to be standard in the realm of three-dimensional gravity is to set $S^{(0)} = 0$ for any spin $s \neq 2$ (vacuum expectation values are assumed to vanish), while $S^{(0)} = -\beta/8G$ for spin two (with G the Newton constant in three dimensions). This choice ensures covariance of the on-shell action under modular transformations [2, 16], in analogy with the similar choice in AdS$_3$ [5]. All in all, combining the value of $S^{(0)}$ with the series (11.36) and renaming j into n, one finds that the three-dimensional partition function (11.30) can be written as

$$Z(\beta,\theta) = e^{\delta_{s,2}\frac{\beta c_2}{24}} \prod_{n=s}^{+\infty} \frac{1}{|1-e^{in(\theta+i\epsilon)}|^2}\,, \quad c_2 = 3/G. \tag{11.37}$$

This expression is the flat limit of the analogous higher-spin partition function in AdS_3 [7]. But most importantly for our purposes, taking $s = 2$ in this formula, we recognize the vacuum BMS_3 character (10.66). This is our first key conclusion in this chapter: it confirms that boundary gravitons in flat space form an irreducible unitary representation of the BMS_3 group of the type described in Chap. 10. The case $s > 2$ will be studied in Sect. 11.2, with similar conclusions.

The result (11.37) can be generalized to orbifolds in flat space: upon declaring that the angular coordinate φ of (9.4) is identified as $\varphi \sim \varphi + 2\pi/N$ with some integer $N > 1$, one obtains a flat three-dimensional conical deficit. One can then evaluate heat kernels on that background by computing a sum over images (11.6), where Γ is now a group $\mathbb{Z} \times \mathbb{Z}_N$ whose two generators enforce (i) the thermal identifications (11.7), and (ii) the orbifolding $\varphi \sim \varphi + 2\pi/N$. An important technical subtlety is that, in order to evaluate a partition function with temperature $1/\beta$ and angular potential θ on that background, the angle appearing in (11.7) must be θ/N rather than θ. The rest of the computation is straightforward, and one finds that the one-loop partition function of gravity can be written as

$$Z_N(\beta, \theta) = e^{-\beta p_0} \prod_{n=1}^{+\infty} \frac{1}{|1 - e^{in(\theta + i\epsilon)}|^2} \,,$$

where $p_0 = -c_2/(24N^2)$. Comparing with (10.61), we recognize the (Euclidean) character of a BMS_3 particle with mass $M = \frac{c_2}{24}(1 - 1/N^2)$, which is indeed the mass one would obtain by writing the conical deficit metric in BMS form (9.25). Note in particular that the sum over images of the orbifolding group $\mathbb{Z}_N$ converts the truncated product of (11.37) into a full product $\prod_{n=1}^{+\infty}(\cdots)$.

The result (11.37) first appeared in [2] and parallels earlier observations in AdS_3 [5]. It is tempting to conjecture that formula (11.37) is one-loop exact, since it is the only expression compatible with BMS_3 symmetry. This being said we will not need to assume one-loop exactness in this thesis and we will not attempt to prove it. The remainder of this chapter is devoted to various extensions of this matching.

Remark In [17] it was shown that the one-loop partition function (11.37) with $s = 2$, and hence the vacuum BMS_3 character (10.66), can be reproduced using quantum Regge calculus. In that context the truncation of the product over $n = 2, 3, \ldots$ is a consequence of triangulation-invariance in the bulk.

11.1.4 Relation to Poincaré Characters

We now show that all one-loop partition functions displayed above can be written as exponentials of (sums of) the Poincaré characters of Sect. 4.2. Recall in particular that massive characters are given by Eq. (4.60), where f is the rotation (11.8) and χ_λ is the character of an irreducible representation of the little group $SO(D-1)$ with highest weight λ.

In order to represent a massive relativistic particle with spin s, we choose the weight λ to be $\lambda_s = (s, 0, \ldots, 0)$ in terms of the dual basis of the Cartan subalgebra of $\mathfrak{so}(D-1)$ described above (11.21). With this choice, expressions (4.60) and (4.65) actually appear in the exponent of (11.24): taking $\alpha^0 = i\beta$, we can rewrite the rotating one-loop partition function for a massive field with spin s as the exponential of a sum of Poincaré characters:

$$Z_{M,s}[\beta, \vec{\theta}\,] = \exp\left[\sum_{n=1}^{+\infty} \frac{1}{n}\,\chi_{M,s}[n\vec{\theta}, in\beta]\right]. \tag{11.38}$$

The series in the exponent diverges for real θ_i's, which can be cured by adding suitable imaginary parts to these angles as explained above. This result holds for any space-time dimension D (along with the infrared regularization (4.64)). From a physical perspective, it is the statement that a free field is a collection of harmonic oscillators, one for each value of momentum: the index n then labels the oscillator modes, while the integral over momenta is the one in the Frobenius formula (10.54). In particular, standard, non-rotating one-loop partition functions are exponentials of sums of characters of (Euclidean) time translations. This relation has also been observed in AdS [13, 14, 18]; our partition functions are flat limits of these earlier results, up to the even-dimensional regularization subtlety mentioned below Eq. (11.19). Note that this issue already occurs at the level of characters: although most of (4.65) is a flat limit of an $\mathrm{SO}(D-1, 2)$ character, it is not clear how to regularize the divergences that pop up when one of the angles vanishes so as to recover the regulators (4.64). This problem also appears for odd D when one or more angles are set to zero.

For massless fields, the situation is a bit more complicated. For even D the massless Poincaré character (4.69) is the limit $M \to 0$ of its massive counterpart (4.65), and the one-loop partition function (11.28) can again be written as an exponential (11.38). But in odd space-time dimensions, $\mathrm{SO}(D-2)$ has lower rank than $\mathrm{SO}(D-1)$, so the rotation (11.8) is *not*, in general, conjugate to an element of the massless little group: it has one angle too much, and whenever all angles $\theta_1, \ldots, \theta_r$ are non-zero, the character (4.33) vanishes. The only non-trivial irreducible character arises when at least one of the angles $\theta_1, \ldots, \theta_r$ vanishes, say $\theta_r = 0$, in which case the massless character takes the form (4.71). However, comparison with (11.32) reveals a mismatch: the partition function does *not* take the form (11.38) in terms of the massless characters (4.71); in field theory, all r angles θ_i may be switched on simultaneously! To accommodate for this one can resort to the angle-dependent coefficients $\mathcal{A}_k^r(\vec{\theta})$ introduced in (11.31), whose origin can again be understood through the massless limit of the character (4.60). Using relation (11.33), the product of massless partition functions with spins ranging from zero to s can be written as (11.38), where the characters on the right-hand side are massless limits of massive Poincaré characters. However, it is unclear whether the quantities appearing in the exponent of (11.32) can be related directly to Poincaré characters *without* invoking a massless limit.

The occurrence of Poincaré characters in (11.38) illuminates certain aspects of gravity and higher-spin theories in three dimensions. Indeed, recall expression

(11.30) for the partition function of a field with spin s in three-dimensional thermal, rotating Minkowski space. (The regularization is unimportant for our present argument.) Since the space-time dimension is odd, the terms of the series in the exponential are not quite massless Poincaré characters, but they can still be interpreted as contributions of specific field excitations. Indeed, the terms $e^{isn\theta}$ are due to the heat kernel with spin s, while the terms $-e^{i(s-1)n\theta}$ come from ghosts, with a minus sign due to their fermionic statistics. The difference $e^{isn\theta} - e^{i(s-1)n\theta}$ vanishes when $\theta = 0$, in accordance with the fact that ghosts cancel all would-be local degrees of freedom. However, when $\theta \neq 0$, the cancellation is incomplete because would-be local field excitations and ghosts have different spins (s and $s-1$, respectively). As a result the one-loop partition function is non-trivial despite the absence of physical local degrees of freedom.

11.2 Representations and Characters of Flat $\mathcal{W}_N$

As an application of the results of the previous section, we now explain how certain combinations of one-loop partition functions in three-dimensional flat space reproduce characters of higher-spin asymptotic symmetry algebras at null infinity. As it turns out, the coadjoint representation of standard $\mathcal{W}_N$ algebras [19–21] will play a key role in the analysis, so we first review briefly the analogous situation of higher-spin fields in AdS_3. We then turn to the case of spin 3 in flat space and describe certain irreducible unitary representations of the corresponding asymptotic symmetry group. Upon evaluating their characters thanks to the Frobenius formula, we find that they match suitable products of partition functions. We also extend these observations to arbitrary spin N. The description of the induced modules and quantum algebras that correspond to this construction are relegated to Sect. 11.3.

11.2.1 *Higher Spins in AdS_3 and $\mathcal{W}_N$ Algebras*

As a preparation for flat space computations, we review here the asymptotic symmetries of higher-spin theories in AdS_3. We also describe the corresponding quantum symmetry algebras, their unitary representations and their characters, which match field-theoretic one-loop partition functions.

Asymptotic Symmetries

Asymptotic symmetries of higher-spin theories in three dimensions were first studied in AdS_3 [22–25], and are similar to the Brown–Henneaux asymptotic symmetries of gravity described in Chap. 8. Here we focus on models including fields with spin ranging from 2 to N; this setup can be described as an $\mathfrak{sl}(N,\mathbb{R}) \oplus \mathfrak{sl}(N,\mathbb{R})$ Chern-Simons action with a principally embedded $\mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{sl}(2,\mathbb{R})$ gravitational subalgebra. When $N = 3$, the asymptotic symmetries are generated by gauge transformations specified by four arbitrary, 2π-periodic functions $\big(X(x^+), \xi(x^+)\big)$ and $\big(\bar{X}(x^-), \bar{\xi}(x^-)\big)$ of the light-cone coordinates $x^\pm$ on the boundary of AdS_3. In particular, the functions $X(x^+)$ and $\bar{X}(x^-)$ generate Brown–Henneaux conformal transformations of the type described in Sect. 8.2. Since the results are left-right symmetric, we focus on the left-moving sector. The surface charge associated with a transformation (X, ξ) then generalizes the (left-moving half of the) gravitational expression (8.42) and takes the form [23]

$$Q_{(X,\xi)}[p, \rho] = \frac{1}{2\pi}\int_0^{2\pi} d\varphi\big[X(x^+)p(x^+) + \xi(x^+)\rho(x^+)\big] \tag{11.39}$$

when the normalization is chosen so that pure AdS_3 with all higher-spin fields switched off has vanishing higher-spin charges and negative mass $-1/8G$. Here $\varphi = (x^+ - x^-)/2$, while $p(x^+)$ and $\rho(x^+)$ are two arbitrary, 2π-periodic functions specifying a solution of the field equations. In particular $p(x^+)$ is one of the two functions $\big(p(x^+), \bar{p}(x^-)\big)$ that determine an on-shell AdS_3 metric (8.38) while $\rho(x^+)$, together with its anti-chiral counterpart $\bar{\rho}(x^-)$, specifies an on-shell higher-spin field configuration. The vacuum field configuration (8.47) corresponds to pure AdS_3 with all higher-spin fields set to zero, and is given by $\rho = \bar{\rho} = 0$, $p = \bar{p} = -\ell/16G$.

One can think of the pair (X, ξ) as an element of the asymptotic symmetry algebra, so the charge (11.39) is the pairing between this algebra and its dual space. This generalizes the pairing (6.34) of the Virasoro algebra with CFT stress tensors, and (p, ρ) may be seen as a coadjoint vector of the symmetry algebra. Its infinitesimal transformation law extends (8.39) and turns out to be [23]

$$\delta_{(X,\xi)}p = Xp' + 2X'p - \frac{c}{12}X''' + 2\,\xi\rho' + 3\,\xi'\rho\,, \tag{11.40a}$$

$$\begin{aligned}\delta_{(X,\xi)}\rho = {} & X\rho' + 3X'\rho + 2\xi p''' + 9\xi' p'' + 15\xi'' p' + 10\xi''' p \\ & - \frac{c}{12}\xi^{(5)} - \frac{192}{c}\left(\xi p p' + \xi' p^2\right),\end{aligned} \tag{11.40b}$$

where prime denotes differentiation with respect to x^+, and $c = 3\ell/2G$ is the Brown–Henneaux central charge. Analogous formulas hold in the anti-chiral sector. Since X generates conformal transformations, this implies that p is a (chiral) quasi-primary field with weight 2 while ρ is a primary with weight 3. Together with the surface charges (11.39), these transformation laws yield the Poisson bracket (8.10):

$$\big\{Q_{(X,\xi)}[\,p, \rho], Q_{(Y,\zeta)}[\,p, \rho]\big\} = -\,\delta_{(X,\xi)}\,Q_{(Y,\zeta)}[p, \pi]\,. \tag{11.41}$$

Formula (11.40) turns out to coincide with the coadjoint representation of a Poisson algebra known as the $\mathcal{W}_3$ *algebra*, and indeed one finds that the bracket (11.41) reproduces the non-linear bracket of a $\mathcal{W}_3$ algebra with central charge c (see Eq. (11.43) below). Similar considerations apply to models including fields with spin ranging from 2 to N [23, 25]; the resulting asymptotic symmetry algebra is the direct sum of two copies of $\mathcal{W}_N$.

Quantum $\mathcal{W}_3$ Algebra

Owing to the fact that (11.40) is the coadjoint representation of $\mathcal{W}_3$, the orbit of the AdS$_3$ vacuum $\big(p = \bar{p} = -c/24, \rho = \bar{\rho} = 0\big)$ under asymptotic symmetry transformations is a direct product of two vacuum coadjoint orbits of $\mathcal{W}_N$; these orbits are well-defined infinite-dimensional manifolds even though the definition of finite symmetry transformations is more intricate than in the pure Virasoro case corresponding to $N = 2$ [21]. Putting all mathematical subtleties under the rug, one thus expects the quantization of that orbit to produce the vacuum highest-weight representation of the quantum algebra $\mathcal{W}_N \oplus \mathcal{W}_N$. Accordingly, we now describe the quantum version of the $\mathcal{W}_3$ algebra.

As in the purely gravitational case (8.46), the classical asymptotic symmetry algebra given by the surface charges (11.39) can be written in terms of modes

$$\mathcal{L}_m \equiv Q_{(e^{imx^+},0)}\,, \qquad \mathcal{W}_m \equiv Q_{(0,e^{imx^+})} \tag{11.42}$$

and their barred counterparts in the right-moving sector. The normalization is such that pure AdS$_3$ has all charges vanishing except $\mathcal{L}_0 = \bar{\mathcal{L}}_0 = -c/24$. Using (11.40), one finds that the Poisson brackets (11.41) of the charges (11.42) take the form of a classical $\mathcal{W}_3$ algebra:

$$\begin{aligned} i\{\mathcal{L}_m, \mathcal{L}_n\} &= (m-n)\mathcal{L}_{m+n} + \frac{c}{12}m^3\delta_{m+n,0}\,, \\ i\{\mathcal{L}_m, \mathcal{W}_n\} &= (2m-n)\mathcal{W}_{m+n}\,, \\ i\{\mathcal{W}_m, \mathcal{W}_n\} &= (m-n)(2m^2+2n^2-mn)\mathcal{L}_{m+n} + \frac{96}{c}\Lambda_{m+n} + \frac{c}{12}m^5\delta_{m+n,0}\,, \end{aligned} \tag{11.43}$$

where $\Lambda_m \equiv \sum_{p\in\mathbb{Z}} \mathcal{L}_{m-p}\mathcal{L}_p$ is a non-linear term and the first line is the usual Virasoro algebra. The same brackets hold in the right-moving sector, so as announced the asymptotic symmetry algebra is a direct sum of two classical $\mathcal{W}_3$ algebras. Under quantization the Poisson brackets are turned into commutators according to $\widehat{i\{\cdot,\cdot\}} = [\hat{\cdot},\hat{\cdot}]$ and the charges $\mathcal{L}_m, \mathcal{W}_n$ become operators L_m, W_n which, in any unitary representation, satisfy the Hermiticity conditions

$$L_m^\dagger = L_{-m}\,, \qquad W_m^\dagger = W_{-m}\,.$$

It is also customary to normalize the Virasoro generators L_m so that the vacuum state has vanishing eigenvalue under L_0, i.e. to rename $L_m + \frac{c}{24}\delta_{m,0}$ into L_m. As a result the commutators of the operators L_m, W_n yield the *quantum* $\mathcal{W}_3$ algebra

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n,\,0}\,, \tag{11.44a}$$

$$[L_m, W_n] = (2m-n)W_{m+n}\,, \tag{11.44b}$$

$$[W_m, W_n] = (m-n)(2m^2+2n^2-mn-8)L_{m+n} + \frac{96}{c+22/5}(m-n):\Lambda_{m+n}: + \frac{c}{12}(m^2-4)(m^3-m)\delta_{m+n,0}\,, \tag{11.44c}$$

whose non-linear terms are normal-ordered according to the prescription

$$:\Lambda_m: \equiv \sum_{p\geq -1} L_{m-p}L_p + \sum_{p<-1} L_p L_{m-p} - \frac{3}{10}(m+3)(m+2)L_m\,. \tag{11.45}$$

Here the term linear in L_m ensures that the operator $:\Lambda_m:$ is quasi-primary with respect to the action of L_m's. Note how the denominator of the structure constant of the non-linear term in (11.44c) involves a shifted central charge $c + 22/5$ instead of the classical c in the last line of (11.43). Analogous commutation relations hold in the barred sector.

Unitary Representations and Characters

Unitary representations of the quantum $\mathcal{W}_3$ algebra are obtained analogously to the Virasoro highest-weight representations of Sect. 8.4, and are spanned by the descendants of a highest-weight state annihilated by operators L_m, W_n with $m, n > 0$. At large c, such representations are irreducible. The same is true of $\mathcal{W}_N$ algebras for any finite N, and irreducible unitary representations of $\mathcal{W}_N \oplus \mathcal{W}_N$ are tensor products of individual highest-weight representations of the two $\mathcal{W}_N$ algebras (at large $c, \bar{c}$). Characters of such representations can be evaluated by adapting the counting argument that led to (8.74). In particular the character of a highest-weight representation of $\mathcal{W}_N \oplus \mathcal{W}_N$ with central charges $(c, \bar{c})$, generic highest weights $(h, \bar{h})$, vanishing higher-spin weights and vanishing higher-spin chemical potentials, is

$$\mathrm{Tr}\left(q^{L_0-c/24}\bar{q}^{\bar{L}_0-\bar{c}/24}\right) = q^{h-c/24}\bar{q}^{\bar{h}-\bar{c}/24}\left(\prod_{n=1}^{+\infty}\frac{1}{|1-q^n|^2}\right)^{N-1}\,. \tag{11.46}$$

This reduces to the product of Virasoro characters (10.62) when $N = 2$. The vacuum character of $\mathcal{W}_N \oplus \mathcal{W}_N$ similarly reads

$$\mathrm{Tr}_{\mathrm{vac}}\left(q^{L_0-c/24}\bar{q}^{\bar{L}_0-\bar{c}/24}\right) = \prod_{s=2}^{N}\left(\prod_{n=s}^{+\infty}\frac{1}{|1-q^n|^2}\right), \tag{11.47}$$

where the truncated product arises because the vacuum state is left invariant by the wedge algebra $\mathfrak{sl}(N, \mathbb{R})$. This reduces to the vacuum character (8.79) when $N = 2$.

As mentioned earlier, it was shown in [5] that the one-loop partition function of gravitons in AdS_3 at temperature $1/\beta$ and angular potential θ is a vacuum character

(8.79) with modular parameter (8.78). This result was later extended to higher-spin theories [6, 7], whose one-loop partition functions on thermal AdS_3 coincide with vacuum $\mathcal{W}_N \oplus \mathcal{W}_N$ characters (11.47) upon including the contribution of fields with spins $s = 2, 3, \ldots, N$. These results confirm the interpretation of irreducible unitary representations of asymptotic symmetry groups as particles dressed with boundary degrees of freedom. The purpose of the remainder of this chapter is to describe the similar matching that occurs in asymptotically flat theories.

11.2.2 Flat $\mathcal{W}_3$ Algebra

The asymptotic symmetries of higher-spin theories at null infinity in three-dimensional flat space were described in [1, 26, 27]. Here we focus on the model describing the gravitational coupling of a field of spin 3, which is a three-dimensional Chern-Simons theory whose gauge algebra $\mathfrak{sl}(3,\mathbb{R}) \in \mathfrak{sl}(3,\mathbb{R})_{\text{Ab}}$ is the flat limit of $\mathfrak{sl}(3,\mathbb{R}) \oplus \mathfrak{sl}(3,\mathbb{R})$. The associated asymptotic symmetry generators turn out to be labelled by four arbitrary, 2π-periodic functions $X(\varphi)$, $\xi(\varphi)$, $\alpha(\varphi)$ and $a(\varphi)$ on the celestial circle at (future) null infinity. Of these, $X(\varphi)$ and $\alpha(\varphi)$ generate standard BMS_3 superrotations and supertranslations (respectively), while ξ and a are their higher-spin extensions. The corresponding surface charges extend the gravitational formula (9.31) and read

$$\mathcal{Q}_{(X,\xi,\alpha,a)}[j,\kappa,p,\rho] = \frac{1}{2\pi}\int_0^{2\pi} d\varphi\Big[X(\varphi)j(\varphi) + \xi(\varphi)\kappa(\varphi) + \alpha(\varphi)p(\varphi) + a(\varphi)\rho(\varphi)\Big], \tag{11.48}$$

where the 2π-periodic functions j, κ, p and ρ determine a solution of the equations of motion. In particular $p(\varphi)$ is the Bondi mass aspect (supermomentum) and $j(\varphi)$ is the angular momentum aspect (angular supermomentum) appearing in an asymptotically flat metric (9.25). The functions ρ and κ are analogous quantities for a spin-3 field. As usual, the quadruple (j, κ, p, ρ) may be seen as an element of the dual space of the asymptotic symmetry algebra. In particular, the higher-spin supermomentum (p, ρ) transforms under higher-spin superrotations (X, ξ) as a coadjoint vector of the $\mathcal{W}_3$ algebra, i.e. according to (11.40), albeit with a central charge $c_2 = 3/G$ instead of c.

The Poisson brackets satisfied by the surface charges (11.48) are given as usual by (11.41) and are most easily expressed in terms of generators

$$\mathcal{J}_m \equiv \mathcal{Q}_{(e^{im\varphi},0,0,0)}\,,\quad \mathcal{K}_m \equiv \mathcal{Q}_{(0,e^{im\varphi},0,0)}\,,\quad \mathcal{P}_m \equiv \mathcal{Q}_{(0,0,e^{im\varphi},0)}\,,\quad \mathcal{Q}_m \equiv \mathcal{Q}_{(0,0,0,e^{im\varphi})}\,.$$

Note that with this normalization pure Minkowski space has all its charges vanishing, except $\mathcal{P}_0 = -1/8G$. One then finds that the $\mathcal{J}_m$'s and $\mathcal{P}_m$'s close according to a $\mathfrak{bms}_3$ algebra (9.37) with central charge $c_2 = 3/G$, while brackets involving higher-spin charges take the form

$$i\{\mathcal{J}_m, \mathcal{K}_n\} = (2m-n)\mathcal{K}_{m+n}\,, \tag{11.49a}$$
$$i\{\mathcal{J}_m, \mathcal{Q}_n\} = (2m-n)\mathcal{Q}_{m+n}\,, \tag{11.49b}$$
$$i\{\mathcal{P}_m, \mathcal{K}_n\} = (2m-n)\mathcal{Q}_{m+n}\,, \tag{11.49c}$$
$$i\{\mathcal{P}_m, \mathcal{Q}_n\} = 0\,, \tag{11.49d}$$
$$i\{\mathcal{K}_m, \mathcal{K}_n\} = (m-n)(2m^2+2n^2-mn)\mathcal{J}_{m+n} + \frac{96}{c_2}(m-n)\Omega_{m+n}\,, \tag{11.49e}$$
$$i\{\mathcal{K}_m, \mathcal{Q}_n\} = (m-n)(2m^2+2n^2-mn)\mathcal{P}_{m+n} + \frac{96}{c_2}\Theta_{m+n} + \frac{c_2}{12}m^5\delta_{m+n,0} \tag{11.49f}$$

where the non-linear terms Ω and Θ are given by

$$\Omega_m \equiv \sum_{p\in\mathbb{Z}} \left(\mathcal{P}_{m-p}\mathcal{J}_p + \mathcal{J}_{m-p}\mathcal{P}_p\right)\,, \qquad \Theta_m \equiv \sum_{p\in\mathbb{Z}} \mathcal{P}_{m-p}\mathcal{P}_p\,. \tag{11.50}$$

These formulas show that, up to central terms, the brackets of $(\mathcal{J}, \mathcal{K})$'s with $(\mathcal{P}, \mathcal{Q})$'s take the same form as the brackets of $(\mathcal{J}, \mathcal{K})$'s with themselves. This is the situation described in (9.48) and it implies that, similarly to (9.64), the asymptotic symmetry algebra is an exceptional semi-direct sum

$$\text{“flat } \mathcal{W}_3 \text{ algebra”} \equiv \mathcal{W}_3 \in_{\text{ad}} (\mathcal{W}_3)_{\text{Ab}}\,, \tag{11.51}$$

where $\mathcal{W}_3$ is the classical $\mathcal{W}_3$ algebra (11.43) and $(\mathcal{W}_3)_{\text{Ab}}$ denotes an Abelian Lie algebra isomorphic, as a vector space, to $\mathcal{W}_3$. This algebra is centrally extended, as the bracket between generators of $\mathcal{W}_3$ and those of $(\mathcal{W}_3)_{\text{Ab}}$ includes a central charge c_2; there is also a central charge c_1 specific to the left (non-Abelian) $\mathcal{W}_3$ subalgebra of (11.51), but it is not switched on in parity-preserving theories. We shall return to this structure in Sect. 11.3, upon describing its quantum version.

Induced Representations and Unitarity

Since the flat $\mathcal{W}_3$ algebra (11.51) has the exceptional form $\mathfrak{g} \in \mathfrak{g}_{\text{Ab}}$, with $\mathfrak{g}$ the standard $\mathcal{W}_3$ algebra, its unitary representations should be induced representations labelled by orbits of supermomenta under the coadjoint action of elements of a group whose tangent space at the identity is the $\mathcal{W}_3$ algebra. However, the non-linearities that appear in $\mathcal{W}$ algebras make this step subtle, so one can bypass the need to control the group as follows. Generic $\mathcal{W}$ algebras define a Poisson manifold through (11.41) and one can classify the submanifolds on which the Poisson structure is invertible, i.e. their symplectic leaves in the terminology of Sect. 5.1. In the case of the Virasoro algebra (which corresponds to $\mathcal{W}_N$ with $N=2$) this concept coincides with that of a coadjoint orbit of the Virasoro group. For higher N the symplectic leaves of $\mathcal{W}_N$ algebras are still well defined [21] despite the lack of a straightforward definition of the group that corresponds to the $\mathcal{W}_N$ algebra. These leaves may be seen as intersections of the coadjoint orbits of $\mathfrak{sl}(N)$-Kac Moody algebras with the constraints that implement the Hamiltonian reduction to $\mathcal{W}_N$ algebras. Accordingly, it should be possible to build unitary representations of flat $\mathcal{W}_N$ algebras as Hilbert spaces of wavefunctions defined on their symplectic leaves, which we assume as usual to admit quasi-invariant measures. In most of the remainder of this chapter, we test that proposal by showing how it can be used to evaluate characters that coincide with higher-spin one-loop partition functions. Note that the non-linearities appearing in the brackets of the algebra (11.51) imply an extra complication for representation theory in that one has to devise a suitable normal-ordering prescription. In the standard $\mathcal{W}_3$ case we displayed this normal ordering in (11.45). In the flat case we will address this issue in Sect. 11.3.

The complete classification of the symplectic leaves of the $\mathcal{W}_3$ algebra has been worked out in [20, 21]; according to our proposal this settles the classification of irreducible unitary representations of the flat $\mathcal{W}_3$ algebra, in the same way that Virasoro coadjoint orbits classify BMS_3 particles. Instead of describing the details of this classification, we focus from now on on orbits of constant supermomenta, which can be classified thanks to the infinitesimal transformation laws (11.40) given by the algebra. To describe such an orbit, let us pick a pair (p, ρ) where $p(\varphi) = p_0$ and $\rho(\varphi) = \rho_0$ are constants, and act on it with an infinitesimal higher-spin superrotation (X, ξ). Then, all terms involving derivatives of p or ρ in Eq. (11.40) vanish, and we find

$$\delta_{(X,\xi)} p_0 = 2\, X' p_0 - \frac{c_2}{12} X''' + 3\, \xi' \rho_0\,, \tag{11.52a}$$

$$\delta_{(X,\xi)} \rho_0 = 3\, X' \rho_0 + 10\, \xi''' p_0 - \frac{c_2}{12}\, \xi^{(5)} - \frac{192}{c_2}\, \xi' p_0^2\,. \tag{11.52b}$$

The little group for (p_0, ρ_0) consists of higher-spin superrotations leaving it invariant. Its Lie algebra is therefore spanned by pairs (X, ξ) such that the right-hand sides of Eq. (11.52) vanish:

$$2\, X' p_0 - \frac{c_2}{12} X''' + 3\, \xi' \rho_0 = 0\,, \tag{11.53a}$$

$$3\, X' \rho_0 + 10\, \xi''' p_0 - \frac{c_2}{12}\, \xi^{(5)} - \frac{192}{c_2}\, \xi' p_0^2 = 0\,. \tag{11.53b}$$

The solutions of these equations depend on the values of p_0 and ρ_0. Here we take $\rho_0 = 0$ for simplicity, i.e. we only consider cases where all higher-spin charges are switched off. Then, given p_0, Eq. (11.53) become two decoupled differential equations for the functions $X(\varphi)$ and $\xi(\varphi)$, leading to three different cases:

- For generic values of p_0, the only pairs (X, ξ) leaving $(p_0, 0)$ invariant are constants, and generate a little group $\mathrm{U}(1) \times \mathbb{R}$.
- For $p_0 = -n^2 c_2/96$ where n is a positive odd integer, the pairs (X, ξ) leaving $(p_0, 0)$ invariant take the form

$$X(\varphi) = A, \qquad \xi(\varphi) = B + C \cos(n\varphi) + D \sin(n\varphi), \tag{11.54}$$

where A, B, C and D are real numbers. The corresponding little group is the n-fold cover of $\mathrm{GL}(2, \mathbb{R})$.

- For $p_0 = -n^2 c_2/24 = -(2n)^2 c_2/96$ where n is a positive integer, the Lie algebra of the little group is spanned by

$$\begin{aligned} X(\varphi) &= A + B\cos(n\varphi) + C\sin(n\varphi), \\ \xi(\varphi) &= D + E\cos(n\varphi) + F\sin(n\varphi) + G\cos(2n\varphi) + H\sin(2n\varphi), \end{aligned} \tag{11.55}$$

where $A, B, \ldots, H$ are real coefficients. The little group is thus an n-fold cover of $\mathrm{SL}(3, \mathbb{R})$. In particular, $p_0 = -c_2/24$ realizes the absolute minimum of energy among all supermomenta belonging to orbits with energy bounded from below. It is thus the supermomentum of the vacuum state, and indeed, upon using $c_2 = 3/G$, the field configuration that corresponds to it is the metric of Minkowski space (with the spin-3 field set to zero on account of $\rho_0 = 0$).

These results extend our earlier observations on Virasoro orbits in Sect. 7.1.

11.2.3 Flat $\mathcal{W}_3$ Characters

The information on little groups turns out to be sufficient to evaluate certain characters along the lines of Sect. 10.3. For instance, consider an induced representation based on the orbit $\mathcal{O}_p$ of a generic pair $(p_0, 0)$, and call (s, σ) the spin of the representation $\mathcal{R}$ of the little group $\mathrm{U}(1) \times \mathbb{R}$. Then take a superrotation which is an element of the $\mathrm{U}(1)$ subgroup (i.e. a rotation $f(\varphi) = \varphi + \theta$). The only point on the orbit that is left invariant by the rotation is $(p_0, 0)$, and the whole integral over the orbit in (10.54) localizes to that point. Therefore, in analogy with the BMS_3 example, the detailed knowledge of the orbit is irrelevant to compute the character. In particular, including a higher-spin supertranslation $\big(\alpha(\varphi), a(\varphi)\big)$, the only components of $\alpha(\varphi)$ and $a(\varphi)$ that survive the integration are their zero-modes α^0 and a^0. The character thus takes the form

$$\chi[(\mathrm{rot}_\theta, \alpha, a)] = e^{is\theta} e^{ip_0\alpha^0} \int_{\mathcal{O}_p} d\mu(k)\, \delta(k, \mathrm{rot}_\theta \cdot k) \tag{11.56}$$

where the little group character $e^{is\theta}$ factors out as in (4.55). In writing this we assume the existence of a quasi-invariant measure μ on the orbit, whose precise expression is unimportant since different measures give representations that are unitarily equivalent. Our remaining task is to integrate the delta function. To do so, we use local coordinates on the orbit, which we choose to be the Fourier modes of higher-spin supermomenta as we did in Sect. 10.3. Since p_0 is generic, the non-redundant coordinates on the orbit are the non-zero modes. As in (10.60) the integral is thus

$$\int_{\mathcal{O}_p} d\mu(k)\delta(k, \text{rot}_\theta \cdot k) = \prod_{n \in \mathbb{Z}^*} \left(\int dk_n \delta(k_n - e^{in\theta} k_n) \right) \prod_{m \in \mathbb{Z}^*} \left(\int d\rho_m \delta(\rho_m - e^{im\theta} \rho_m) \right), \tag{11.57}$$

where we call k_n the Fourier modes of the standard (spin 2) supermomentum, while ρ_m are the modes of its higher-spin counterpart. Performing the integrals over Fourier modes and adding small imaginary parts $i\epsilon$ to θ as in (10.56), we obtain

$$\chi[(\text{rot}_\theta, \alpha, a)] = e^{is\theta} e^{ip_0\alpha^0} \left(\prod_{n=1}^{+\infty} \frac{1}{|1 - e^{in(\theta+i\epsilon)}|^2} \right)^2. \tag{11.58}$$

This is a natural spin-3 extension of the spin-2 (BMS_3) massive character (10.61), in the same way that (11.46) generalizes Virasoro characters. It is also a flat limit of (11.46) for $N = 3$.

A similar computation can be performed for orbits of other higher-spin supermomenta $(p_0, 0)$. The only subtlety is that, for the values of p_0 for which the little group is larger than $\text{U}(1) \times \mathbb{R}$, the orbit has higher codimension in $\mathcal{W}_3^*$ than the generic orbit just discussed. Accordingly, there are fewer coordinates on the orbit and the products of integrals (11.57) are truncated. For instance, when $p_0 = -n^2 c_2 = /24$ with n a positive integer, the little group is generated by pairs (X, ξ) of the form (11.55), so that the Fourier modes providing non-redundant local coordinates on the orbit (in a neighbourhood of $(p_0, 0)$) are the modes k_m with $m \notin \{-n, 0, n\}$ and the higher-spin modes ρ_m with $m \notin \{-2n, -n, 0, n, 2n\}$. Assuming that the representation $\mathcal{R}$ of the little group is trivial, this produces a character

$$\chi[(\text{rot}_\theta, \alpha, a)] = e^{-in^2 c_2 \alpha^0/24} \left(\prod_{\substack{m=1,\\ m \neq n}}^{+\infty} \frac{1}{|1 - e^{im(\theta+i\epsilon)}|^2} \right) \cdot \left(\prod_{\substack{m=1,\\ m \neq n,\\ m \neq 2n}}^{+\infty} \frac{1}{|1 - e^{im(\theta+i\epsilon)}|^2} \right). \tag{11.59}$$

The choice $n = 1$ specifies the vacuum representation of the flat $\mathcal{W}_3$ algebra; taking α to be a Euclidean time translation by $i\beta$, we get

$$\chi_{\text{vac}}[(\text{rot}_\theta, \alpha = i\beta, a = 0)] = e^{\beta c_2/24} \left(\prod_{n=2}^{+\infty} \frac{1}{|1 - e^{in(\theta+i\epsilon)}|^2} \right) \cdot \left(\prod_{n=3}^{+\infty} \frac{1}{|1 - e^{in(\theta+i\epsilon)}|^2} \right). \tag{11.60}$$

This is one of our key results in this chapter. Indeed, comparing with Eq. (11.37), we recognize the product of the (suitably regularized) rotating one-loop partition functions of massless fields with spins two and three in three-dimensional flat space. It provides a first non-trivial check of our proposal for the construction of unitary representations of flat $\mathcal{W}_N$ algebras.

All the induced representations described above are unitary by construction, provided one can define (quasi-invariant) measures on the corresponding orbits. In analogy with representations of the $\mathfrak{bms}_3$ algebra, they can also be described as induced modules that generalize those of Sect. 10.2; we will turn to them in Sect. 11.3.

11.2.4 Flat $\mathcal{W}_N$ Algebras

The considerations of the previous pages can be generalized to higher-spin theories in flat space with spins ranging from 2 to N. In AdS$_3$ the asymptotic symmetries of models with this field content consist of two copies of a $\mathcal{W}_N$ algebra, so it is natural to anticipate that the corresponding theory in flat space will have an asymptotic symmetry algebra of the exceptional form

$$\text{“flat } \mathcal{W}_N \text{ algebra”} \equiv \mathcal{W}_N \in_{\text{ad}} (\mathcal{W}_N)_{\text{Ab}}\,, \tag{11.61}$$

in analogy with (11.51). The surface charges generating these symmetries should coincide with the pairing of the Lie algebra of (11.61) with its dual space, and they are likely to satisfy a centrally extended algebra. Since the presence of higher-spin fields does not affect the value of the central charge in three-dimensional AdS gravity [22, 23], one expects the central charge in this case to be the usual $c_2 = 3/G$ appearing in mixed brackets. This structure was indeed observed for $N = 4$ in [1]. We now argue that this proposal must hold for any N by showing that the vacuum character of (11.61), computed along the lines followed above for flat $\mathcal{W}_3$, reproduces the product of one-loop partition functions of fields of spin $2, 3, \ldots, N$.

According to our proposal for the characterization of the representations of semi-direct sums of the type (11.61), unitary representation of flat $\mathcal{W}_N$ algebras are classified by their symplectic leaves, that is, by orbits of higher-spin supermomenta $(p_1, \ldots, p_{N-1})$. (Here $p_1(\varphi)$ is the supermomentum that we used to write as $p(\varphi)$, while $p_2(\varphi)$ is what we called $\rho(\varphi)$ for $N = 3$.) The infinitesimal transformations that generalize (11.40) and that define these orbits locally can be found for instance in [25]. Here we focus on the vacuum orbit where we set all higher-spin charges to zero and take only $p_1 = -c_2/24$ to be non-vanishing. This particular supermomentum is left fixed by higher-spin superrotations of the form

$$X_i(\varphi) = A_i + \sum_{j=1}^{i} \big(B_{ij}\cos(j\varphi) + C_{ij}\sin(j\varphi)\big), \qquad i = 1, \ldots, N-1, \tag{11.62}$$

where the coefficients A_i, B_{ij}, C_{ij} are real. In principle one can obtain such symmetry generators by looking for the little group of the vacuum as in (11.53), using for instance the explicit formulas of [25]. Yet, a simpler way to derive the same result is to look for the higher-spin isometries of the vacuum in the first-order formulation, in which the theory is described by a Chern-Simons action with gauge algebra $\mathfrak{sl}(N, \mathbb{R}) \in_{\text{ad}} (\mathfrak{sl}(N, \mathbb{R}))_{\text{Ab}}$ (see e.g. [1, 27, 28]). In this language, and in terms of retarded Bondi coordinates (r, φ, u), the vacuum field configuration takes the form

$$A_\mu(x) = b(r)^{-1} g(u,\varphi)^{-1} \partial_\mu \left[g(u,\varphi) b(r) \right], \qquad b(r) = \exp\left[\frac{r}{2} P_{-1}\right], \tag{11.63}$$

where $g(u,\varphi)$ is a field valued in $\mathrm{SL}(N,\mathbb{R}) \ltimes \mathfrak{sl}(N,\mathbb{R})$, given by

$$g(u,\varphi) = \exp\left[\left(P_1 + \frac{1}{4}P_{-1}\right)u + \left(J_1 + \frac{1}{4}J_{-1}\right)\varphi\right] \tag{11.64}$$

in terms of Poincaré generators that satisfy the commutation relations (10.26). The isometries of this field configuration are generated by gauge parameters of the form $(g \cdot b)^{-1}T_a(g \cdot b)$, where T_a is any of the basis elements of the gauge algebra. Upon expanding $g^{-1}T_a g$ as a position-dependent linear combination of gauge algebra generators, the function multiplying the lowest weight generator coincides with the corresponding asymptotic symmetry parameter (see e.g. [23] for details). The latter can be obtained as follows.

For convenience, we diagonalize the Lorentz piece of the group element (11.64) as

$$\exp\left[\left(J_1 + \frac{1}{4}J_{-1}\right)\varphi\right] = Se^{iJ_0\varphi}S^{-1} \tag{11.65}$$

where S is some $\mathrm{SL}(2,\mathbb{R})$ matrix. Then the gauge parameters that generate the little group of the vacuum configuration can be written as

$$\exp\left[-\left(J_1 + \frac{1}{4}J_{-1}\right)\varphi\right]\sum_{m=-\ell}^{\ell}\alpha^m W_m^{(\ell)} \exp\left[\left(J_1 + \frac{1}{4}J_{-1}\right)\varphi\right] \tag{11.66a}$$

$$= S\,e^{-iJ_0\varphi}\sum_{m=-\ell}^{\ell}\alpha^m\, S^{-1}\, W_m^{(\ell)}\, S\,e^{iJ_0\varphi}\,S^{-1}, \tag{11.66b}$$

where the α^m's are certain real coefficients, while the $W_m^{(\ell)}$'s (with $2 \leq \ell \leq N$ and $-\ell \leq m \leq \ell$) generate the $\mathfrak{sl}(N,\mathbb{R})$ algebra (including $J_m \equiv W_m^{(2)}$). Note that the matrix S preserves the conformal weight since it is an exponential of $\mathfrak{sl}(2,\mathbb{R})$ generators, so that

$$\sum_{m=-\ell}^{\ell}\alpha^m\, S\, W_m^{(\ell)}\, S^{-1} = \sum_{m=-\ell}^{\ell}\tilde{\alpha}^m W_m^{(\ell)} \tag{11.67}$$

for some coefficients $\tilde{\alpha}^j$ obtained by acting on the α^m's with an invertible linear map. Since each generator $W_m^{(\ell)}$ has weight m under J_0, Eq. (11.66b) can be rewritten as

$$\sum_{m=-\ell}^{\ell}e^{im\varphi}\tilde{\alpha}^m\, S\, W_m^{(\ell)}\, S^{-1} = \sum_{m,n=-\ell}^{\ell}\beta^{mn}W_n^{(\ell)}e^{ij\varphi} = \sum_{m=-\ell}^{\ell}e^{im\varphi}\beta^{m\ell}W_\ell^{(\ell)} + \cdots \tag{11.68}$$

for some coefficients β^{mn}. In the last step we omitted all terms proportional to $W_m^{(\ell)}$'s with $m < \ell$; the important piece is the term that multiplies the highest-weight generator $W_\ell^{(\ell)}$: it is the function on the circle that generates the asymptotic symmetry corresponding to the generator $\sum_{m=-\ell}^{\ell}\alpha^m W_m^{(\ell)}$ that we started with in (11.66a). Since the $\beta^{m\ell}$'s are related to the α^m's by an invertible linear map, and since there are $2\ell+1$ linearly independent generators of this type, the isometries of the vacuum exactly span the set of functions of the form (11.62). This is what we wanted to prove; there are N^2-1 linearly independent asymptotic symmetry generators of this form, and they span the Lie algebra $\mathfrak{sl}(N,\mathbb{R})$.

The character associated with the vacuum representation of (11.61) can then be worked out exactly as in the cases $N=2$ and $N=3$ discussed above: using the Fourier modes of the $N-1$ components of supermomentum as coordinates on the orbit, we need to mod out the redundant modes. For the vacuum orbit, these are the modes ranging from $-(s-1)$ to $(s-1)$ for the s^{th} component. The integral over the localizing delta function in the Frobenius formula (4.33) then produces a character

$$\boxed{\chi[(\text{rot}_\theta, a_1 = i\beta)] = e^{\beta c_2/24}\prod_{s=2}^{N}\left(\prod_{n=s}^{+\infty}\frac{1}{|1-e^{in(\theta+i\epsilon)^2}|}\right)} \tag{11.69}$$

where we implicitly set to zero all higher-spin supertranslations except the gravitational one, $a_1 = i\beta$. Comparing with (11.37), we recognize the product of one-loop partition functions of massless higher-spin fields with spins ranging from 2 to N, including a classical contribution. This result confirms, on the one hand, our conjecture (11.61) for the asymptotic symmetry algebras of generic higher-spin theories in three-dimensional flat space, and on the other hand it provides another consistency check of our proposal for the characterization of unitary representations of flat $\mathcal{W}_N$ algebras. It is also a flat limit of the vacuum $\mathcal{W}_N \oplus \mathcal{W}_N$ character displayed in (11.47).

11.3 Flat $\mathcal{W}_3$ Modules

We now turn to the algebraic analogue of the above considerations, i.e. we describe induced modules of flat $\mathcal{W}_N$ algebras along the lines of Sect. 10.2. For simplicity we focus on the case $N=3$ but the construction also applies to other higher-spin extensions of $\mathfrak{bms}_3$. Due to the non-linearities of $\mathcal{W}_3$, our plan in this section is slightly different from that of Sect. 10.2. Namely, we start by describing the quantum flat $\mathcal{W}_3$ algebra as an ultrarelativistic limit of the direct sum of two quantum $\mathcal{W}_3$ algebras, which produces a specific ordering of operators in the non-linear terms of commutators. We then move on to the description of induced modules of the ultrarelativistic quantum flat $\mathcal{W}_3$ algebra, and show that the ultrarelativistic normal ordering is defined with respect to a rest frame vacuum. Along the way we compare our results to those of the non-relativistic limit described in [1], and point out that the two limits lead to different quantum algebras.

11.3.1 Ultrarelativistic and Non-relativistic Limits of $\mathcal{W}_3$

The flat $\mathcal{W}_3$ algebra (11.51) can be obtained as an Inönü-Wigner contraction of the direct sum of two $\mathcal{W}_3$ algebras. This flat limit was discussed at the semiclassical level in [26, 27], and a Galilean limit of the quantum algebra was described in [1]. Here we are interested instead in an ultrarelativistic limit of $\mathcal{W}_3 \oplus \mathcal{W}_3$. The key difference between the Galilean and ultrarelativistic contractions is that the latter mixes generators with positive and negative mode numbers, while the former does not. For linear algebras such as Virasoro, this makes no difference and the two contractions yield identical quantum algebras, namely $\mathfrak{bms}_3 \cong \mathfrak{gca}_2$. When non-linear terms are involved in the contraction, however, Galilean and ultrarelativistic limits generally give different quantum algebras, as we now explain.

Ultrarelativistic Contraction

The quantum $\mathcal{W}_3$ algebra is spanned by two sets of generators L_m and W_m ($m \in \mathbb{Z}$) whose commutation relations were displayed in (11.44). Consider now a direct sum $\mathcal{W}_3 \oplus \mathcal{W}_3$, where the generators and the central charge of the other copy of $\mathcal{W}_3$ will be denoted with a bar on top ($\bar{L}_m$, $\bar{W}_m$ and $\bar{c}$). Introducing a length scale ℓ to be interpreted as the AdS$_3$ radius, we define new generators P_m and J_m as in (10.32), as well as

$$K_m \equiv W_m - \bar{W}_{-m} , \qquad Q_m \equiv \frac{1}{\ell}\left(W_m + \bar{W}_{-m}\right) . \tag{11.70}$$

We also define central charges c_1 and c_2 as in (9.93). In the limit $\ell \to \infty$, and provided the central charges scale in such a way that both c_1 and c_2 are finite, one finds that J_m and P_m satisfy the $\mathfrak{bms}_3$ brackets (10.41) together with

$$[J_m,\ K_n] = (2m-n)K_{m+n} , \qquad [J_m,\ Q_n] = (2m-n)Q_{m+n} , \tag{11.71a}$$

$$[P_m,\ K_n] = (2m-n)Q_{m+n} , \qquad [P_m,\ Q_n] = 0 . \tag{11.71b}$$

The remaining brackets involving higher-spin generators are

$$[K_m, K_n] = (m-n)(2m^2+2n^2-mn-8)J_{m+n} + \frac{96}{c_2}(m-n)\Omega_{m+n} \tag{11.71c}$$

$$- \frac{96\,c_1}{c_2^2}(m-n)\Theta_{m+n} + \frac{c_1}{12}(m^2-4)(m^3-m)\delta_{m+n,\,0} , \tag{11.71d}$$

$$[K_m, Q_n] = (m-n)(2m^2+2n^2-mn-8)P_{m+n} + \frac{96}{c_2}(m-n)\Theta_{m+n}$$

$$+ \frac{c_2}{12}(m^2-4)(m^3-m)\delta_{m+n,\,0} , \tag{11.71e}$$

$$[Q_m,\ Q_n] = 0 , \tag{11.71f}$$

where the non-linear terms Ω_m and Θ_m are quadratic operators given by (11.50), with the exact same ordering (and calligraphic letters replaced by usual capital letters):

$$\Omega_m \equiv \sum_{p\in\mathbb{Z}} \left(P_{m-p}J_p + J_{m-p}P_p\right) , \qquad \Theta_m \equiv \sum_{p\in\mathbb{Z}} P_{m-p}P_p . \tag{11.72}$$

The commutation relations (11.71) are quantum analogues of the Poisson brackets (11.49), including a central charge c_1 and with operators normalized so that the vacuum has zero eigenvalue under P_0. One can check that with the definition (11.50), the brackets given by (10.41) and (11.71) satisfy Jacobi identities, so the generators J_m, K_m, P_n, Q_n span a well-defined non-linear Lie algebra. We call it the *quantum flat* $\mathcal{W}_3$ *algebra*. In any unitary representation, its generators satisfy the Hermiticity conditions

$$(Q_m)^\dagger = Q_{-m} , \qquad (K_m)^\dagger = K_{-m} . \tag{11.73}$$

supplemented with (10.27) for $m \in \mathbb{Z}$.

The expressions (11.72) for the quadratic terms follow from the identities

$$:\Lambda_m: + :\bar{\Lambda}_m: = \frac{\ell^2}{2}\,\Theta_m + \mathcal{O}(\ell) , \qquad :\Lambda_m: - :\bar{\Lambda}_m: = \frac{\ell}{2}\,\Omega_m + \mathcal{O}(1) \tag{11.74}$$

where $:\Lambda_m:$ is the normal-ordered quadratic term (11.45) of the quantum $\mathcal{W}_3$ algebra, while $:\bar{\Lambda}_m:$ is its right-moving counterpart. Note, in particular, that both the linear term in (11.45) and the mixing between positive and negative modes in (10.32)–(11.70) are necessary to reorganize the sum of quadratic terms with the precise order of (11.72). We shall see in Sect. 11.3.2 that (11.50) is a normal-ordered polynomial with respect to the natural vacuum in induced modules of the quantum flat $\mathcal{W}_3$ algebra.

Galilean Contraction

In order to compare the quantum flat $\mathcal{W}_3$ algebra (11.71) with other results in the literature [1], we now consider the non-relativistic limit of the quantum direct sum $\mathcal{W}_3 \oplus \mathcal{W}_3$. It is obtained by defining central charges $\tilde{c}_1$ and $\tilde{c}_2$ as in (10.53), introducing new generators $\tilde{J}_m$ and $\tilde{P}_m$ as in (10.52) and writing

$$\tilde{K}_m \equiv \bar{W}_m + W_m , \qquad \tilde{Q}_m \equiv \frac{1}{\ell}\left(\bar{W}_m - W_m\right) . \tag{11.75}$$

Note the difference with respect to (11.70). In the limit $\ell \to +\infty$ one obtains brackets of the same form as in (11.71) upon putting tildes on top of all generators, but there are two important differences: (i) the coefficient in front of Θ_{m+n} in (11.71d) contains a shifted central charge $\tilde{c}_1 + 44/5$, and (ii) the quadratic term $\tilde{\Omega}_m$ reads

$$\tilde{\Omega}_m = \sum_{p\geq -1}\left(\tilde{P}_{m-p}\tilde{J}_p + \tilde{J}_{m-p}\tilde{P}_p\right) + \sum_{p<-1}\left(\tilde{P}_p\tilde{J}_{m-p} + \tilde{J}_p\tilde{P}_{m-p}\right) - \frac{3}{5}\,(m+3)(m+2)\tilde{P}_m \tag{11.76}$$

instead of (11.72). The non-linear term $\tilde{\Theta}_m$ remains the same (up to tildes) since the generators $\tilde{P}_m$ commute in the large ℓ limit. The quadratic combinations $\tilde{\Theta}_m$ and

$\tilde{\Omega}_m$ can then be interpreted as normal-ordered operators with respect to a Galilean highest-weight vacuum defined by conditions of the type (10.49) with $\tilde{M} = \tilde{s} = 0$. These differences show that the two contractions lead to different *quantum* algebras, despite the fact that the corresponding classical algebras coincide.[3] Thus, in the presence of higher-spin fields, the difference between ultrarelativistic and Galilean limits manifests itself directly in the symmetry algebras and not only at the level of the representations that survive in the limit.

In the following we restrict attention to irreducible unitary representations of the ultrarelativistic quantum algebra (11.71), built once again according to the induced module prescription of Sect. 10.2. On the other hand, highest-weight representations of Galilean contractions of two copies of non-linear $\mathcal{W}$ algebras were discussed in [1], where it was shown that unitary representations with higher-spin states do not exist.

11.3.2 Induced Modules for the Flat $\mathcal{W}_3$ Algebra

According to our proposal of Sect. 11.2.2, the Hilbert space of any unitary representation of the flat $\mathcal{W}_3$ algebra consists of wavefunctions on the orbit of a higher-spin supermomentum $\big(p(\varphi), \rho(\varphi)\big)$. Assuming that the orbit admits a quasi-invariant measure, a basis of the Hilbert space is provided by plane waves (10.13) with definite supermomentum. For definiteness, let us focus on an orbit containing a constant higher-spin supermomentum (p_0, ρ_0); this is to say that the representation admits a rest frame. There is a corresponding plane wave $\Psi_{(p_0,\rho_0)}$, and any other plane wave can be obtained by acting on $\Psi_{(p_0,\rho_0)}$ with a higher-spin superrotation.

Massive modules

Let us take $p_0 = M - c_2/24$ with $M > 0$; this corresponds to a massive representation of the flat $\mathcal{W}_3$ algebra. Assuming also that ρ_0 is generic, the little group is $\mathrm{U}(1) \times \mathbb{R}$ and the spin of the representation is therefore a pair $(s, \sigma) \in \mathbb{R}^2$. Now, the plane wave at rest $\Psi_{(p_0,\rho_0)} \equiv |M, \rho_0\rangle$ is a state that satisfies

$$P_m|M, \rho_0\rangle = 0\,, \qquad Q_m|M, \rho_0\rangle = 0 \qquad \text{for } m \neq 0\,, \tag{11.77a}$$

and is an eigenstate of zero-mode charges:

$$P_0|M, \rho_0\rangle = M|M, \rho_0\rangle\,, \qquad J_0|M, \rho_0\rangle = s|M, \rho_0\rangle\,, \tag{11.77b}$$

$$Q_0|M, \rho_0\rangle = \rho_0|M, \rho_0\rangle\,, \qquad K_0|M, \rho_0\rangle = \sigma|M, \rho_0\rangle\,. \tag{11.77c}$$

Here M and s are the mass and spin labels encountered in (10.42), while ρ_0 and σ are their spin-3 counterparts. As before we call $|M, \rho_0\rangle$ the *rest frame state* of the

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[3] An interesting problem is to understand if these algebras are merely different because of an unfortunate choice of basis, or if they are genuinely distinct in the sense that they are not isomorphic. We will not address this issue here.

representation and we normalize the operator P_0 so that the vacuum has vanishing P_0 eigenvalue.

The conditions (11.77) define a one-dimensional representation of the subalgebra spanned by $\{P_m, Q_m, J_0, W_0\}$. They can be used to define an induced module $\mathscr{H}$ with basis elements

$$K_{k_1} \dots K_{k_m} J_{l_1} \dots J_{l_n} |M, \rho_0\rangle \,, \tag{11.78}$$

where $k_1 \leq \cdots \leq k_m$ and $l_1 \leq \cdots \leq l_n$ are non-zero integers. This provides an explicit representation of the quantum flat $\mathcal{W}_3$ algebra. Note that the presence of non-linearities in the commutators (11.71) does not affect the construction of the induced module, which involves the universal enveloping algebra anyway.

As usual, unitarity is somewhat hidden in the induced module picture but can be recognized in the fact that the state $|M, \rho_0\rangle$ is a plane wave, and that acting on it with finite higher-spin superrotations generates an orthonormal basis of plane wave states for the carrier space of the representation. Irreducibility can be inferred from the same argument that we used for $\mathfrak{bms}_3$: by construction, a supermomentum orbit is a homogeneous space for the action of superrotations, and this carries over to the higher-spin setting. This implies that $\mathcal{W}_3$ superrotations can map any plane wave state on any other one, which in turn implies that the space of the representation has no non-trivial invariant subspace.

Vacuum Module

The vacuum module of the flat $\mathcal{W}_3$ algebra can be built in direct analogy to its $\mathfrak{bms}_3$ counterpart discussed around (10.44). The only subtlety is the enhancement of the little group, which leads to additional conditions on superrotations. Indeed the vacuum state $|0\rangle$ is now an eigenstate of all modes P_m and Q_m with zero eigenvalue, and satisfies in addition

$$J_n|0\rangle = 0 \quad \text{for } n = -1, 0, 1, \qquad K_m|0\rangle = 0 \quad \text{for } m = -2, -1, 0, 1, 2. \tag{11.79}$$

These conditions ensure that the vacuum is invariant under the $\mathfrak{sl}(3, \mathbb{R})$ wedge algebra of the $\mathcal{W}_3$ subalgebra (which includes in particular the Lorentz algebra). The corresponding module can then be built as usual by acting with higher-spin superrotation generators on the vacuum state and producing states of the form (11.78), where now all l_i's must be different from $-1, 0, 1$ and all k_i's must be different from $-2, -1, 0, 1, 2$. We stress that the l_i's and k_i's can be positive or negative, in sharp contrast to the non-relativistic modules investigated in [1].

The definition of the flat $\mathcal{W}_3$ vacuum allows us to interpret the quadratic terms (11.72) as being normal-ordered. Indeed, their expectation values vanish in the vacuum $|0\rangle$:

$$\langle 0|\Theta_n|0\rangle = \langle 0|\Omega_n|0\rangle = 0. \tag{11.80}$$

These considerations appear to be a robust feature of "flat $\mathcal{W}$ algebras": ultrarelativistic contractions of $\mathcal{W}_N \oplus \mathcal{W}_N$ algebras always take the form

$$\text{“flat } \mathcal{W}_N\text{”} = \mathcal{W}_N \in_{\text{ad}} (\mathcal{W}_N)_{\text{Ab}} \tag{11.81}$$

and therefore contain an Abelian ideal, where the semi-direct sum ensures that the structure constants of the non-linear terms are always proportional to inverse powers of the central charge. Indeed, for a non-linear operator of n^{th} order the structure constants are of order $\frac{1}{c^{n-1}}$ at large c. When expanding them in powers of the contraction parameter ℓ, this implies that the leading term is proportional to ℓ^{1-n} thanks to (9.93). In order to obtain a finite expression, it is thus necessary that the resulting non-linear operator consists of at least $n-1$ Abelian generators. Terms of this kind always have a vanishing expectation value in the rest frame vacuum state, although the precise ordering in the polynomial should be fixed by other means, e.g. by defining the algebra via a contraction of the quantum algebra or by imposing Jacobi identities. Thus the conditions (11.77) with $M = \rho_0 = s = \sigma = 0$, together with (11.79), provide a valid definition of the vacuum for all quantum flat $\mathcal{W}_N$ algebras.

By contrast, for a highest-weight vacuum of the type (10.49), the quadratic operators Ω_m given by (11.72) generally have non-vanishing vacuum expectation values. Thus the extra non-linear structure introduced by higher spins exhibits the fact that the natural representations in the ultrarelativistic limit are the induced ones discussed above, rather than the highest-weight ones of [1, 29]. This difference emphasizes the physical distinction between ultrarelativistic and Galilean limits: the former is adapted to gravity, and more generally to models of fundamental interactions, where unitarity is a key requirement. In particular, flat space holography (at least in the framework of Einstein gravity) is expected to rely on the unitary construction described in this thesis. By contrast, the Galilean viewpoint is suited to condensed matter applications, and more generally to situations where unitarity need not hold — as was indeed argued in [29]. We stress that this difference is a genuine quantum higher-spin effect: it is not apparent at the classical level, and it does not occur in pure gravity either.

11.4 Super-BMS$_3$ and Flat Supergravity

This section is devoted to supersymmetric extensions of the BMS$_3$ group, to their representations, and to their characters. Accordingly we start by describing rotating one-loop partition functions of fermionic fields in flat space, along the same lines as in Sect. 11.1. Upon confirming that they take the form of exponentials of Poincaré characters (11.38), we specialize to $D = 3$ space-time dimensions. There we describe supersymmetric BMS$_3$ groups and their unitary representations, and note that super BMS$_3$ multiplets contain towers of infinitely many particles with increasing spins. Finally, we show that the resulting characters match suitable combinations of bosonic and fermionic one-loop partition functions.

11.4.1 Fermionic Higher Spin Partition Functions

We wish to evaluate the partition function (11.1) of a free fermionic field ψ with spin $s + 1/2$ (where s is a non-negative integer) and mass $M > 0$. Its Euclidean action can be presented either (i) using a symmetric, γ-traceless spinor field with s space-time indices and a set of auxiliary fields with no gauge symmetry [30] or (ii) using a set of symmetric spinor fields with $s, s-1, \ldots, 0$ space-time indices and vanishing triple γ-trace, subject to a gauge symmetry generated by γ-traceless parameters with $s-1, \ldots, 0$ space-time indices [31]. In the latter case, just as for bosons, the action is given by a sum of actions for massless fields of each of the involved spins, plus a set of cross-coupling terms proportional to the mass. In the limit $M \to 0$ the quadratic couplings vanish and one is left with a sum of decoupled Fang-Fronsdal actions [32]

$$S[\psi, \bar{\psi}] = \int d^D x\, \bar{\psi}^{\mu_1 \cdots \mu_s} \left(\mathcal{S}_{\mu_1 \ldots \mu_s} - \frac{1}{2} \gamma_{(\mu_1} \not{\mathcal{S}}_{\mu_2 \cdots \mu_s)} - \frac{1}{2} \delta_{(\mu_1 \mu_2} \mathcal{S}_{\mu_3 \cdots \mu_s) \lambda}{}^{\lambda} + \text{h.c.} \right), \tag{11.82}$$

where space-time indices are raised and lowered thanks to the Euclidean metric (and "h.c." means "Hermitian conjugate"). We use the same symmetrization conventions as in Sect. 11.1 and

$$\mathcal{S}_{\mu_1 \ldots \mu_s} = \left(\not{\partial} \psi_{\mu_1 \ldots \mu_s} - \partial_{(\mu_1} \not{\psi}_{\mu_2 \ldots \mu_s)} \right) . \tag{11.83}$$

The slash notation means $\not{\mathcal{V}} \equiv \gamma^\mu \mathcal{V}_\mu$, where the γ^μ's are Dirac matrices satisfying the anticommutation relations $\{\gamma^\mu, \gamma^\nu\} = \delta^{\mu\nu}$.

To compute the partition function for ψ, $\bar{\psi}$ one has to evaluate a path integral (11.1) with the integration measure $\mathcal{D}\psi\mathcal{D}\bar{\psi}$ and S the action (11.82) or its massive analogue. The fermionic fields live on $\mathbb{R}^D/\mathbb{Z}$ as defined by the group action (11.7), but in contrast to bosons, they satisfy *anti*periodic boundary conditions along the thermal cycle. For a massive field, one thus finds that the partition function is given by

$$\log Z = \frac{1}{2} \log\det(-\Delta^{(s+1/2)} + M^2) - \frac{1}{2} \log\det(-\Delta^{(s-1/2)} + M^2)\,, \tag{11.84}$$

where $\Delta^{(s+1/2)}$ is the Laplacian acting on antiperiodic, symmetric, γ-traceless spinor fields with s indices on $\mathbb{R}^D/\mathbb{Z}$. For massless fields, the gauge symmetry enhancement requires gauge-fixing and ghosts, leading to [33]

$$\log Z = \frac{1}{2} \log\det(-\Delta^{(s+1/2)}) - \log\det(-\Delta^{(s-1/2)}) + \frac{1}{2} \log\det(-\Delta^{(s-3/2)})\,. \tag{11.85}$$

Equations (11.84) and (11.85) are fermionic analogues of the bosonic formulas (11.11) and (11.25). To evaluate the functional determinants, we rely once more on heat kernels and the method of images described in Sect. 11.1.1.

The heat kernel $\mathcal{K}^{AB}{}_{\mu_s,\nu_s}$ associated with the operator $(-\Delta^{(s+1/2)}+M^2)$ on $\mathbb{R}^D$ is the unique solution of

$$(\Delta_{(s+1/2)}-M^2-\partial_t)\mathcal{K}^{AB}{}_{\mu_s,\nu_s}=0\,,\quad \mathcal{K}^{AB}{}_{\mu_s,\,\nu_s}(t=0,x,x')=\mathbb{I}^{(F)}_{\mu_s,\,\nu_s}\mathbf{1}^{AB}\delta^{(D)}(x-x')\,. \tag{11.86}$$

Here $\mathcal{K}^{AB}{}_{\mu_s,\nu_s}$ is a bispinor in the indices A and B, and a symmetric bitensor in the indices μ_s and ν_s. (We use again the shorthand μ_s to denote a set of s symmetrized indices.) It is also γ-traceless in the sense that

$$\gamma^\mu \mathcal{K}_{\mu_s,\,\nu_s}=\mathcal{K}_{\mu_s,\,\nu_s}\gamma^\nu=0\,. \tag{11.87}$$

The solution of (11.86) satisfying this requirement is

$$\mathcal{K}_{\mu_s,\,\nu_s}(t,x,x')=\frac{1}{(4\pi t)^{D/2}}\,e^{-M^2t-\frac{1}{4t}|x-x'|^2}\,\mathbb{I}^{(F)}_{\mu_s,\,\nu_s}\,, \tag{11.88}$$

where $\mathbb{I}^{(F)}_{\mu_s,\nu_s}$ is the following bisymmetric, γ-traceless tensor:

$$\mathbb{I}^{(F)}_{\mu_s,\,\nu_s}=\sum_{k=0}^{\lfloor\frac{s}{2}\rfloor}\frac{(-1)^k2^kk!\,[D+2(s-k-1)]!!}{s!\,[D+2(k-1)]!!}\left(\delta^k_{\mu\mu}\delta^{s-2k}_{\mu\nu}\delta^s_{\nu\nu}-\frac{\delta^s_{\mu\mu}\delta^{s-2k-1}_{\mu\nu}\delta^s_{\nu\nu}\gamma_\mu\gamma_\nu}{D+2(s-k-1)}\right). \tag{11.89}$$

Up to the replacement of $\mathbb{I}$ by $\mathbb{I}^{(F)}$, the fermionic heat kernel (11.88) is the same as the bosonic one in Eq. (11.14). In particular, $\mathbb{I}^{(F)}$ carries all its tensor and spinor indices.

To evaluate the determinant of $(-\Delta^{(s+1/2)}+M^2)$ on $\mathbb{R}^D/\mathbb{Z}$, we use once more the method of images (11.6). As before, we need to keep track of the non-trivial index structure of $\mathcal{K}^{AB}{}_{\mu_s,\nu_s}$, which leads to

$$\mathcal{K}^{\mathbb{R}^D/\mathbb{Z}}_{\mu_s,\,\alpha_s}(t,x,x')=\sum_{n\in\mathbb{Z}}(-1)^n(J^n)_\alpha{}^\beta\dots(J^n)_\alpha{}^\beta\,U^n\,\mathcal{K}_{\mu_s,\,\beta_s}\big(t,x,\gamma^n(x')\big)\,, \tag{11.90}$$

where the factor $(-1)^n$ comes from antiperiodic boundary conditions, J is the matrix (11.8), and U is a $2^{\lfloor D/2\rfloor}\times 2^{\lfloor D/2\rfloor}$ matrix acting on spinor indices in such a way that

$$J^\alpha{}_\beta\gamma^\beta=U\gamma^\alpha U^{-1}\,. \tag{11.91}$$

In other words, U is the matrix corresponding to the transformation (11.8) in the spinor representation of SO(D), and it can be written as

$$U=\exp\left[\frac{1}{4}\sum_{j=1}^{\lfloor(D-1)/2\rfloor}\theta_j[\gamma_{2j-1},\gamma_{2j}]\right].$$

In particular, a rotation by 2π around any given axis maps the field ψ on $-\psi$, in accordance with the fact that spinors represent SO(D) up to a sign. Note that, using an explicit D-dimensional representation of the γ matrices, one gets

$$\mathrm{Tr}(U^n) = 2^{\lfloor D/2 \rfloor} \prod_{i=1}^{r} \cos(n\theta_i/2)\,. \tag{11.92}$$

Now, plugging (11.90) into formula (11.3) for the determinant of $-\Delta^{(s+1/2)}$, one obtains a sum of integrals which can be evaluated exactly as in the bosonic case. The only difference with respect to bosons comes from the spin structure, and the end result is

$$-\log\det(-\Delta^{(s+1/2)} + M^2) = \sum_{n\,\in\,\mathbb{Z}^*} \frac{(-1)^n}{|n|} \frac{\chi_s^{(F)}[n\vec{\theta}, \vec{\epsilon}\,]}{\prod_{j=1}^{r} |1 - e^{in(\theta_j + i\epsilon_j)}|^2} \times \begin{cases} e^{-|n|\beta M} & D \text{ odd} \\ \frac{ML}{\pi} K_1(|n|\beta M) & D \text{ even} \end{cases} \tag{11.93}$$

where we have discarded a volume divergence independent of all chemical potentials (as in Eq. (11.20)), and where

$$\chi_s^{(F)}[n\vec{\theta}, \vec{\epsilon}\,] = (J^{\mu\alpha})^s \, \mathrm{Tr}\left[\mathbb{I}^{(F)}_{\mu_s, \alpha_s}\right] \tag{11.94}$$

is the fermionic analogue of (11.19), with the same rough regularization as in Eq. (11.20) (a more careful regularization will be described below for $D = 3$). This result takes the same form as (11.20), up to the replacement of χ_s by $\chi_s^{(F)}$ and the occurrence of $(-1)^n$ due to antiperiodicity. In Appendices sections "Mixed Traces and Symmetric Polynomials" and "Symmetric Polynomials and SO(D) Characters", we show that

$$\chi_s^{(F)}[n\vec{\theta}] \overset{11.\mathrm{B}.1\&11.B.2}{=} \begin{cases} \chi^{(D)}_{\lambda_s^{(F)}}[n\vec{\theta}\,] & \text{for odd } D, \\ \chi^{(D)}_{\lambda_s^{(F)}}[n\vec{\theta}, 0] & \text{for even } D, \end{cases} \tag{11.95}$$

where the term on the right-hand side is the character of an irreducible representation of SO(D) with highest weight $\lambda_s^{(F)} = (s + 1/2, 1/2, \ldots, 1/2)$, written here in the dual basis of the Cartan subalgebra of $\mathfrak{so}(D)$ described above (11.21).

Having computed the required functional determinants on $\mathbb{R}^D/\mathbb{Z}$, we can now write down the partition functions given by (11.84) and (11.85). In the massive case, the difference of Laplacians acting on fields with spins $(s + 1/2)$ and $(s - 1/2)$ produces the difference of two factors (11.95), with labels s and $s - 1$. It turns out that formula (11.23) still holds if we replace λ_s and λ_{s-1} by their fermionic counterparts, $\lambda_s^{(F)}$ and $\lambda_{s-1}^{(F)}$. (The proof of this statement follows the exact same steps as in the bosonic case described in Appendix section "Differences of SO(D) Characters", up to obvious replacements that account for the change in the highest weight vector.) Accordingly, the rotating one-loop partition function of a massive field with spin $s + 1/2$ is

$$Z(\beta,\vec{\theta}) = \exp\left[\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\frac{\chi^{(D-1)}_{\lambda_s^{(F)}}[n\vec{\theta},\vec{\epsilon}\,]}{\prod_{j=1}^{r}|1-e^{in(\theta_j+i\epsilon_j)}|^2}\times\begin{cases}e^{-n\beta M} & (D\text{ odd})\\ \frac{ML}{\pi}K_1(n\beta M) & (D\text{ even})\end{cases}\right]. \tag{11.96}$$

In the massless case we must take into account one more difference of characters, namely (11.27) with λ_s replaced by $\lambda_s^{(F)}$. For $D \geq 4$, this difference can be written as a combination of $\mathrm{SO}(D-2)$ characters (the proof is essentially the same as in Appendix section "Differences of $\mathrm{SO}(D)$ Characters"), and the partition function of a massless field with spin $s+1/2$ exactly takes the form (11.28) or (11.32) (for D even or odd, respectively) with an additional factor of $(-1)^{n+1}$ in the sum over n, and the replacement of λ_s by $\lambda_s^{(F)}$. One can also verify that relation (11.34) remains true for fermionic partition functions.

For $D=3$, differences of SO(2) characters cannot be reduced any further (recall the discussion surrounding (11.29)), so the best one can do is to write the partition function of a massless field with spin $s+1/2$ as

$$Z(\beta,\theta) = \exp\left[\sum_{n=1}^{+\infty}\frac{(-1)^{n+1}}{n}\frac{1}{|1-e^{in(\theta+i\epsilon)}|^2}\left(e^{i(s+1/2)n(\theta+i\epsilon)}-e^{i(s-1/2)n(\theta+i\epsilon)}+\text{c.c.}\right)\right] \tag{11.97}$$

provided $s \geq 1$. (For $s=0$ the exponentials in the summand reduce to $e^{in(\theta+i\epsilon)/2}+$ c.c., without any negative contribution.) Here we are using once more the crude regularization described around (11.20); a more careful prescription, motivated by the bosonic combination (11.35), consists in regulating the sum of exponentials in the summand according to

$$e^{i(s+1/2)n(\theta+i\epsilon)} - e^{i(s-1/2)n\theta-(s+3/2)n\epsilon} + \text{c.c.} \tag{11.98}$$

Upon using this expression in the summand of (11.97) instead of the naive combination of exponentials written there, the series in the exponential becomes

$$\begin{aligned}\sum_{n=1}^{+\infty}\frac{(-1)^{n+1}}{n}\frac{q^{n(s+1/2)}-q^{n(s+1/2)}\bar{q}^n+\text{c.c.}}{|1-q^n|^2} &= \sum_{n=1}^{+\infty}\left(\frac{(-1)^{n+1}}{n}\frac{q^{n(s+1/2)}}{1-q^n}+\text{c.c.}\right)\\ &= \sum_{j=s}^{+\infty}\log(1+q^{j+1/2})+\text{c.c.}\end{aligned}$$

in terms of $q=e^{i(\theta+i\epsilon)}$. As in the bosonic case (11.36), the regularization (11.98) has ensured that $\log Z$ splits as the sum of a chiral and an anti-chiral function of q. After renaming j into n, the end result is the following expression for the partition function of a field with spin $s+1/2$ in three dimensions:

$$Z = \prod_{n=s}^{+\infty} \left|1 + e^{i(n+1/2)(\theta+i\epsilon)}\right|^2, \tag{11.99}$$

which can also be recovered as the flat limit of the corresponding AdS result [33]. The remainder of this chapter is devoted to relating this partition function to the vacuum characters of various supersymmetric extensions of the BMS$_3$ group.

11.4.2 Supersymmetric BMS$_3$ Groups

The supersymmetric BMS$_3$ groups describe the symmetries of three-dimensional, asymptotically flat supergravity [34–38]. Here we briefly review some background on super Lie groups and the super Virasoro algebra, which we then use to provide a definition of various supersymmetric extensions of BMS$_3$. The corresponding unitary representations and characters will be investigated in Sect. 11.4.3.

Supersymmetric Induced Representations

A super Lie group is a pair (Γ_0, γ) where Γ_0 is a Lie group in the standard sense, while γ is a super Lie algebra whose even part coincides with the Lie algebra of Γ_0, and whose odd part is a Γ_0-module such that the differential of the Γ_0 action be the bracket between even and odd elements of γ [39]. Then a *super semi-direct product* is a super Lie group of the form [40, 41]

$$\big(G \ltimes_\sigma A, \mathfrak{g} \Subset (A + \mathcal{A})\big), \tag{11.100}$$

where $G \ltimes A$ is a standard (bosonic) semi-direct product group with Lie algebra $\mathfrak{g} \Subset A$, while $\mathfrak{g} \Subset (A + \mathcal{A})$ is a super Lie algebra whose odd subalgebra $\mathcal{A}$ is a G-module such that the bracket between elements of $\mathfrak{g}$ and elements of $\mathcal{A}$ be the differential of the action of G on $\mathcal{A}$, and such that $[A, \mathcal{A}] = 0$ and $\{\mathcal{A}, \mathcal{A}\} \subseteq A$. By virtue of this definition, the action of G on $\mathcal{A}$ is compatible with the super Lie bracket:

$$\{g \cdot S, g \cdot T\} = \sigma_g \{S, T\} \qquad \forall\, S, T \in \mathcal{A}, \tag{11.101}$$

where σ is the action of G on A.

It was shown in [40, 41] that all irreducible, unitary representations of a super semi-direct product are induced in essentially the same sense as for standard, bosonic groups. In particular, they are classified by the orbits and little groups of $G \ltimes_\sigma A$, as explained in Sect. 4.1. However, there are two important differences with respect to the purely bosonic case:

1. Unitarity rules out all orbits on which energy can be negative, so that the momentum orbits giving rise to unitary representations of the supergroup form a subset of the full menu of orbits available in the purely bosonic case. More precisely, given a momentum $p \in A^*$, it must be such that

$$\langle p, \{S, S\}\rangle \geq 0 \qquad \forall S \in \mathcal{A}. \tag{11.102}$$

When this condition is not satisfied, the representations of (11.100) associated with the orbit $\mathcal{O}_p$ are not unitary. The momenta satisfying condition (11.102) are said to be *admissible*. Note that admissibility is a G-invariant statement: if $f \in G$ and if p is admissible, then so is $f \cdot p$, by virtue of (11.101). For instance, the only admissible momenta for the super Poincaré group are those of massive or massless particles with positive energy (and the trivial momentum $p = 0$).

2. Given an admissible momentum p, the odd piece $\mathcal{A}$ of the supersymmetric translation algebra produces a (generally degenerate) Clifford algebra

$$\mathcal{C}_p = T(\mathcal{A})/\left\{S^2 - \langle p, \{S, S\}\rangle \mid S \in \mathcal{A}\right\}, \tag{11.103}$$

where $T(\mathcal{A})$ is the tensor algebra of $\mathcal{A}$. Quotienting this algebra by its ideal generated by the radical of $\mathcal{A}$, one obtains a non-degenerate Clifford algebra $\bar{\mathcal{C}}_p$. Since $\mathcal{A}$ is a G-module, there exists an action of the little group G_p on $\bar{\mathcal{C}}_p$; let us denote this action by $a \mapsto g \cdot a$ for $a \in \bar{\mathcal{C}}_p$ and $g \in G_p$. To obtain a representation of the full supergroup (11.100), one must find an irreducible representation τ of $\bar{\mathcal{C}}_p$ and a representation $\mathcal{R}_0$ of G_p acting in the same space, and compatible with τ in the sense that

$$\tau[g \cdot a] = \mathcal{R}_0[g] \cdot \tau[a] \cdot (\mathcal{R}_0[g])^{-1}. \tag{11.104}$$

For finite-dimensional groups, the pair $(\tau, \mathcal{R}_0)$ turns out to be unique up to multiplication of $\mathcal{R}_0$ by a character of G_p (and possibly up to parity-reversal). Given such a pair, we call it the *fundamental representation* of the supersymmetric little group.

The Clifford algebra (11.103) leads to a replacement of the irreducible, "spin" representations of the little group, by generally *reducible* representations $\mathcal{R}_0 \otimes \mathcal{R}$. This is the multiplet structure of supersymmetry: the restriction of an irreducible unitary representation of a supergroup to its bosonic subgroup is generally reducible, and the various irreducible components account for the combination of spins that gives rise to a SUSY multiplet. In the Poincaré group, an irreducible supermultiplet contains finitely many spins; by contrast, we will see below that super-BMS$_3$ multiplets contain infinitely many spins. Apart from this difference, the structure of induced representations of super semi-direct products is essentially the same as in the bosonic case: they consist of wavefunctions on an orbit, taking their values in the space of the representation $\mathcal{R}_0 \otimes \mathcal{R}$. In particular, the Frobenius formula (10.54) for characters remains valid, up to the replacement of $\mathcal{R}$ by $\mathcal{R}_0 \otimes \mathcal{R}$.

Supersymmetric Virasoro Algebra

As a preparation for super BMS$_3$, let us first recall the definition of the super Virasoro algebra. The latter is built by adding to Vect(S^1) an odd subalgebra $\mathcal{F}_{-1/2}(S^1)$ of $-1/2$-densities on the circle [42, 43]. This produces a Lie superalgebra, isomorphic to Vect$(S^1) \oplus \mathcal{F}_{-1/2}(S^1)$ as a vector space, which we shall write as $\mathfrak{s}$Vect(S^1). Its elements are pairs (X, S), where $X = X(\varphi)\partial/\partial\varphi$ and $S = S(\varphi)(d\varphi)^{-1/2}$, and the super Lie bracket is defined as

$$\big[(X, S), (Y, T)\big\} \equiv \Big([X, Y] + S \otimes T, X \cdot T - Y \cdot S\Big). \tag{11.105}$$

Here $[X, Y]$ is the standard Lie bracket of vector fields and the dot denotes the natural action of vector fields on $\mathcal{F}_{-1/2}(S^1)$, so that $X \cdot T$ is the $-1/2$-density with component

$$X \cdot T \equiv XT' - \frac{1}{2}X'T. \tag{11.106}$$

(This is formula (6.32) with $h = -1/2$.) Upon expanding the functions $X(\varphi)$ and $S(\varphi)$ in Fourier modes, one recovers the standard $\mathcal{N} = 1$ supersymmetric extension of the Witt algebra. Choosing $S(\varphi)$ to be periodic or antiperiodic leads to the Ramond or the Neveu–Schwarz sector of the superalgebra, respectively.

The central extension of $\mathfrak{s}\mathrm{Vect}(S^1)$ is the super Virasoro algebra, $\mathfrak{svir}$. Its elements are triples (X, S, λ) where $(X, S) \in \mathfrak{s}\mathrm{Vect}(S^1)$ and $\lambda \in \mathbb{R}$, with a super Lie bracket

$$\big[(X, S, \lambda), (Y, T, \mu)\big\} \equiv \Big([X, Y] + S \otimes T, X \cdot T - Y \cdot S, \mathsf{c}(X, Y) + \mathsf{h}(S, T)\Big), \tag{11.107}$$

where c is the Gelfand–Fuks cocycle (6.43) while h is its supersymmetric cousin,

$$\mathsf{h}(S, T) \equiv \frac{1}{12\pi}\int_0^{2\pi} d\varphi\, S'T'\,. \tag{11.108}$$

By expanding the functions X and S in Fourier modes, one obtains the usual commutation relations of $\mathcal{N} = 1$ super Virasoro. Explicitly, defining the generators

$$\mathcal{L}_m \equiv \big(e^{im\varphi}\partial_\varphi, 0, 0\big), \qquad \mathcal{Q}_r \equiv \big(0, e^{ir\varphi}(d\varphi)^{-1/2}, 0\big), \qquad \mathcal{Z} \equiv (0, 0, 1),$$

one finds that (11.107) yields the super Lie brackets

$$\begin{aligned} i[\mathcal{L}_m, \mathcal{L}_n\} &= (m - n)\mathcal{L}_{m+n} + \frac{\mathcal{Z}}{12}m^3\delta_{m+n,0}\,, \\ i[\mathcal{L}_m, \mathcal{Q}_r\} &= \left(\frac{m}{2} - r\right)\mathcal{Q}_{m+r}\,, \\ [\mathcal{Q}_r, \mathcal{Q}_s\} &= \mathcal{L}_{r+s} + \frac{\mathcal{Z}}{6}r^2\delta_{r+s,0}\,. \end{aligned} \tag{11.109}$$

The super Virasoro algebra is (half of) the asymptotic symmetry algebra of three-dimensional supergravity with Brown–Henneaux boundary conditions [44, 45] (see also [46]). In that context the vector field X is one of the components of an asymptotic Killing vector field (8.30), while S is one of the components of an asymptotic Killing spinor. The fact that the quantization of three-dimensional supergravity produces super Virasoro representations was verified in [6] by showing that the one-loop partition function of supergravity on thermal AdS_3 coincides with the vacuum character of two super Virasoro algebras. In the remainder of this section our goal is to describe the flat analogue of these results.

Supersymmetric BMS_3 Groups

Equipped with the definition of super semi-direct products and that of the super Virasoro algebra, we can now define the $\mathcal{N} = 1$ super BMS_3 group [34, 35]: it is a super semi-direct product (11.100) whose even piece is the BMS_3 group (9.58), and whose odd subspace is the space of densities $\mathcal{F}_{-1/2}(S^1)$ with the bracket $\{S, T\} = S \otimes T$. In other words, the (centreless) super $\mathfrak{bms}_3$ algebra is a super semi-direct sum

$$\mathfrak{sbms}_3 = \text{Vect}(S^1) \in \left(\text{Vect}(S^1)_{\text{Ab}} \oplus \mathcal{F}_{-1/2}\right), \tag{11.110}$$

where $\text{Vect}(S^1)_{\text{Ab}} \oplus \mathcal{F}_{-1/2}$ may be seen as an Abelian version of $\mathfrak{s}\text{Vect}(S^1)$. Again, choosing periodic/antiperiodic boundary conditions for $\mathcal{F}_{-1/2}$ yields the Ramond/Neveu–Schwarz sector of the theory (respectively). Central extensions can be included as in (9.62) and lead to a supersymmetric version of the centrally extended algebra (9.64). The elements of the resulting super Lie algebra $\widehat{\mathfrak{sbms}}_3$ are 5-tuples $(X, \lambda; \alpha, S, \mu)$, where (X, α, S) belongs to $\mathfrak{sbms}_3$ and λ, μ are real numbers, with a super Lie bracket that extends (9.46):

$$\begin{aligned}&\Big[(X, \lambda; \alpha, S, \mu), (Y, \kappa; \beta, T, \nu)\Big\} = \\ &= \Big([X, Y], \mathsf{c}(X, Y); [X, \beta] - [Y, \alpha], X \cdot T - Y \cdot S; \mathsf{c}(X, \beta) - \mathsf{c}(Y, \alpha) + \mathsf{h}(S, T)\Big).\end{aligned} \tag{11.111}$$

Here c is again the Gelfand–Fuks cocycle (6.43) while h is given by (11.108). Upon introducing generators analogous to (9.67), the central charges (9.47) and $\mathcal{Q}_r \equiv (0, 0; 0, e^{ir\varphi}(d\varphi)^{-1/2}, 0)$, one finds the brackets (9.68) supplemented with

$$i[\mathcal{J}_m, \mathcal{Q}_r\} = \left(\frac{m}{2} - r\right)\mathcal{Q}_{m+r}\,, \tag{11.112a}$$

$$i[\mathcal{P}_m, \mathcal{Q}_r\} = 0\,, \tag{11.112b}$$

$$[\mathcal{Q}_r, \mathcal{Q}_s\} = \mathcal{P}_{r+s} + \frac{\mathcal{Z}_2}{6}\, r^2 \delta_{r+s,0}\,. \tag{11.112c}$$

The indices r, s are integers/half-integers in the Ramond/Neveu–Schwarz sector, respectively. Note that the centrally extended bracket of supercharges only involves the central charge $\mathcal{Z}_2$ that pairs superrotations with supertranslations.

In the gravitational context, the functions X and α generate superrotations and supertranslations, while $S(\varphi)$ generates local supersymmetry transformations that become global symmetries upon enforcing suitable boundary conditions on the fields. The surface charge associated with (X, α, S) then takes the form [34]

$$\mathcal{Q}_{(X,\alpha,S)}[j,p,\psi]=\frac{1}{2\pi}\int_0^{2\pi}d\varphi\Big[X(\varphi)j(\varphi)+\alpha(\varphi)p(\varphi)+S(\varphi)\psi(\varphi)\Big], \quad (11.113)$$

where j and p are the angular momentum and Bondi mass aspects of (9.25), while $\psi(\varphi)$ is one of the subleading components of the gravitino at null infinity. The triple (j,p,ψ) is a coadjoint vector for the (centrally extended) super BMS$_3$ group. In particular (j,p) are quadratic densities, while $\psi(\varphi)$ has weight 3/2 on the circle. Upon using formula (11.41), the charges (11.113) satisfy the algebra (11.111) with definite values $\mathcal{Z}_1=0$, $\mathcal{Z}_2=c_2=3/G$ for the central charges. Note that the gravitino naturally satisfies Neveu–Schwarz boundary conditions on the celestial circle, as it represents Lorentz transformations up to a sign.

The construction of the super BMS$_3$ group can be generalized in a straightforward way. Indeed, let G be a (bosonic) group, $\mathfrak{g}$ its Lie algebra, $\mathfrak{sg}$ a super Lie algebra whose even subalgebra is $\mathfrak{g}$. Then one can associate with G a (bosonic) exceptional semi-direct product $G\ltimes\mathfrak{g}$ — the even $\widehat{\text{BMS}}_3$ group (9.62) is of that form, with G the Virasoro group. Now let $\mathfrak{sg}_{\text{Ab}}$ denote the "Abelian" super Lie algebra which is isomorphic to $\mathfrak{sg}$ as a vector space, but where all brackets involving elements of $\mathfrak{g}$ are set to zero. One may then define a super semi-direct product

$$\big(G\ltimes\mathfrak{g},\mathfrak{g}\Subset\mathfrak{sg}_{\text{Ab}}\big) \quad (11.114)$$

where we use the notation (11.100). This structure appears to be ubiquitous in three-dimensional, asymptotically flat supersymmetric higher-spin theories.

11.4.3 Supersymmetric BMS$_3$ Particles

Unitary representations of the super BMS$_3$ group can be classified along the lines explained in Chap. 10. In the remainder of this section we describe this classification and use it to evaluate characters of the centrally extended super BMS$_3$ group. We conclude with the observation that these characters reproduce one-loop partition functions of three-dimensional asymptotically flat supergravity and hypergravity.

Admissible super BMS$_3$ Orbits

The unitary representations of super BMS$_3$ are classified by the same supermomentum orbits as in the purely bosonic case, i.e. coadjoint orbits of the Virasoro group. However, supermomenta that do not satisfy condition (11.102) are forbidden, so our first task is to understand which orbits are admissible. To begin, recall that the admissibility condition (11.102) is invariant under superrotations. Thus, if we consider a supermomentum orbit containing a constant p_0 say, the supermomenta on the orbit will be admissible if and only if p_0 is. Including the central charge c_2, we ask: which pairs (p_0,c_2) are such that

$$\langle(p_0,c_2),\{S,S\}\rangle\geq 0 \qquad \text{for any } S\in\mathcal{F}_{-1/2}(S^1)\,? \quad (11.115)$$

Here $\langle\cdot,\cdot\rangle$ is the pairing (6.111) of centrally extended supermomenta with centrally extended supertranslations. Using the super Lie bracket (11.111), we find

$$\langle(p_0, c_2), \{S, S\}\rangle = \frac{1}{2\pi}\int_0^{2\pi} d\varphi \left(p_0(S(\varphi))^2 + \frac{c_2}{6}(S'(\varphi))^2\right). \tag{11.116}$$

Since the term involving $(S')^2$ can be made arbitrarily large while keeping S^2 arbitrarily small, a necessary condition for (p_0, c_2) to be admissible is that c_2 be non-negative. Already note that this condition did not arise in the bosonic BMS_3 group. The admissibility condition on p_0, on the other hand, depends on the sector under consideration:

- In the Ramond sector, $S(\varphi)$ is a periodic function on the circle. In particular, $S(\varphi) = \text{const.}$ is part of the supersymmetry algebra, so for expression (11.116) to be non-negative for any S, we must impose $p_0 \geq 0$.
- In the Neveu–Schwarz sector, $S(\varphi)$ is antiperiodic (i.e. $S(\varphi+2\pi) = -S(\varphi)$) and can be expanded in Fourier modes as

$$S(\varphi) = \sum_{n\in\mathbb{Z}} s_{n+1/2}\, e^{i(n+1/2)\varphi}. \tag{11.117}$$

  Then expression (11.116) becomes

$$\langle(p_0, c_2), \{S, S\}\rangle = \sum_{n\in\mathbb{Z}}\left[p_0 + \frac{c_2}{6}(n+1/2)^2\right]|s_{n+1/2}|^2, \tag{11.118}$$

  and the admissibility condition amounts to requiring all coefficients in this series to be non-negative, which gives

$$p_0 \geq -\frac{c_2}{24}. \tag{11.119}$$

These bounds are consistent with earlier observations in three-dimensional supergravity [34], according to which Minkowski space-time (corresponding to $p_0 = -c_2/24$) realizes the Neveu–Schwarz vacuum, while the Ramond vacuum is realized by the null orbifold (corresponding to $p_0 = 0$). Analogous results hold in AdS_3 [46]. More general admissibility conditions can presumably be worked out for *non-constant* supermomenta by adapting the proof of the positive energy theorem of Sect. 7.3, but we will not address this question here.

Super BMS_3 Multiplets

As explained around (11.103), any unitary representation of super BMS_3 based on a supermomentum orbit $\mathcal{O}_p$ comes equipped with a representation τ of the Clifford algebra

$$\mathcal{C}_p = T\left(\mathcal{F}_{-1/2}(S^1)\right)/\left\{S^2 - \langle(p, c_2), \{S, S\}\rangle\right\}. \tag{11.120}$$

Let us build such a representation. For definiteness we work in the Neveu–Schwarz sector and take p to be a constant admissible supermomentum $p_0 = M - c_2/24$ with $M > 0$, whose little group is U(1). Then the bilinear form (11.118) is non-degenerate and the representation τ of the Clifford algebra (11.120) must be such that

$$\tau[\mathcal{Q}_r]\cdot\tau[\mathcal{Q}_s] + \tau[\mathcal{Q}_s]\cdot\tau[\mathcal{Q}_r] = \left(\frac{c_2}{6}(r^2 - 1/4) + M\right)\delta_{r+s,0}\,, \quad r, s \in \mathbb{Z} + 1/2. \tag{11.121}$$

In order to make τ irreducible, we start with a highest-weight state $|0\rangle$ such that $\tau[\mathcal{Q}_r]|0\rangle = 0$ for $r > 0$, and generate the space of the representation by its "descendants" $\tau[\mathcal{Q}_{-r_1}]...\tau[\mathcal{Q}_{-r_n}]|0\rangle$, with $0 < r_1 < \cdots < r_n$. It follows from the Lie brackets (11.112) that each descendant state has spin $s + \sum_{i=1}^{n} r_i$, where s is the spin of the state $|0\rangle$; this observation uniquely determines the little group representation $\mathcal{R}_0$ satisfying (11.104). Thus, a super BMS$_3$ particle consists of infinitely many particles with spins increasing from s to infinity.

A similar construction can be carried out for the vacuum supermomentum at $M = 0$, with the subtlety that the Clifford algebra (11.120) (or equivalently (11.121)) is degenerate. As explained below (11.103), one needs to quotient (11.120) by the radical of the bilinear form (11.118), resulting in a non-degenerate Clifford algebra $\bar{\mathcal{C}}_p$. In the case at hand this algebra is generated by supercharges $\mathcal{Q}_r$ with $|r| > 1$, and the representation τ must satisfy (11.121) with $M = 0$ and $|r|, |s| > 1$. The remainder of the construction is straightforward: starting from a state $|0\rangle$ with, say, vanishing spin, one generates the space of the representation by acting on it with $\tau[\mathcal{Q}_{-r}]$'s, where $r > 1$. The vacuum representation of super BMS$_3$ thus contains infinitely many "spinning vacua" with increasing spins.

Characters

The Fock space representations just described can be used to evaluate characters. For example, in the massive case with spin s one finds the supersymmetric little group character

$$\operatorname{tr}\left[e^{i\theta J_0}\right] = e^{is\theta}\left[1 + e^{i\theta/2} + e^{3i\theta/2} + e^{2i\theta} + \cdots\right] = e^{is\theta}\prod_{n=1}^{+\infty}\left(1 + e^{i(n-1/2)(\theta+i\epsilon)}\right), \tag{11.122}$$

where we have added a small imaginary part to θ to ensure convergence of the product; the trace is taken in the fermionic Fock space associated with the "highest-weight state" $|0\rangle$. The vacuum case is similar, except that the product would start at $n = 2$ rather than $n = 1$ (and $s = 0$). Note that (11.122) explicitly breaks parity invariance; this can be fixed by replacing the parity-breaking Fock space representations τ described above by parity-invariant tensor products $\tau \otimes \bar{\tau}$, where $\bar{\tau}$ is the same as τ with the replacement of $\mathcal{Q}_r$ by $\mathcal{Q}_{-r}$. The trace of a rotation operator in the space of $\tau \otimes \bar{\tau}$ then involves the norm squared of the product appearing in (11.122).

As explained at the beginning of Sect. 11.4.2, the character of an induced representation of a super semi-direct product takes the same form (4.33) as in the bosonic case, but with the character of $\mathcal{R}$ replaced by that of a (reducible) representation $\mathcal{R}_0 \otimes \mathcal{R}$ compatible with the Clifford algebra representation τ. We thus find that the character of a rotation by θ (together with a Euclidean time translation by β), in the parity-invariant vacuum representation of the $\mathcal{N} = 1$, Neveu–Schwarz super BMS$_3$ group, reads

$$\chi_{\text{vac}}^{\text{super BMS}}[(\text{rot}_\theta, i\beta)] = \chi_{\text{vac}}^{\text{BMS}}[(\text{rot}_\theta, i\beta)] \cdot \prod_{n=2}^{+\infty} |1 + e^{i(n-1/2)(\theta+i\epsilon)}|^2$$
$$= e^{\beta c_2/24} \prod_{n=2}^{+\infty} \frac{|1 + e^{i(n-1/2)(\theta+i\epsilon)}|^2}{|1 - e^{in(\theta+i\epsilon)}|^2}\,. \qquad (11.123)$$

Comparing with (11.37) and (11.99), we recognize the product of the (suitably regularized) partition functions of two massless fields with spins 2 and 3/2, that is, the one-loop partition function of $\mathcal{N} = 1$ supergravity in three-dimensional flat space.

Higher-Spin Supersymmetry and Hypergravity

In [36, 37], the authors considered a three-dimensional hypergravity theory consisting of a metric coupled to a single field with half-integer spin $s + 1/2$, with s larger than one. Upon imposing suitable asymptotically flat boundary conditions, they found that the asymptotic symmetry algebra spans a superalgebra that extends the bosonic $\mathfrak{bms}_3$ algebra by generators $\mathcal{Q}_r$ of spin $s + 1/2$. The one-loop partition function of that system is the product of the graviton partition function (see Eq. (11.37) for $s = 2$) with the fermionic partition function (11.99). We now show that this partition function coincides with the vacuum character of the corresponding asymptotic symmetry group (in the Neveu–Schwarz sector).

The irreducible, unitary representations of the asymptotic symmetry group of [37] are classified by the same orbits and little groups as for the standard BMS$_3$ group. In particular, we can consider the orbit of a constant supermomentum $p_0 = M - c_2/24$; the associated Clifford algebra representation τ mentioned below (11.103) then satisfies a natural generalization of Eq. (11.121) (see Eq. (7.23) in [37]):

$$\tau[\mathcal{Q}_r]\tau[\mathcal{Q}_\ell] + \tau[\mathcal{Q}_\ell]\tau[\mathcal{Q}_r] = \prod_{j=0}^{s-1} \left(\frac{c_2}{6} \left(r^2 - \frac{(2j+1)^2}{4} \right) + M \right) \delta_{r+\ell,0}\,, \quad (11.124)$$

where r and ℓ are integers or half-integers, depending on the sector under consideration (Ramond or Neveu–Schwarz, respectively). In order for the orbit to be admissible in the sense of (11.102), the value of M must be chosen so as to ensure that all coefficients on the right-hand side of (11.124) are non-negative. In particular, the vacuum value $M = 0$ is admissible in the Neveu–Schwarz sector, in which case the anticommutators $\{\tau[\mathcal{Q}_r], \tau[\mathcal{Q}_{-r}]\}$ vanish for $|r| = 1/2, \ldots, s - 1/2$. Thus, in the Neveu–Schwarz vacuum, the Clifford algebra (11.124) degenerates and τ must really be seen as a representation of the non-degenerate subalgebra generated by the $\mathcal{Q}_r$'s with $|r| \geq s$. The corresponding Fock space representation can be built as explained below (11.121), and the spins of the basis states in this representation are uniquely determined by the fact that the $\mathcal{Q}_r$'s have spin $s + 1/2$. The corresponding Fock space character is thus

$$\text{tr}\left[e^{i\theta J_0}\right] = \prod_{n=s+1}^{+\infty} \left(1 + e^{i(n-1/2)(\theta+i\epsilon)}\right), \tag{11.125}$$

which generalizes (11.122). The character for $\tau \otimes \bar{\tau}$ is the squared norm of this expression, and the resulting vacuum character of the hypersymmetric BMS$_3$ group is

$$\chi_{\text{vac}}^{\text{hyper BMS}}[(\text{rot}_\theta, i\beta)] = e^{\beta c_2/24} \frac{\prod_{n=s}^{+\infty} |1 + e^{i(n+1/2)(\theta+i\epsilon)}|^2}{\prod_{m=2}^{+\infty} |1 - e^{im(\theta+i\epsilon)}|^2}. \tag{11.126}$$

As announced earlier, this coincides with the (suitably regularized) one-loop partition function of asymptotically flat gravity coupled to a massless field with spin $s + 1/2$. We have thus completed our overview of the relation between BMS$_3$ characters and one-loop partition functions in three dimensions.

Appendix A: From Mixed Traces to Bosonic Characters*

This section and the next one are technical appendices that describe various computations concerned with characters of highest-weight representations of SO(n). These considerations are useful for Sects. 11.1.2 and 11.4.1. Other than that, they may be skipped on a first reading.

Mixed Traces and Symmetric Polynomials

In this part of the appendix we prove that the mixed trace (11.19) of $\mathbb{I}_{\mu_s,\alpha_s}$ in D dimensions coincides with a certain difference of complete homogeneous symmetric polynomials in the traces of J^n as given by

$$\chi_s[n\vec{\theta}\,] = h_s(J^n) - h_{s-2}(J^n), \tag{11.127}$$

where

$$h_s(J^n) = \sum_{\substack{m_1,\ldots,m_s \in \mathbb{N} \\ m_1+2m_2+\cdots+sm_s=s}} \left[\prod_{k=1}^{s} \frac{\left(\text{Tr}[(J^n)^k]\right)^{m_k}}{m_k! k^{m_k}}\right]. \tag{11.128}$$

By definition, the *complete homogeneous symmetric polynomial* of degree s in D complex variables $\lambda_1, \ldots, \lambda_D$ is

$$h_s(\lambda_1, \ldots, \lambda_D) = \sum_{\substack{\ell_1,\ldots,\ell_D=0 \\ \ell_1+\cdots+\ell_D=s}}^{s} \lambda_1^{\ell_1}\lambda_2^{\ell_2}\ldots\lambda_D^{\ell_D} = \sum_{1\le\ell_1\le\ell_2\le\ldots\le\ell_s\le D} \lambda_{\ell_1}\lambda_{\ell_2}\ldots\lambda_{\ell_s}. \tag{11.129}$$

Using the variant of Newton's identities

$$h_s(\lambda_1, \ldots, \lambda_D) = \frac{1}{s}\sum_{N=1}^{s} h_{s-N}(\lambda_1, \ldots, \lambda_D)(\lambda_1^N + \cdots + \lambda_D^N)\,, \tag{11.130}$$

one can show by recursion (see e.g. [47, p. 24f]) that the polynomial (11.129) can equivalently be written as in (11.128):

$$h_s(\lambda_1, \ldots, \lambda_D) = \sum_{\substack{m_1,\ldots,m_s\in\mathbb{N} \\ m_1+2m_2+\cdots+sm_s=s}} \prod_{k=1}^{s} \frac{(\lambda_1^k + \cdots + \lambda_D^k)^{m_k}}{m_k!\,k^{m_k}}\,. \tag{11.131}$$

We shall use this relation later. To prove (11.127), we start with the following:

Lemma Let J be a complex $D \times D$ matrix with eigenvalues $\lambda_1, \ldots, \lambda_D$. Then,

$$(\delta^{\mu\alpha})^s \frac{1}{s!}\left(J_{\mu\alpha}\right)^s = h_s(\lambda_1, \lambda_2, \ldots, \lambda_D)\,, \tag{11.132}$$

where we use the same notation for contracting symmetrized indices as in (11.13).

Proof The left-hand side of (11.132) can be seen as a trace over symmetric tensor powers of J. Indeed, $\delta^{\mu\alpha} J_{\mu\alpha} = \mathrm{Tr}(J)$ is clear; as for $\frac{1}{2}(\delta^{\mu\alpha})^2\left(J_{\mu\alpha}\right)^2$, one gets

$$\frac{1}{2}(\delta^{\mu\alpha})^2\left(J_{\mu\alpha}\right)^2 = \frac{1}{2}\Big(\mathrm{Tr}(J)^2 + \mathrm{Tr}\Big(J^2\Big)\Big) = \mathrm{Tr}\Big(S^2(J)\Big) = \frac{1}{2}\sum_{i=1}^{2}\mathrm{Tr}\Big(J^i\Big)\,\mathrm{Tr}\Big(S^{2-i}(J)\Big)\,, \tag{11.133}$$

where $S^k(J)$ is the k^{th} symmetric tensor power of J. One then defines recursively

$$\frac{1}{s!}(\delta^{\mu\alpha})^s\left(J_{\mu\alpha}\right)^s = \mathrm{Tr}\left(S^s(J)\right) = \frac{1}{s}\sum_{i=1}^{s}\mathrm{Tr}\left(J^i\right)\mathrm{Tr}\left(S^{s-i}(J)\right)\,, \tag{11.134}$$

so that $\frac{1}{s!}(\delta^{\mu\alpha})^s\left(J_{\mu\alpha}\right)^s$ is just a trace in the s^{th} symmetric tensor power of the D-dimensional vector space V on which $J_{\mu\alpha}$ acts as a linear operator. Now consider an eigenbasis $\{e_1, \ldots, e_D\}$ for $J_{\mu\alpha}$, with $J \cdot e_k = \lambda_k e_k$. Since $\frac{1}{s!}\left(J_{\mu\alpha}\right)^s$ is the s^{th} symmetric tensor power of $J_{\mu\alpha}$ one can construct an eigenbasis for $\frac{1}{s!}\left(J_{\mu\alpha}\right)^s$ by symmetrizing $e_{k_1} \otimes e_{k_2} \otimes \ldots \otimes e_{k_D}$, with $k_1 \le k_2 \le \cdots \le k_D$. These eigenvectors have eigenvalues $\lambda_{l_1}\lambda_{l_2}\ldots\lambda_{l_D}$, and since $(\delta^{\mu\alpha})^s \frac{1}{s!}\left(J_{\mu\alpha}\right)^s$ is the trace of $\frac{1}{s!}\left(J_{\mu\alpha}\right)^s$, relation (11.132) follows upon using the second expression of $h_s(\lambda_1, \ldots, \lambda_D)$ in (11.129). ∎

We can now turn to the proof of (11.127). To this end we fix conventionally the number of terms entering the contraction of two symmetrized expressions as follows. Objects with *lower* indices are symmetrized with the minimum number of terms required and without overall normalization factor, while objects with *upper* indices are not symmetrized at all, since the symmetrization is induced by the contraction. This specification is needed because terms with lower and upper indices in a contraction may have a different index structure and therefore the number of terms needed for their symmetrization may be different. For instance

$$A^{\mu}B^{\mu}C^{\mu}D_{\mu\mu}E_{\mu} \equiv A^{\mu}B^{\nu}C^{\rho}\left(D_{\mu\nu}E_{\rho}+D_{\nu\rho}E_{\mu}+D_{\rho\mu}E_{\nu}\right)$$
$$=\frac{1}{2}\left(A^{\mu}B^{\nu}C^{\rho}+A^{\nu}B^{\rho}C^{\mu}+A^{\rho}B^{\mu}C^{\nu}+A^{\mu}B^{\rho}C^{\nu}+A^{\rho}B^{\nu}C^{\mu}+A^{\nu}B^{\mu}C^{\rho}\right)D_{\mu\nu}E_{\rho}\,. \tag{11.135}$$

In order to simplify computations, we define

$$T_{\mu_s,\,\alpha_s} \equiv J_{\mu\alpha}\dots J_{\mu\alpha}\,, \qquad T^{[s]} \equiv T_{\mu_s,\,\alpha_s}\,(\delta^{\mu\alpha})^s\,, \tag{11.136}$$

which implies the contraction rules

$$\delta^{\mu\mu}T_{\mu_s,\,\alpha_s} = 2\,\delta_{\alpha\alpha}T_{\mu_{s-2},\,\alpha_{s-2}}\,, \quad \delta^{\alpha\alpha}T_{\mu_s,\alpha_s} = 2\,\delta_{\mu\mu}T_{\mu_{s-2},\alpha_{s-2}}\,. \tag{11.137}$$

In terms of the tensors T_{μ_s,α_s}, the mixed trace (11.19) can be written as

$$\chi_s[n\vec{\theta}] = \frac{1}{s!}\,T_{\mu_s,\beta_s}\Bigg[\left(\delta^{\mu\beta}\right)^s+\sum_{m=1}^{\lfloor \frac{s}{2}\rfloor}\frac{(-1)^m\,s!\,[D+2\,(s-m-2)]!!}{2^m m!\,(s-2m)!\,[D+2\,(s-2)]!!}\times \tag{11.138}$$
$$\times\,(\delta^{\mu\mu})^m\,(\delta^{\mu\beta})^{s-2m}\,(\delta^{\beta\beta})^m\Bigg]$$

$$\overset{(11.137)}{=}\frac{1}{s!}\,T^{[s]}+\sum_{m=1}^{[\frac{s}{2}]}\frac{(-1)^m\,[D+2\,(s-m-2)]!!}{2^{m-1}m!\,(s-2m)!\,[D+2\,(s-2)]!!}\times \tag{11.139}$$
$$\times\,(\delta^{\mu\mu})^m\,(\delta^{\mu\beta})^{s-2m}\,(\delta^{\beta\beta})^{m-1}\delta_{\mu\mu}T_{\mu_{s-2},\beta_{s-2}}\,. \tag{11.140}$$

To compute the trace of the $(\delta^{\mu\mu})^m\,(\delta^{\mu\beta})^{s-2m}\,(\delta^{\beta\beta})^{m-1}$ terms, we first change our symmetrization from $\delta_{\mu\mu}T_{\mu_{s-2},\beta_{s-2}}$ (which contains $\frac{s!}{2(s-2)!}$ terms) to the aforementioned product of δ's. In doing so one has to introduce a factor accounting for the number of terms in each structure as

$$\delta_{\mu\mu}T_{\mu_{s-2},\,\beta_{s-2}} \rightsquigarrow \frac{s!}{2(s-2)!}\ \text{terms}, \tag{11.141a}$$

$$(\delta^{\mu\mu})^{mu}\left(\delta^{\mu\beta}\right)^{s-2m}\left(\delta^{\beta\beta}\right)^{m-1} \rightsquigarrow \frac{s!}{2^m m!}\times\frac{(s-2)!}{2^{m-1}(m-1)!(s-2m)!}\ \text{terms}, \tag{11.141b}$$

which implies

$$\chi_s[n\vec{\theta}] = \frac{1}{s!}T^{[s]} + \sum_{m=1}^{\lfloor\frac{s}{2}\rfloor} \frac{(-1)^m\, 2^{m-1}(m-1)!\,[D+2\,(s-m-2)]!!}{[(s-2)!]^2\,[D+2\,(s-2)]!!}\times \\ \times\delta^m_{\mu\mu}\delta^{s-2m}_{\mu\beta}\delta^{m-1}_{\beta\beta}\delta^{\mu\mu}T^{\mu_{s-2},\beta_{s-2}}. \tag{11.142}$$

Taking into account the correct combinatorial factors one obtains

$$\delta^m_{\mu\mu}\delta^{s-2m}_{\mu\beta}\delta^{m-1}_{\beta\beta}\delta^{\mu\mu} = [D+2(s-m-1)]\,\delta^{m-1}_{\mu\mu}\delta^{s-2m}_{\mu\beta}\delta^{m-1}_{\beta\beta} + 2m\,\delta^m_{\mu\mu}\delta^{s-2m-2}_{\mu\beta}\delta^m_{\beta\beta}\,, \tag{11.143}$$

which then yields

$$\chi_s[n\vec{\theta}] = \frac{1}{s!}\,T^{[s]} + \left(\sum_{m=1}^{\lfloor\frac{s}{2}\rfloor} \frac{(-1)^m\, 2^{m-1}(m-1)!\,[D+2\,(s-m-1)]!!}{[D+2\,(s-2)]!!}\delta^{m-1}_{\mu\mu}\delta^{s-2m}_{\mu\beta}\delta^{m-1}_{\beta\beta} \right. \\ \left. + \sum_{m=1}^{\lfloor\frac{s}{2}\rfloor-1} \frac{(-1)^m\, 2^m m!\,[D+2\,(s-m-2)]!!}{[D+2\,(s-2)]!!}\delta^m_{\mu\mu}\delta^{s-2m-2}_{\mu\beta}\delta^m_{\beta\beta}\right)\frac{1}{[(s-2)!]^2}\,T^{\mu_{s-2},\beta_{s-2}}. \tag{11.144}$$

Shifting $m \to m+1$ in the upper sum one can see that both sums are identical apart from the overall sign and the lower extremum. Thus (11.144) boils down to

$$\chi_s[n\vec{\theta}] = \frac{1}{s!}\,T^{[s]} - \frac{1}{[(s-2)!]^2}\,\delta^{s-2}_{\mu\beta}T^{\mu_{s-2},\beta_{s-2}} = \frac{1}{s!}\,T^{[s]} - \frac{1}{(s-2)!}\,T^{[s-2]}\,. \tag{11.145}$$

Now using (11.136) and (11.132) one obtains

$$\chi_s[n\vec{\theta}] = \frac{1}{s!}\,T^{[s]} - \frac{1}{(s-2)!}\,T^{[s-2]} = h_s(\lambda_1,\lambda_2,\ldots,\lambda_D) - h_{s-2}(\lambda_1,\lambda_2,\ldots,\lambda_D)\,, \tag{11.146}$$

where $\lambda_1,\ldots,\lambda_D$ are the eigenvalues of J^n. (These eigenvalues are $e^{\pm in\theta_j}$ for $j = 1,\ldots,r$, and one or two unit eigenvalues depending on whether D is odd or even, respectively.) This leads to the desired result: since traces of powers of J^n can be written as

$$\mathrm{Tr}[(J^n)^k] = \lambda_1^k + \cdots + \lambda_D^k \tag{11.147}$$

in terms of the eigenvalues of J^n, the complete homogeneous symmetric polynomials expressed as (11.131) exactly coincide with the combination (11.128), and equation (11.146) coincides with (11.127).

Symmetric Polynomials and SO(D) Characters

In this part of the appendix we review the relation between complete homogeneous symmetric polynomials and characters of orthogonal groups. Most of the explicit proofs can be found in [48], Chap. 24, to which we refer for details on our arguments below. We study separately the cases of odd and even D and let $r \equiv \lfloor (D-1)/2 \rfloor$, with $\theta_1, \ldots, \theta_r$ the non-vanishing angles appearing in the rotations (11.8).

Odd D

We consider the Lie algebra $\mathfrak{so}(D) = \mathfrak{so}(2r+1)$, with rank r. Choosing a basis of $\mathbb{C}^{2r+1}$ such that the Lie algebra $\mathfrak{so}(2r+1)_{\mathbb{C}}$ can be written in terms of complex matrices, we may choose the Cartan subalgebra to be the subalgebra $\mathfrak{h}$ of $\mathfrak{so}(2r+1)_{\mathbb{C}}$ consisting of diagonal matrices. As a basis of $\mathfrak{h}$ we choose the matrices H_i whose entries all vanish, except the (i,i) and $(r+i, r+i)$ entries which are 1 and -1, respectively (with $i = 1, \ldots, r$). In our convention (11.7), the operator H_i generates rotations in the plane (x_i, y_i). Then, calling L_i the elements of the dual basis (such that $\langle L_i, H_j \rangle = \delta_{ij}$), a dominant weight is one of the form $\lambda = \lambda_1 L_1 + \cdots \lambda_r L_r \equiv (\lambda_1, \ldots, \lambda_r)$ with $\lambda_1 \geq \cdots \geq \lambda_r \geq 0$.

Let λ be a dominant weight for $\mathfrak{so}(2r+1)$. According to formula (24.28) in [48], the character of the irreducible representation of $\mathfrak{so}(2r+1)$ with highest weight λ is

$$\chi_\lambda^{(2r+1)}[q_1, \ldots, q_r] = \mathrm{Tr}_\lambda \left[q_1^{H_1} \cdots q_r^{H_r} \right] = \frac{\left| q_j^{\lambda_i + r - i + \frac{1}{2}} - q_j^{-(\lambda_i + r - i + \frac{1}{2})} \right|}{\left| q_j^{r - i + \frac{1}{2}} - q_j^{-(r - i + \frac{1}{2})} \right|}, \tag{11.148}$$

where $q_1, \cdots q_r$ are arbitrary complex numbers,[4] Tr_λ denotes a trace taken in the space of the representation, and $\left| A_{ij} \right|$ denotes the determinant of the matrix A with rows i and columns j. This expression is a corollary of the Weyl character formula. Using proposition A.60 and Corollary A.46 of [48], it can be rewritten as

$$\chi_\lambda^{(2r+1)}[q_1, \ldots, q_r] = \left| h_{\lambda_i - i + j} - h_{\lambda_i - i - j} \right|, \tag{11.149}$$

where $h_j = h_j \left(q_1, \ldots, q_n, q_1^{-1}, \ldots, q_n^{-1}, 1 \right)$ is a complete homogeneous symmetric polynomial of degree j in $2r+1$ variables. In particular, for a highest weight $\lambda_s = (s, 0, \ldots, 0)$ (where s is a non-negative integer), the matrix appearing on the right-hand side of (11.149) is upper triangular, with the entry at $i = j = 1$ given by $h_s - h_{s-2}$ and all other entries on the main diagonal equal to one. Accordingly, the determinant in (11.149) boils down to $h_s - h_{s-2}$ in that simple case. For the rotation (11.8) we may identify $q_j = e^{in\theta_j}$, and we conclude that

$$\chi_{\lambda_s}^{(2r+1)}[n\vec{\theta}] = \frac{\left| \sin\left[\left(\lambda_i + r - i + \frac{1}{2} \right) n\theta_j \right] \right|}{\left| \sin\left[\left(r - i + \frac{1}{2} \right) n\theta_j \right] \right|} = h_s(J^n) - h_{s-2}(J^n), \tag{11.150}$$

where $\lambda_i = s\, \delta_{i1}$. Thus for odd D the difference of symmetric polynomials in (11.127) is just a character of SO(D).

[4] Eventually these numbers will be exponentials of angular potentials, so they are fugacities associated with the rotation generators H_i.

Even D

We now turn to the Lie algebra $\mathfrak{so}(2r+2)$, with rank $r+1$. As in the odd case we choose a basis of $\mathbb{C}^{2r+2}$ such that we can write the Lie algebra $\mathfrak{so}(2r+2)$ in terms of complex matrices and the Cartan subalgebra is generated by $r+1$ diagonal matrices H_i whose entries all vanish, except $(H_i)_{ii} = 1$ and $(H_i)_{r+1+i,r+1+i} = -1$. We call L_i the elements of the dual basis, and with these conventions a weight $\lambda = \lambda_1 L_1 + \ldots + \lambda_{r+1} L_{r+1} \equiv (\lambda_1, \ldots, \lambda_{r+1})$ is dominant if $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r \geq |\lambda_{r+1}|$.

Let λ be a dominant weight for $\mathfrak{so}(2r+2)$. Then formula (24.40) in [48] gives the character of the associated highest-weight representation as

$$\begin{aligned}\chi_\lambda^{(2r+2)}[q_1, \ldots, q_{r+1}] &= \mathrm{Tr}_\lambda \left[q_1^{H_1} \cdots q_{r+1}^{H_{r+1}} \right] \\ &= \frac{\left| q_j^{\lambda_i + r+1-i} + q_j^{-(\lambda_i + r+1-i)} \right| + \left| q_j^{\lambda_i + r+1-i} - q_j^{-(\lambda_i + r+1-i)} \right|}{\left| q_j^{r+1-i} + q_j^{-(r+1-i)} \right|}, \end{aligned} \tag{11.151}$$

where we use the same notations as in (11.148), except that now $i, j = 1, \ldots, r+1$. Note that the second term in the numerator of this expression vanishes whenever $\lambda_{r+1} = 0$ (because the $(r+1)^{\text{th}}$ row of the matrix $q_j^{\lambda_i+r+1-i} - q_j^{-(\lambda_i+r+1-i)}$ vanishes). Since this is the case that we will be interested in, we may safely forget about that second term from now on. Alternatively, for the mixed traces (11.19) that we need, we may take $q_j = e^{in\theta_j}$ for $j = 1, \ldots, r$ and $q_{r+1} = 1$ without loss of generality, so that this second term vanishes again. Using proposition A.64 of [48], one can then rewrite (11.151) as

$$\chi_\lambda^{(2r+2)}[q_1, \ldots, q_r, 1] = \left| h_{\lambda_i - i + j} - h_{\lambda_i - i - j} \right|, \tag{11.152}$$

where $h_j = h_j \left(q_1, \ldots, q_r, 1, q_1^{-1}, \ldots, q_r^{-1}, 1 \right)$. Finally, using the same arguments as for odd D, one easily verifies that the determinant on the right-hand side of (11.152) reduces once more to $h_s - h_{s-2}$ for a highest weight $\lambda_s = (s, 0, \ldots, 0)$. Writing again $q_j = e^{in\theta_j}$, one concludes that, for even D,

$$\chi_{\lambda_s}^{(2r+2)} \left[n\theta_1, \ldots, n\theta_r, n\theta_{r+1} = 0 \right] = \left. \frac{\left| \cos\left[(\lambda_i + r + 1 - i)\, n\theta_j \right] \right|}{\left| \cos\left[(r+1-i)\, n\theta_j \right] \right|} \right|_{\theta_{r+1}=0} = h_s(J^n) - h_{s-2}(J^n), \tag{11.153}$$

where $\lambda_i = s\, \delta_{i1}$. This concludes the proof of (11.21). Note that, for *non-vanishing* θ_{r+1}, the quotient of denominators in the middle of (11.153) is actually the character $\chi_{\lambda_s}^{(2r+2)}(n\theta_1, \ldots, n\theta_r, n\theta_{r+1})$. This detail will be useful in Appendix section "Differences of SO(D) Characters".

Differences of SO(D) Characters

In this part of the appendix we prove the following relations between characters of orthogonal groups:

$$\chi^{(2r+1)}_{\lambda_s}[\vec{\theta}\,] - \chi^{(2r+1)}_{\lambda_{s-1}}[\vec{\theta}\,] = \chi^{(2r)}_{\lambda_s}[\vec{\theta}\,], \tag{11.154a}$$

$$\chi^{(2r)}_{\lambda_s}[\vec{\theta}\,] - \chi^{(2r)}_{\lambda_{s-1}}[\vec{\theta}\,] = \sum_{k=1}^{r} \mathcal{A}^r_k[\vec{\theta}\,]\chi^{(2r-1)}_{\lambda_s}[\theta_1, \dots, \widehat{\theta_k}, \dots, \theta_r]. \tag{11.154b}$$

Here $\vec{\theta} = (\theta_1, .., \theta_r)$, λ_s is the weight with components $(s, 0, \dots, 0)$ in the basis defined above Eqs. (11.148) and (11.151), and the hat denotes omission of an argument, while the coefficients $\mathcal{A}^r_k$ are the quotients of determinants defined in (11.31). Note that, when one of the angles $\theta_1, \dots, \theta_r$ vanishes, say $\theta_\ell = 0$, then $\mathcal{A}^r_k = \delta_{k\ell}$ and relation (11.154b) reduces to

$$\chi^{(2r)}_{\lambda_s}[\vec{\theta}\,]\Big|_{\theta_\ell=0} - \chi^{(2r)}_{\lambda_{s-1}}[\vec{\theta}\,]\Big|_{\theta_\ell=0} = \chi^{(2r-1)}_{\lambda_s}[\theta_1, \dots, \widehat{\theta_\ell}, \dots, \theta_r]. \tag{11.155}$$

Proof of (11.154) We start by defining the matrices

$$(A^r)_{ij} = \sin\left[(r - i + \tfrac{1}{2})\theta_j\right], \qquad (B^r)_{ij} = \cos\left[(r - i)\theta_j\right], \tag{11.156}$$

so that in particular

$$\mathcal{A}^r_k(\vec{\theta}) = \frac{|B^r|_{\theta_k=0}}{|B^r|}. \tag{11.157}$$

We shall also use the shorthand notation

$$M^r[\theta_k] \equiv |M_{ij}(\theta_1, \dots, \theta_{k-1}, \theta_{k+1}, \dots, \theta_{r+1})| \tag{11.158}$$

to denote the determinant of the $r \times r$ matrix missing the angle θ_k of any of the matrices defined in (11.156). As a preliminary step towards the proof, we list the four following identities:

$$\frac{|A^r|}{\prod_{j=1}^r \sin\left(\theta_j/2\right)} = 2^{r-1}|B^r|, \tag{11.159a}$$

$$|\cos\left[(r - i)\theta_j\right]| = 2^{\frac{(r-1)(r-2)}{2}} \prod_{1\le i<j\le r} (\cos(\theta_i) - \cos(\theta_j)), \tag{11.159b}$$

$$\frac{|B^r|_{\theta_k=0}}{A^{r-1}[\theta_k]} = 2^{r-1}(-1)^{k+1}\prod_{\substack{j=1\\ j\neq k}}^{r} \sin\left(\theta_j/2\right), \tag{11.159c}$$

$$|B^r| = \sum_{k=1}^{r} |B^r|_{\theta_k=0}\,. \tag{11.159d}$$

Here (11.159a) can be proven by induction on r upon expanding the determinant $|A^r(\vec{\theta})|$ along the first line of the matrix A^r. Property (11.159b) can be shown by observing that

$$\cos[(r-i)\theta_j] = 2^{r-i-1}\cos^{r-i}(\theta_j) + \sum_{k=1}^{r-i-1} c_k \cos(k\theta_j) \tag{11.160}$$

with some irrelevant real coefficients c_k, and that the contribution of the second term of this expression to the determinant $|\cos[(r-i)\theta_j]|$ vanishes by linear dependence. Equation (11.159c) then follows from (11.159a) and (11.159b), while property (11.159d) can again be proved by induction on r.

Thanks to Eq. (11.159), we can tackle the proof of (11.154). Equation (11.154a) is easy: using expression (11.150) for the character $\chi^{(2r+1)}_{\lambda_s}$, we can write the difference of characters on the left-hand side of (11.154a) as

$$\chi^{(2r+1)}_{\lambda_s} - \chi^{(2r+1)}_{\lambda_{s-1}} = \frac{\sum_{k=1}^{r}(-1)^{k+1} 2\cos[(s+r-1)\theta_k]\sin(\theta_k/2)\, A^{r-1}[\theta_k]}{|A^r|}. \tag{11.161}$$

Property (11.159a) then allows us to reduce this expression to the quotient of denominators appearing in the middle of Eq. (11.153) (with the replacement of $r+1$ by r and all angles non-zero), which is indeed the sought-for character $\chi^{(2r)}_{\lambda_s}[\vec{\theta}\,]$.

Equation (11.154b) requires more work. Using once more the expression in the middle of (11.153), we first rewrite the left-hand side of (11.154b) as

$$\chi^{(2r)}_{\lambda_s} - \chi^{(2r)}_{\lambda_{s-1}} = \frac{\sum_{k=1}^{r}(-1)^{k+1}(-2\sin[(s+r-\frac{3}{2})\theta_k]\sin(\theta_k/2)\, B^{r-1}[\theta_k]}{|B^r|}. \tag{11.162}$$

Let us now recover this expression as a combination of characters of SO$(2r-1)$: using formula (11.150) and the identities (11.159), one finds

$$\begin{aligned}
&\sum_{k=1}^{r}\chi^{(2r-1)}_{\lambda_s}[\theta_1,\ldots,\widehat{\theta_k},\ldots,\theta_r]|B^r|_{\theta_k=0}\\
\overset{(11.159c)}{=}\;&\sum_{k=1}^{r}(-1)^{k+1}2^{r-1}\prod_{\substack{j=1\\ j\neq k}}^{r}\sin\big(\theta_j/2\big)\,\times\\
&\times\Bigg[\sum_{j=1}^{k-1}(-1)^{j+1}\sin[(s+r-\tfrac{3}{2})\theta_j]A^{r-2}[\theta_j,\theta_k]\\
&+\sum_{j=k+1}^{r}(-1)^{j}\sin[(s+r-\tfrac{3}{2})\theta_j]A^{r-2}[\theta_j,\theta_k]\Bigg]\\
\overset{(11.159a)}{=}\;&\sum_{k=1}^{r}(-1)^{k+1}2^{2r-4}\sin[(s+r-\tfrac{3}{2})\theta_k]\sin(\theta_k/2)\,\times
\end{aligned}$$

$$\times \left[\sum_{j=1}^{k-1} (-1)^j B^{r-2}[\theta_j, \theta_k] \prod_{\substack{i=1 \\ i \notin \{j,k\}}}^{r} \sin^2(\theta_i/2) \right.$$

$$\left. + \sum_{j=k+1}^{r} (-1)^{j+1} B^{r-2}[\theta_j, \theta_k] \prod_{\substack{i=1 \\ i \notin \{j,k\}}}^{r} \sin^2(\theta_i/2) \right]$$

$$\stackrel{(11.159b)}{=} \sum_{k=1}^{r} (-1)^{k+1}(-2) \sin[(s+r-\tfrac{3}{2})\theta_k] \sin(\theta_k/2)$$

$$\left[\sum_{j=1}^{k-1} (-1)^j \; B^{r-1}[\theta_k]\Big|_{\theta_j=0} + \sum_{j=k+1}^{r} B^{r-1}[\theta_k]\Big|_{\theta_j=0} \right]$$

$$\stackrel{(11.159d)}{=} \sum_{k=1}^{r} (-1)^{k+1}(-2) \sin[(s+r-\tfrac{3}{2})\theta_k] \sin(\theta_k/2)\, B^{r-1}[\theta_k]. \tag{11.163}$$

This coincides with the numerator of the right-hand side of (11.162), so identity (11.154b) follows with $\mathcal{A}_k^r$ given by (11.157). ■

From SO(D) to SO$(D-1)$

In this appendix we prove relation (11.33) between characters of SO(D) and SO$(D-1)$:

Lemma One has the following relations:

$$\chi^{(2r+1)}_{\lambda_s}[\theta_1, \ldots, \theta_r] = \sum_{j=0}^{s} \chi^{(2r)}_{\lambda_j}(\theta_1, \ldots, \theta_r), \tag{11.164a}$$

$$\chi^{(2r)}_{\lambda_s}[\theta_1, \ldots, \theta_r] = \sum_{j=0}^{s} \sum_{k=1}^{r} \mathcal{A}_k^r(\vec{\theta}) \chi^{(2r-1)}_{\lambda_j}[\theta_1, \ldots, \widehat{\theta_k}, \ldots, \theta_r]. \tag{11.164b}$$

Here λ_j is the weight $(j, 0, \ldots, 0)$ as explained above (11.21) or below (11.149), and $\mathcal{A}_k^r(\vec{\theta})$ is the quotient (11.31) or (11.157). Since the proofs of these two identities are very similar, we will only display the proof of (11.164a).

Proof of (11.164a) Equation (11.164a) can be written as

$$\frac{\sum_{k=1}^{r} (-1)^{k+1} \sin[(s+r-\frac{1}{2})\theta_k] A^{r-1}[\theta_k]}{|A^r|} \stackrel{(11.159a)}{=} \sum_{j=0}^{s} \frac{\sum_{k=1}^{r} (-1)^{k+1} \cos[(j+r-1)\theta_k] B^{r-1}[\theta_k]}{|B^r|}, \tag{11.165}$$

where we used formulas (11.150) and (11.153) for the characters, as well as the definition (11.156) of A^r and B^r. One can then use identities (11.159a) and (11.159c) to match the right-hand side of this expression with the left-hand side, proving the desired identity. ■

Appendix B: From Mixed Traces to Fermionic Characters*

This appendix is the fermionic (half-integer spin) analogue of section "From Mixed Traces to Bosonic Characters". It may be skipped in a first reading.

Mixed Traces and Symmetric Polynomials

Our goal here is to prove the first equality of (11.95), following the same method as in Appendix section "Mixed Traces and Symmetric Polynomials" for the bosonic case. First, using the definition (11.91) of U and the contraction rules (11.137), one can write (11.94) as

$$\chi_s^{(F)}[n\vec{\theta}] = \left[\frac{1}{s!}T^{[s]} + \sum_{m=1}^{\lfloor \frac{s}{2} \rfloor} \frac{(-1)^m [D+2(s-m-1)]!!}{2^{m-1} m!(s-2m)![D+2(s-1)]!!} \times \right.$$
$$\left. \times\, (\delta^{\mu\mu})^m (\delta^{\mu\beta})^{s-2m} (\delta^{\beta\beta})^{m-1} \delta_{\mu\mu} T_{\mu_{s-2},\beta_{s-2}} \right] \mathrm{Tr}[U^n]$$
$$+ \sum_{m=0}^{\lfloor \frac{s-1}{2} \rfloor} \frac{(-1)^{m+1}[D+2(s-m-2)]!!}{2^m m!(s-2m-1)![D+2(s-1)]!!} \times$$
$$\times\, \mathrm{Tr}[T_{\mu_{s-1},\beta_{s-1}} \gamma_\mu \gamma^\mu (\delta^{\mu\mu})^m (\delta^{\mu\beta})^{s-2m-1} (\delta^{\beta\beta})^m U^n], \tag{11.166}$$

where $T^{[s]}$ is the notation (11.136). In the first term of this expression, we shift the symmetrization on the δ's i.e. we exchange upper and lower indices while taking into account the change in multiplicities of the terms involved; in all other terms, we compute one contraction with $\delta^{\beta\beta}$. Equation (11.166) then simplifies to

$$\chi_s^{(F)}[n\vec{\theta}] = \frac{1}{s!}T^{[s]}\mathrm{Tr}[U^n] - \frac{1}{[(s-1)!]^2[D+2(s-1)]}\mathrm{Tr}[T^{\mu_{s-1},\beta_{s-1}}\gamma_\mu\gamma^\mu\delta_{\mu\beta}^{s-1}U^n]$$
$$+ \sum_{m=1}^{\lfloor \frac{s}{2} \rfloor} \left[\frac{(-1)^m 2^{m-1}(m-1)![D+2(s-m-1)]!!}{[(s-2)!]^2[D+2(s-1)]!!} T^{\mu_{s-2},\beta_{s-2}} \delta^{\mu\mu} \delta_{\mu\mu}^m \delta_{\mu\beta}^{s-2m} \delta_{\beta\beta}^{m-1} \mathrm{Tr}[U^n] \right.$$
$$\left. + \frac{(-1)^{m+1} 2m-1(m-1)![D+2(s-m-2)]!!}{[(s-2)!]^2[D+2(s-1)]!!} \mathrm{Tr}[T^{\mu_{s-2},\beta_{s-2}} \delta^{\mu\mu} \delta_{\mu\mu}^m \delta_{\mu\beta}^{s-2m-1} \delta_{\beta\beta}^m \gamma_\mu \gamma_\beta U^n] \right]. \tag{11.167}$$

The γ traces and mixed traces can now be evaluated using

$$\gamma^\mu \gamma_\mu \delta_{\mu\beta}^{s-1} = [D+2(s-1)]\delta_{\mu\beta}^{s-1} - \gamma_\mu\gamma_\beta\delta_{\mu\beta}^{s-2}, \tag{11.168a}$$

$$\delta^{\mu\mu}\delta_{\mu\mu}^m \delta_{\mu\beta}^{s-2m} \delta_{\beta\beta}^{m-1} = [D+2(s-m-1)]\delta_{\mu\mu}^{m-1}\delta_{\mu\beta}^{s-2m}\delta_{\beta\beta}^{m-1} + 2m\,\delta_{\mu\mu}^m \delta_{\mu\beta}^{s-2m-2}\delta_{\beta\beta}^m, \tag{11.168b}$$

$$\delta^{\mu\mu}\delta_{\mu\mu}^m \delta_{\mu\beta}^{s-2m-1} \delta_{\beta\beta}^{m-1}\gamma_\mu\gamma_\beta = [D+2(s-m-1)]\delta_{\mu\mu}^{m-1}\delta_{\mu\beta}^{s-2m-1}\delta_{\beta\beta}^{m-1}\gamma_\mu\gamma_\beta + 4m\,\delta_{\mu\mu}^m\delta_{\mu\beta}^{s-2m-2}\delta_{\beta\beta}^m + 2m\,\delta_{\mu\mu}^m\delta_{\mu\beta}^{s-2m-3}\delta_{\beta\beta}^m\gamma_\mu\gamma_\beta, \tag{11.168c}$$

which yields

$$\begin{aligned}\chi_s^{(F)}[n\vec{\theta}] = &\left[\frac{1}{s!}T^{[s]} - \frac{1}{(s-1)!}T^{[s-1]}\right]\mathrm{Tr}[U^n]\\ &+ \frac{1}{[(s-1)!]^2[D+2(s-1)]}\mathrm{Tr}[T^{\mu_{s-1},\beta_{s-1}}\delta^{s-2}_{\mu\beta}\gamma_\mu\gamma_\beta U^n]\\ &- \frac{D+2(s-2)}{[(s-2)!]^2[D+2(s-1)]}T^{\mu_{s-2},\beta_{s-2}}\delta^{s-2}_{\mu\beta}\mathrm{Tr}[U^n]\\ &+ \frac{1}{[(s-2)!]^2[D+2(s-1)]}\mathrm{Tr}[T^{\mu_{s-2},\beta_{s-2}}\delta^{s-3}_{\mu\beta}\gamma_\mu\gamma_\beta U^n].\end{aligned} \tag{11.169}$$

Using (11.146) and the definition (11.91) of U, together with some careful counting, one verifies that this expression matches $\left[h_s(J^n) - h_{s-1}(J^n)\right]\mathrm{Tr}[U^n]$, which was to be proven.

Symmetric Polynomials and SO(D) Characters

In this part of the appendix we prove the second equality in (11.95), following essentially the same steps as in Appendix section "Mixed Traces and Symmetric Polynomials". We refer again to [48] for details, and we write the components of weights in the dual basis of the Cartan subalgebra described above (11.148) and (11.151). We will consider separately odd and even space-time dimensions.

Odd D

The character of a half-spin representation of $\mathfrak{so}(2r+1)$ with a dominant highest weight $\lambda = (\lambda_1 + \frac{1}{2}, \lambda_2 + \frac{1}{2}, \ldots, \lambda_r + \frac{1}{2})$ is [49, p.258f]

$$\chi_\lambda^{(2r+1)}[\theta_1, \ldots, \theta_r] = \frac{\left|\sin\left[(\lambda_i + r - i + 1)\,\theta_j\right]\right|}{\left|\sin\left[\left(r - i + \frac{1}{2}\right)\theta_j\right]\right|} = \left(\prod_{i=1}^r 2\cos\left(\frac{\theta_i}{2}\right)\right)\frac{\left|\sin\left[(\lambda_i + r - i + 1)\,\theta_j\right]\right|}{\left|\sin\left[(r - i + 1)\,\theta_j\right]\right|}. \tag{11.170}$$

Owing to expression (11.92) for the trace of U^n, the second equality in (11.95) is equivalent to

$$h_s(J) - h_{s-1}(J) = \frac{\left|\sin\left[(\lambda_i + r - i + 1)\,\theta_j\right]\right|}{\left|\sin\left[(r - i + 1)\,\theta_j\right]\right|} \tag{11.171}$$

for $\lambda_i = s\delta_{i1}$. To prove this, consider the difference of the bosonic character (11.150) and the right-hand side of (11.171):

$$\frac{\left|\sin[(\lambda_i + r - i + \frac{1}{2})\theta_j]\right|}{\left|\sin[(r - i + \frac{1}{2})\theta_j]\right|} - \frac{\left|\sin[(\lambda_i + r - i + 1)\theta_j]\right|}{\left|\sin[(r - i + 1)\theta_j]\right|}. \tag{11.172}$$

Introducing the notation

$$(\mathcal{A}^r)_{ij} = 2\sin\left[(r-i+1)\theta_j\right], \quad (\mathcal{B}^r)_{ij} = 2\cos\left[(r-i+\tfrac{1}{2})\theta_j\right] \qquad (11.173)$$

and in terms of (11.156), this difference can be written as

$$\frac{\sum_{k=1}^{r}(-1)^{k+1}\sin[(s+r-\frac{1}{2})\theta_k]A^{r-1}[\theta_k]}{|A^r|} - \frac{\sum_{k=1}^{r}(-1)^{k+1}2\sin[(s+r)\theta_k]\mathcal{A}^{r-1}[\theta_k]}{|\mathcal{A}^r|} \qquad (11.174)$$

upon expanding the determinants along the first row. Now it turns out that[5]

$$2^r\left|A^r\right|\prod_{i=1}^{r}2\cos\left(\theta_i/2\right) = |\mathcal{A}^r|, \quad 2^{r-1}\left|B^r\right|\prod_{i=1}^{r}2\cos\left(\theta_i/2\right) = |\mathcal{B}^r|, \qquad (11.175)$$

and plugging this property in (11.174) one sees that (11.172) is just $h_{s-1}(J) - h_{s-2}(J)$. Since the first term of (11.172) equals $h_s(J) - h_{s-2}(J)$ by virtue of (11.150), this proves (11.171).

Even D

The character of an irreducible representation of $\mathfrak{so}(2r+2)$ with (dominant) highest-weight $\lambda = (\lambda_1 + 1/2, \ldots, \lambda_{r+1} + 1/2)$ can be written as [49, p. 258–259]

$$\chi_\lambda^{(2r+2)}[\theta_1,\ldots,\theta_r] = \frac{\left|\cos\left[\left(\lambda_i + r - i + \frac{3}{2}\right)\theta_j\right]\right|}{\left|\cos\left[(r-i+1)\,\theta_j\right]\right|} = \prod_{i=1}^{r+1}2\cos\left(\frac{\theta_i}{2}\right)\frac{\left|\cos\left[\left(\lambda_i + r - i + \frac{3}{2}\right)\theta_j\right]\right|}{\left|\cos\left[\left(r - i + \frac{3}{2}\right)\theta_j\right]\right|}, \qquad (11.176)$$

where we are including the possibility of a non-zero angle θ_{r+1} (while in (11.95) we take $\theta_{r+1} = 0$). Taking into account (11.92), proving the second equality in (11.95) amounts to showing that

$$h_s(J) - h_{s-1}(J) = \left.\frac{\left|\cos\left[\left(\lambda_i + r - i + \frac{3}{2}\right)\theta_j\right]\right|}{\left|\cos\left[\left(r-i+\frac{3}{2}\right)\theta_j\right]\right|}\right|_{\theta_{r+1}=0} \qquad (11.177)$$

for $\lambda_i = s\delta_{i1}$. To prove this we proceed as in the odd-dimensional case: the difference of the bosonic character (11.153) and the right-hand side of (11.177),

$$\left.\frac{\left|\cos[(\lambda_i + r - i + 1)\theta_j]\right|}{\left|\cos[(r-i+1)\theta_j]\right|}\right|_{\theta_{r+1}=0} - \left.\frac{\left|\cos[(\lambda_i + r - i + \frac{3}{2})\theta_j]\right|}{\left|\cos[(r-i+\frac{3}{2})\theta_j]\right|}\right|_{\theta_{r+1}=0}, \qquad (11.178)$$

[5] See e.g. [49, p.259].

can be written as

$$\left[\frac{\sum_{k=1}^{r+1}(-1)^{k+1}\cos[(s+r)\theta_k]B^r[\theta_k]}{|B^{r+1}|} - \frac{\sum_{k=1}^{r+1}(-1)^{k+1}2\cos[(s+r+\frac{1}{2})\theta_k]\mathcal{B}^r[\theta_k]}{|\mathcal{B}^{r+1}|}\right]_{\theta_{r+1}=0} \tag{11.179}$$

upon expanding the determinants along the first row and using the notation (11.156)–(11.173). One can then verify that this reduces to $h_{s-1}(J) - h_{s-2}(J)$ by the same argument as in the odd-dimensional case. By virtue of the second equality in (11.153), this proves (11.177).

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Chapter 12
Conclusion

We have now completed our survey of the group-theoretic aspects of three-dimensional gravity, and in particular of BMS symmetry in three dimensions. In this conclusion we take one last look at what we have achieved.

Quantum Symmetries

The overarching theme of this thesis has been group theory and its application to quantum systems with symmetries. Accordingly, parts I and II of this thesis were devoted to a broad overview of group representations and geometry. In particular we have motivated and introduced central extensions, defined induced representations, applied them to semi-direct products and relativistic symmetry groups, and explained how classical mechanical systems with symmetries become symmetric quantum systems upon "replacing Poisson brackets by commutators". We have also applied some of these tools to the Virasoro group — the symmetry group of two-dimensional conformal field theories — and used it to analyse certain properties of three-dimensional gravity on Anti-de Sitter backgrounds.

$\mathbf{BMS_3}$ Particles

Part III of the thesis was devoted to the application of group theoretic methods to the study of asymptotically flat quantum gravity in three dimensions. In that context our weapon of choice has been the BMS group in three dimensions, which we have introduced as an asymptotic symmetry group in Chap. 9, before working out its abstract definition independently of gravity. We have seen in particular that it enjoys an exceptional structure of the type $G \ltimes \mathfrak{g}$, where G is the Virasoro group spanned by superrotations while $\mathfrak{g}$ is its Lie algebra, spanned by supertranslations. A crucial implication of this structure was that irreducible unitary representations of the BMS_3 group, i.e. BMS_3 particles, have supermomenta that span coadjoint orbits of the Virasoro group. This observation has allowed us to classify all such representations thanks to the classification of Virasoro orbits exposed earlier, in Chap. 7.

B. Oblak, *BMS Particles in Three Dimensions*, Springer Theses,
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We also observed that supermomentum orbits have a straightforward interpretation in gravity, since supermomenta coincide with Bondi mass aspects of asymptotically flat space-time metrics. As a result we interpreted BMS_3 particles in two equivalent ways: (i) as quantizations of orbits of asymptotically flat metrics under BMS_3 transformations, and (ii) as relativistic particles dressed with gravitational boundary degrees of freedom. These topological degrees of freedom are the three-dimensional analogue of soft gravitons, so a BMS_3 particle is effectively a particle dressed with soft gravitons.

As a confirmation of this interpretation, we evaluated BMS_3 characters and showed that they coincide with gravitational one-loop partition functions. On the group-theoretic side this computation involves the Frobenius character formula (4.33), which is essentially an integral of little group characters over a supermomentum orbit. Remarkably, for non-zero angular potentials, we found that the integral localizes to a single point on the supermomentum orbit, which allowed us to evaluate characters exactly. On the field-theoretic side the one-loop partition function was evaluated using heat kernel methods which turn out, unsurprisingly, to be more tractable in flat space than in Anti-de Sitter space.

Higher Spins and Supergravity

The study of thermodynamics has led us into higher-spin theories, whose partition functions in flat space could be computed with little extra effort compared to the gravitational case. We have used this computation as an excuse to investigate the unitary representations of asymptotic symmetry algebras that occur in that context, with methods and results very similar to those of the purely gravitational setting. A striking aspect of these considerations was the fact that it enabled us to compare ultrarelativistic and non-relativistic limits of conformal higher spins in a way that pure gravity does not allow. In doing so we uncovered a sharp difference between the two limits at the quantum level. As a corollary we concluded that flat space holography should not be described as a Galilean conformal field theory.

We have similarly studied the supersymmetric generalization of BMS_3 symmetry, whose unitary representations are essentially super BMS_3 multiplets consisting of an infinite tower of BMS_3 particles with increasing spins.

Is This Quantum Gravity?

In the introduction of the thesis we motivated the study of BMS symmetry by presenting it as a way to tackle the quantization of gravity. This is a good moment to ask to what extent this proposal has succeeded. To begin, we should realize that what we have done *is*, indeed, a partial quantization of gravity: we have described almost explicitly a family of Hilbert spaces endowed with operator algebras inherited from gravitational symmetries, and we have used these Hilbert spaces to compute concrete quantities such as partition functions. In this sense we have actually worked with a partial version of quantum gravity.

This being said, one should not be overly enthusiastic about what we have achieved: in essence we have worked out the flat space analogue of results that were mostly already known in the framework of AdS_3/CFT_2. In fact, in many cases our results were flat limits of their AdS peers, although the BMS_3 approach often led us to consider slightly different questions and use somewhat different methods than those suggested by conformal symmetries. Thus we have indeed made progress in our understanding of flat space holography, but whether this opens new doors towards quantum gravity is a whole other matter.

There are two simple arguments that show why our work is not *quite* quantum gravity. First, what we have studied are *irreducible* unitary representations of asymptotic symmetry groups, while realistic gravitational systems are expected to form highly reducible representations. In essence, saying that we have studied quantum gravity by studying irreducible representations of BMS_3 would be tantamount to saying that relativistic one-particle quantum mechanics is the same as quantum field theory, which is of course untrue. Secondly, one should realize that our study of the symmetries of gravity hasn't taught us anything about the microscopic details of gravity itself. For instance, the fact that the phase space of gravity forms the coadjoint representation of the asymptotic symmetry group is merely a restatement of the fact that momentum maps belong to the coadjoint representation, which is a robust feature of all symmetric phase spaces; it does not tell us anything about the details of gravity.

A Look Forward

Our observations on BMS particles in three dimensions have allowed us to describe dressed particles in a group-theoretic framework. While we haven't described interacting particles in this thesis, it is likely that our methods do apply to such cases as well. In particular, describing scattering phenomena in terms of BMS particles instead of standard (naked) particles should incorporate soft graviton contributions. In three dimensions this could presumably be used to describe, say, the merger of two particles into a flat space cosmology; optimistically, BMS_3 representations might then even account for gravitational quantum corrections to such amplitudes! In four dimensions the situation is much less well understood, for reasons that we alluded to earlier. In that case the problem is much more basic, since the very definition of BMS symmetry is elusive — let alone its quantum representations.

A related project is the description of BMS world lines. The reader may recall that we described in Chap. 5 a general procedure for building world line actions associated with arbitrary Lie groups, and that the application of this method to the Poincaré group resulted in relativistic world lines. It is tempting to ask what happens when that approach is applied to the BMS_3 group; the answer is very natural: the resulting action principle describes world lines propagating in the space of supertranslations, which can equivalently be seen as relativistic world lines dressed with gravitational degrees of freedom. Yet another way to think of these world-lines is to interpret them as two-dimensional field theories dual to three-dimensional asymptotically flat gravity, and indeed one finds that their partition functions coincide with gravitational (one-loop) partition functions. These considerations should appear soon in a separate publication [1]. Note that, as before, the application of these ideas to BMS in *four* dimensions is much more problematic due to the lack of a proper definition of BMS_4 symmetry.

There is undoubtedly much more to be done in the future, both in toy models such as three-dimensional gravity and in real-world, four-dimensional systems. In the wake of the experimental observation of gravitational waves [2], it is likely that new tools and methods will soon be required to understand and study gravity, both classically and quantum-mechanically. On a more philosophical note, it is remarkable that a question seemingly as simple as "what is a particle?" has an answer as intricate and rich as what we have been attempting to describe in this thesis. We hope to have contributed to a partial solution to the problem, and look forward to investigating some of its future applications.

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Index

B. Oblak, *BMS Particles in Three Dimensions*, Springer Theses,
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